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TENSE POLYADIC $n \times m-$VALUED ŁUKASIEWICZ–MOISIL ALGEBRAS

Abstract

In 2015, A.V. Figallo and G. Pelaitay introduced tense $n \times m$-valued Łukasiewicz-Moisil algebras, as a common generalization of tense Boolean algebras and tense n-valued Łukasiewicz-Moisil algebras. Here we initiate an investigation into the class $\mathbf{tpLM}_{n \times m}$ of tense polyadic $n \times m$-valued Łukasiewicz-Moisil algebras. These algebras constitute a generalization of tense polyadic Boolean algebras introduced by Georgescu in 1979, as well as the tense polyadic n-valued Łukasiewicz-Moisil algebras studied by Chiriţă in 2012. Our main result is a representation theorem for tense polyadic $n \times m$-valued Łukasiewicz-Moisil algebras.

1. Introduction

In 1962, polyadic Boolean algebras were defined by Halmos as algebraic structures of classical predicate logic. One of the main results in the theory of polyadic Boolean algebras is Halmos representation theorem (see [22]). This result is the algebraic counterpart of Gödel's completeness theorem for predicate logic. This subject caused great interest and led several authors to deepen and generalized the algebras defined by Halmos, to such an extent that research is still being conducted in this direction. For instance, the classes of polyadic Heyting algebras ([25]), polyadic MV-algebras ([30]), polyadic BL-algebras ([12]), polyadic θ-valued Łukasiewicz-Moisil algebras ([1]), polyadic GMV-algebras ([23]), to mention a few.

Tense classical logic is an extension of the classical logic obtained by adding to the bivalent logic the tense operators G (it is always going to be the case that) and H (it has always been the case that). Taking into account that tense algebras (or tense Boolean algebras) constitute the algebraic basis for the tense bivalent logic (see [4]), Georgescu introduced in [21] the tense polyadic algebras as algebraic structures for tense classical predicate logics. They are obtained by endowing a polyadic Boolean algebra with the tense operators G and H. On the other hand, the study of tense Łukasiewicz-Moisil algebras (or tense LM_n-algebras) and tense MV-algebras introduced by Diaconescu and Georgescu in [11] has been proven of importance (see [2, 5, 7, 8, 9, 15, 6, 16, 19]). In particular, in [8], Chiriţă, introduced tense θ-valued Łukasiewicz-Moisil algebras and proved a representation theorem which allowed to show the completeness of the tense θ-valued Moisil logic (see [7]). In [11], the authors formulated an open problem about representation of tense MV-algebras, this problem was solved in [26, 3] for semisimple tense MV-algebras. Also, in [2], were studied tense basic algebras which are an interesting generalization of tense MV-algebras.

Tense MV-algebras and tense LM_n-algebras can be considered the algebraic framework for some tense many-valued propositional calculus (tense Łukasiewicz logic and tense Moisil logic). Another open problem proposed in [11] is to develop the corresponding predicate logics and to study their algebras. On the other hand, polyadic MV-algebras, introduce in [30] (resp. polyadic LM_n-algebras [1]), constitute the algebraic counterpart of Łukasiewicz predicate logic (resp. Moisil predicate logic). Then, we can define tense polyadic MV-algebras (resp. tense polyadic LM_n-algebras [10]) as algebraic structures corresponding to tense Łukasiewicz predicate logic (resp. tense Moisil predicate logic).

In 1975 W. Suchoń ([31]) defined matrix Łukasiewicz algebras so generalizing n-valued Łukasiewicz algebras without negation ([24]). In 2000, A. V. Figallo and C. Sanza ([13]) introduced $n \times m$-valued Łukasiewicz algebras with negation which are both a particular case of matrix Łukasiewicz algebras and a generalization of n-valued Łukasiewicz-Moisil algebras ([1]). It is worth noting that unlike what happens in n-valued Łukasiewicz-Moisil algebras, generally the De Morgan reducts of $n \times m$-valued Łukasiewicz algebras with negation are not Kleene algebras. Furthermore, in [28] an important example which legitimated the study of this new class of algebras is provided. Following the terminology established in [1], these algebras were called $n \times m$– valued Łukasiewicz-Moisil algebras (or $LM_{n \times m}$-algebras for

short). $\mathrm{LM}_{n \times m}$-algebras were studied in [17, 27, 28, 29] and [14]. In particular, in [17] the authors introduced the class of monadic $n \times m$-valued Łukasiewicz-Moisil algebras, namely $n \times m$-valued Łukasiewicz-Moisil algebras endowed with a unary operation called *existential quantifier*. These algebras constitute a commom generalization of monadic Boolean algebras and monadic n-valued Łukasiewicz-Moisil algebras ([20]).

On the other hand, an important question proposed in [11] is to investigate the representation of tense polyadic LM_n-algebras and the completeness of their logical system. Taking into acount these problems, in the present paper, we introduce and investigate tense polyadic $n \times m$-valued Łukasiewicz-Moisil algebras, structures that generalize the tense polyadic Boolean algebras, as well as the tense polyadic n-valued Łukasiewicz-Moisil algebras. Our main result is a representation theorem for tense polyadic $n \times m$-valued Łukasiewicz-Moisil algebras.

The paper is organized as follows: in section 2, we briefly summarize the main definitions and results needed throughout the paper. In section 3, we define the class of polyadic $n \times m$-valued Łukasiewicz-Moisil algebras. The main result of this section is a representation theorem for polyadic $n \times m$-valued Łukasiewicz-Moisil algebras. In section 4, we introduced the class of tense polyadic $n \times m$-valued Łukasiewicz-Moisil algebras as a common generalization of tense polyadic Boolean algebras and tense polyadic n-valued Łukasiewicz-Moisil algebras. Finally, in section 5, we give a representation theorem for tense polyadic $n \times m$-valued Łukasiewicz-Moisil algebras. It extends the representation theorem for tense polyadic Boolean algebras, as well as the representation theorem for tense n-valued Łukasiewicz-Moisil algebras.

2. Preliminaries

2.1. $n \times m$-valued Łukasiewicz-Moisil algebras

In this subsection we recall the definition of $n \times m$-valued Łukasiewicz-Moisil algebras and some constructions regarding the relationship between these algebras and Boolean algebras.

In [28], $n \times m$-valued Łukasiewicz-Moisil algebras (or $\mathrm{LM}_{n \times m}$-algebras), in which n and m are integers, $n \geq 2$, $m \geq 2$, were defined as algebras

$$\mathcal{L} = \langle L, \vee, \wedge, \sim, (\sigma_{ij})_{(i,j) \in (n \times m)}, 0_L, 1_L \rangle$$

where $(n \times m)$ is the cartesian product $\{1, \ldots, n-1\} \times \{1, \ldots, m-1\}$, the reduct $\langle L, \vee, \wedge \sim, 0_L, 1_L \rangle$ is a De Morgan algebra and $(\sigma_{ij})_{(i,j) \in (n \times m)}$ is a family of unary operations on L verifying the following conditions for all $(i, j), (r, s) \in (n \times m)$ and $x, y \in L$:

(C1) $\sigma_{ij}(x \vee y) = \sigma_{ij}x \vee \sigma_{ij}y$,
(C2) $\sigma_{ij}x \leq \sigma_{(i+1)j}x$,
(C3) $\sigma_{ij}x \leq \sigma_{i(j+1)}x$,
(C4) $\sigma_{ij}\sigma_{rs}x = \sigma_{rs}x$,
(C5) $\sigma_{ij}x = \sigma_{ij}y$ for all $(i, j) \in (n \times m)$ imply $x = y$,
(C6) $\sigma_{ij}x \vee \sim \sigma_{ij}x = 1_L$,
(C7) $\sigma_{ij}(\sim x) = \sim \sigma_{(n-i)(m-j)}x$.

DEFINITION 2.1. Let $\mathcal{L} = \langle L, \vee, \wedge, \sim, (\sigma_{ij})_{(i,j) \in (n \times m)}, 0_L, 1_L \rangle$ be an $\mathrm{LM}_{n \times m}$-algebra. We say that L is *complete* if the lattice $\langle L, \vee, \wedge, 0_L, 1_L \rangle$ is complete.

DEFINITION 2.2. Let $\mathcal{L} = \langle L, \vee, \wedge, \sim, (\sigma_{ij})_{(i,j) \in (n \times m)}, 0_L, 1_L \rangle$ be an $\mathrm{LM}_{n \times m}$-algebra. We say that L is *completely chrysippian* if, for every $\{x_k\}_{k \in K}$ ($x_k \in L$ for all $k \in K$) such that $\bigwedge_{k \in K} x_k$ and $\bigvee_{k \in K} x_k$ exist, the following properties hold: $\sigma_{ij}(\bigwedge_{k \in K} x_k) = \bigwedge_{k \in K} \sigma_{ij}(x_k)$, $\sigma_{ij}(\bigvee_{k \in K} x_k) = \bigvee_{k \in K} \sigma_{ij}(x_k)$ (for all $(i, j) \in (n \times m)$).

Let $\mathcal{L} = \langle L, \vee, \wedge, \sim, (\sigma_{ij})_{(i,j) \in (n \times m)}, 0_L, 1_L \rangle$ be an $\mathrm{LM}_{n \times m}$-algebra. We will denote by $C(L)$ the set of the complemented elements of L. In [28], it was proved that $C(L) = \{x \in L \mid \sigma_{ij}(x) = x, \text{ for any } (i, j) \in (n \times m)\}$. These elements will play an important role in what follows.

DEFINITION 2.3. Let $\mathcal{L}_1 = \langle L_1, \vee, \wedge, \sim, (\sigma_{ij})_{(i,j) \in (n \times m)}, 0_{L_1}, 1_{L_1} \rangle$ and $\mathcal{L}_2 = \langle L_2, \vee, \wedge, \sim, (\sigma_{ij})_{(i,j) \in (n \times m)}, 0_{L_2}, 1_{L_2} \rangle$ be two $\mathrm{LM}_{n \times m}$−algebras. A *morphism* of $\mathrm{LM}_{n \times m}$-algebras is a function $f : L_1 \longrightarrow L_2$ such that, for all $x, y \in L_1$ and $(i, j) \in (n \times m)$, we have

(a) $f(0_{L_1}) = 0_{L_2}$, $f(1_{L_1}) = 1_{L_2}$,
(b) $f(x \vee y) = f(x) \vee f(y)$, $f(x \wedge y) = f(x) \wedge f(y)$,
(c) $f \circ \sigma_{ij} = \sigma_{ij} \circ f$,
(d) $f(\sim x) = \sim f(x)$.

REMARK 2.4. Let us observe that condition (d) in Definition 2.3 is a direct consequence of (C5), (C7) and the conditions (a) to (c).

EXAMPLE 2.5. Let $\mathcal{B} = \langle B, \vee, \wedge, \neg, 0_B, 1_B \rangle$ be a Boolean algebra. The set $B \uparrow^{(n \times m)} = \{ f \mid f : (n \times m) \longrightarrow B$ such that for arbitraries i, j if $r \leq s$, then $f(r, j) \leq f(s, j)$ and $f(i, r) \leq f(i, s) \}$ of increasing functions in each component from $(n \times m)$ to B can be made into an $\text{LM}_{n \times m}$-algebra

$$D(B) = \langle B \uparrow^{(n \times m)}, \vee, \wedge, \sim, (\sigma_{ij})_{(i,j) \in (n \times m)}, 0_{B \uparrow^{(n \times m)}}, 1_{B \uparrow^{(n \times m)}} \rangle$$

where $0_{B \uparrow^{(n \times m)}}, 1_{B \uparrow^{(n \times m)}} : (n \times m) \longrightarrow B$ are defined by $0_{B \uparrow^{(n \times m)}}(i, j) = 0_B$ and $1_{B \uparrow^{(n \times m)}}(i, j) = 1_B$, for every $(i, j) \in (n \times m)$, the operations of the lattice $\langle B \uparrow^{(n \times m)}, \vee, \wedge \rangle$ are defined pointwise and $(\sigma_{ij} f)(r, s) = f(i, j)$ for all $(r, s) \in (n \times m)$, $(\sim f)(i, j) = \neg f(n - i, m - j)$ for all $(i, j) \in (n \times m)$ (see [28, Proposition 3.2.]).

Let $\mathcal{B}, \mathcal{B}'$ be two Boolean algebras, $g : B \longrightarrow B'$ be a Boolean morphism and $D(B)$ and $D(B')$ be the corresponding $\text{LM}_{n \times m}$-algebras. We define the function $D(g) : D(B) \longrightarrow D(B')$ in the following way: $D(g)(u) = g \circ u$, for every $u \in D(B)$. Then, the function $D(g) : D(B) \longrightarrow D(B')$ is a morphism of $\text{LM}_{n \times m}$-algebras. We will denote by **B** the category of Boolean algebras and by $\mathbf{LM_{n \times m}}$ the category of $\text{LM}_{n \times m}$-algebras. Then, the assignment $B \mapsto D(B)$, $g \mapsto D(g)$ defines a covariant functor $D : \mathbf{B} \longrightarrow \mathbf{LM_{n \times m}}$.

DEFINITION 2.6. Let $\mathcal{L} = \langle L, \vee, \wedge, \sim, (\sigma_{ij})_{(i,j) \in (n \times m)}, 0_L, 1_L \rangle$ be an $\text{LM}_{n \times m}$-algebra. A non-empty subset M of L is an $n \times m$-*ideal* of L, if M is an ideal of the lattice $\langle L, \vee, \wedge, 0_L, 1_L \rangle$ which verifies this condition: $x \in M$ implies $\sigma_{11}(x) \in M$.

2.2. Tense Boolean algebras

Tense Boolean algebras are algebraic structures for tense classical propositional logic. In this logic there exist two tense operators G (*it is always going to be the case that*) and H (*it has always been the case that*). We will recall the basic definitions of tense Boolean algebras (see [21, 9]).

DEFINITION 2.7. A *tense Boolean algebra* is a triple $(\mathcal{B}, G, H)$ such that $\mathcal{B} = \langle B, \vee, \wedge, \neg, 0_B, 1_B \rangle$ is a Boolean algebra and G and H are two unary operations on B such that:

1. $G(1_B) = 1_B$, $H(1_B) = 1_B$,
2. $G(x \wedge y) = G(x) \wedge G(y)$, $H(x \wedge y) = H(x) \wedge H(y)$.

DEFINITION 2.8. Let $\mathcal{B} = \langle B, \vee, \wedge, \neg, G, H, 0_B, 1_B \rangle$ and $\mathcal{B}' = \langle B', \vee, \wedge, \neg, G', H', 0_{B'}, 1_{B'} \rangle$ be two tense Boolean algebras. A function $f : B \longrightarrow B'$ is a *morphism of tense Boolean algebras* if f is a Boolean morphism and it satisfies the following conditions: $f(G(x)) = G'(f(x))$ and $f(H(x)) = H'(f(x))$, for any $x \in B$.

2.3. Tense Polyadic Boolean algebras

The tense polyadic Boolean algebras were introduced in [21] as algebraic structures for tense classical predicate logic.

Let U be a non-empty set throughout this paper.

DEFINITION 2.9. A *tense polyadic Boolean algebra* is a sextuple $(\mathcal{B}, U, S, \exists, G, H)$ such that the following properties hold:

(i) $(\mathcal{B}, U, S, \exists)$ is a polyadic Boolean algebra (see [22]),
(ii) $(\mathcal{B}, G, H)$ is a tense Boolean algebra (see Definition 2.7),
(iii) $S(\tau)(G(p)) = G(S(\tau)(p))$, for any $\tau \in U^U$ and $p \in B$,
(iv) $S(\tau)(H(p)) = H(S(\tau)(p))$, for any $\tau \in U^U$ and $p \in B$.

We shall recall now the construction of the example of tense polyadic Boolean algebra from [21].

DEFINITION 2.10. A *tense system* has the form $\mathcal{T} = (T, (X_t)_{t \in T}, R, Q, 0)$, where

(i) T is an arbitrary non-empty set,
(ii) R and Q are two binary relations on T,
(iii) $0 \in T$,
(iv) X_t is a non-empty set for every $t \in T$, with the following property:
      If tRs or tQs, then $X_t \subseteq X_s$ for every $t, s \in T$.

Recall that the algebra $\mathbf{2} = (\{0, 1\}, \vee = \max, \wedge = \min, \neg, 0, 1) = (\{0, 1\}, \rightarrow, \neg, 1)$, where $\neg x = 1 - x$, $x \rightarrow y = \max(\neg x, y)$, for $x, y \in \{0, 1\}$ is a Boolean algebra, called the standard Boolean algebra (see [21]).

Let $\mathcal{T}$ be a tense system and $\mathbf{2}$ be the standard Boolean algebra with two elements. We denote by

$$F_T^U = \{(f_t)_{t \in T} \mid f_t : X_t^U \longrightarrow \mathbf{2}, \text{ for every } t \in T\}.$$

On F_T^U we will consider the following operations:

(pb1) $(f_t)_{t \in T} \to (g_t)_{t \in T} = (f_t \to g_t)_{t \in T}$, where $(f_t \to g_t)(x) = f_t(x) \to g_t(x)$,

for all $x \in X_t^U$,

(pb2) $\neg(f_t)_{t \in T} = (\neg f_t)_{t \in T}$, where $(\neg f_t)(x) = \neg(f_t(x))$, for all $x \in X_t^U$,

(pb3) $1^{\mathcal{T}} = (1_t)_{t \in T}$, where $1_t : X_t^U \longrightarrow \mathbf{2}$, $1_t(x) = 1$, for all $t \in T$ and $x \in X_t^U$.

LEMMA 2.11. (Georgescu [21]) $\mathcal{F}_T^U = (F_T^U, \to, \neg, 1^{\mathcal{T}})$ *is a Boolean algebra.*

On F_T^U we consider the tense operators G and H, by:

(pb4) $G((f_t)_{t \in T}) = (g_t)_{t \in T}$, $g_t : X_t^U \longrightarrow \mathbf{2}$, $g_t(x) = \bigwedge\{f_s(i \circ x) \mid tRs, s \in T\}$,

(pb5) $H((f_t)_{t \in T}) = (h_t)_{t \in T}$, $h_t : X_t^U \longrightarrow \mathbf{2}$, $h_t(x) = \bigwedge\{f_s(i \circ x) \mid tQs, s \in T\}$,

where $i : X_t \longrightarrow X_s$ is the inclusion map.

LEMMA 2.12. (Georgescu [21]) $(\mathcal{F}_T^U, G, H)$ *is a tense Boolen algebra.*

On $\mathcal{F}_T^U$ we shall consider now the following functions.

(pb6) For any $\tau \in U^U$, we define $S(\tau) : F_T^U \longrightarrow F_T^U$ by $S(\tau)((f_t)_{t \in T}) = (g_t)_{t \in T}$, where $g_t : X_t^U \longrightarrow \mathbf{2}$, $g_t(x) = f_t(x \circ \tau)$, for every $t \in T$ and $x \in X_t^U$,

(pb7) For any $J \subseteq U$, we consider the function $\exists(J) : F_T^U \longrightarrow F_T^U$, defined by

$\exists(J)((f_t)_{t \in T}) = (g_t)_{t \in T}$, where $g_t : X_t^U \longrightarrow \mathbf{2}$ is defined by:

$g_t(x) = \bigvee\{f_t(y) \mid y \in X_t^U, y \mid_{U \setminus J} = x \mid_{U \setminus J}\}$, for every $x \in X_t^U$.

LEMMA 2.13. (Georgescu [21]) $(\mathcal{F}_T^U, U, S, \exists, G, H)$ *is a tense polyadic Boolean algebra.*

DEFINITION 2.14. Let $(\mathcal{B}, U, S, \exists, G, H)$ be a tense polyadic Boolean algebra. A subset J of U is a *support* of $p \in B$ if $\exists (U \setminus J) p = p$. The intersection of the supports of an element $p \in B$ will be denoted by J_p. A tense polyadic Boolean algebra is *locally finite* if every element has a finite support. The *degree* of $(\mathcal{B}, U, S, \exists, G, H)$ is the cardinality of U.

THEOREM 2.15. (Georgescu [21]) *Let $(\mathcal{B}, U, S, \exists, G, H)$ be a locally finite tense polyadic Boolean algebra of infinite degree and Γ be a proper filter of B such that $J_p = \emptyset$, for any $p \in \Gamma$. Then there exist a tense system $\mathcal{T} = (T, (X_t)_{t \in T}, R, Q, 0)$ and a morphism of tense polyadic Boolean algebras $\Phi : B \longrightarrow F_T^U$, such that, for every $p \in \Gamma$, we have: $\Phi(p) = (f_t)_{t \in T}$ implies $f_0(x) = 1$, for all $x \in X_t^U$.*

2.4. Tense $n \times m$-valued Łukasiewicz-Moisil algebras

The tense $n \times m$-valued Łukasiewicz-Moisil algebras were introduced by A. V. Figallo and G. Pelaitay in [18], as a common generalization of tense Boolean algebras [21] and tense n-valued Łukasiewicz-Moisil algebras [10].

DEFINITION 2.16. A *tense $n \times m$-valued Łukasiewicz-Moisil algebra* (or tense $\mathrm{LM}_{n \times m}$-algebra) is a triple$(\mathcal{L}, G, H)$ such that $\mathcal{L} = \langle L, \vee, \wedge, \sim, (\sigma_{ij})_{(i,j) \in (n \times m)}, 0_L, 1_L \rangle$ is an $\mathrm{LM}_{n \times m}$-algebra and for all $x, y \in L$,

1. $G(1_L) = 1_L$, $H(1_L) = 1_L$,
2. $G(x \wedge y) = G(x) \wedge G(y)$, $H(x \wedge y) = H(x) \wedge H(y)$,
3. $G(\sigma_{ij}(x)) = \sigma_{ij}(G(x))$, $H(\sigma_{ij}(x)) = \sigma_{ij}(H(x))$, for any $(i, j) \in (n \times m)$.

DEFINITION 2.17. Let $(\mathcal{L}, G, H)$ and $(\mathcal{L}', G, H)$ be two tense $\mathrm{LM}_{n \times m}$-algebras. A function $f : L \longrightarrow L'$ is a *morphism of tense $\mathrm{LM}_{n \times m}$-algebras* if f is a $\mathrm{LM}_{n \times m}$-morphism and it satisfies the following conditions: $f(G(x)) = G'(f(x))$ and $f(H(x)) = H'(f(x))$, for any $x \in L$.

3. Polyadic $n \times m$-valued Łukasiewicz-Moisil algebras

In this section we will introduce the polyadic $\mathrm{LM}_{n \times m}$-algebras as a common generalization of polyadic Boolean algebras and polyadic LM_n-algebras. We will recall from [17] the definition of monadic $n \times m$-valued Łukasiewicz-Moisil algebras which we will use in this section.

DEFINITION 3.1. A *monadic $n \times m$-valued Łukasiewicz-Moisil algebra* (or monadic $\mathrm{LM}_{n \times m}$-algebra) is a pair $(\mathcal{L}, \exists)$ where $\mathcal{L} = \langle L, \vee, \wedge, \sim, \{\sigma_{ij}\}_{(i,j) \in (n \times m)}, 0_L, 1_L \rangle$ is an $\mathrm{LM}_{n \times m}$-algebra and $\exists$ is a unary operation on L verifying the following conditions for all $(i, j) \in (n \times m)$ and $x, y \in L$:

(E1) $\exists 0 = 0$,
(E2) $x \wedge \exists x = x$,
(E3) $\exists (x \wedge \exists y) = \exists x \wedge \exists y$,
(E4) $\sigma_{ij}(\exists x) = \exists(\sigma_{ij} x)$.

REMARK 3.2. These algebras, for the case $m = 2$, they coincide with monadic n-valued Łukasiewicz-Moisil algebras introduced by Georgescu and Vraciu in [20].

DEFINITION 3.3. A *polyadic $n \times m$-valued Łukasiewicz-Moisil algebra* (or polyadic $\mathrm{LM}_{n \times m}$-algebra) is a quadruple $(\mathcal{L}, U, S, \exists)$ where $\mathcal{L} = \langle L, \vee, \wedge, \sim, \{\sigma_{ij}\}_{(i,j) \in (n \times m)}, 0_L, 1_L \rangle$ is an $\mathrm{LM}_{n \times m}$-algebra, S is a function from U^U to the set of endomorphisms of L and $\exists$ is a function from $\mathcal{P}(U)$ to L^L, such that the following axioms hold:

(i) $S(1_U) = \mathbf{1_{LL}}$,
(ii) $S(\rho \circ \tau) = S(\rho) \circ S(\tau)$, for every $\rho, \tau \in U^U$,
(iii) $\exists(\emptyset) = \mathbf{1_{LL}}$,
(iv) $\exists(J \cup J') = \exists(J) \circ \exists(J')$, for every $J, J' \subseteq U$,
(v) $S(\rho) \circ \exists(J) = S(\tau) \circ \exists(J)$, for every $J \subseteq U$ and for every $\rho, \tau \in U^U$ such that $\rho \mid_{U \setminus J} = \tau \mid_{U \setminus J}$,
(vi) $\exists(J) \circ S(\rho) = S(\rho) \circ \exists(\rho^{-1}(J))$ such that $J \subseteq U$ and for every $\rho \in U^U$ such that $\rho \mid_{\rho^{-1}(J)}$ is injective,
(vii) for every $J \subseteq U$, the pair $(\mathcal{L}, \exists(J))$ is a monadic $\mathrm{LM}_{n \times m}$-algebra.

DEFINITION 3.4. Let $(\mathcal{L}, U, S, \exists)$ and $(\mathcal{L}', U, S, \exists)$ be two polyadic $\mathrm{LM}_{n \times m}$-algebras. A function $f : L \longrightarrow L'$ is a *morphism of polyadic $\mathrm{LM}_{n \times m}$-algebras* if f is a morphism of $\mathrm{LM}_{n \times m}$-algebras and $f \circ S(\rho) = S(\rho) \circ f$, $f \circ \exists(J) = \exists(J) \circ f$, for every $\rho \in U^U$ and $J \subseteq U$.

REMARK 3.5. If $(\mathcal{L}, U, S, \exists)$ is a polyadic $\mathrm{LM}_{n \times m}$-algebra, then $C(L)$ can be endowed with a canonical structure of polyadic Boolean algebra. Every

polyadic $LM_{n \times m}$-morphism $f : (\mathcal{L}, U, S, \exists) \longrightarrow (\mathcal{L}', U, S, \exists)$ induces a morphism of polyadic Boolean algebras $C(f) : (C(L), U, S, \exists) \longrightarrow (C(L'), U, S, \exists)$. In this way we have defined a functor from the category $\mathbf{PLM}_{n \times m}$ of polyadic $LM_{n \times m}$-algebras to the category $\mathbf{PB}$ of polyadic Boolean algebras.

REMARK 3.6. The notion of polyadic $LM_{n \times m}$-subalgebra is defined in a natural way.

DEFINITION 3.7. Let $(\mathcal{L}, U, S, \exists)$ be a polyadic $LM_{n \times m}$-algebra and $a \in L$. A subset J of U is a *support* of a if $\exists(U \setminus J)a = a$. A polyadic $LM_{n \times m}$-algebra is *locally finite* if every element has a finite support. The *degree* of $(\mathcal{L}, U, S, \exists)$ is the cardinality of U.

LEMMA 3.8. *Let $(\mathcal{L}, U, S, \exists)$ be a polyadic $LM_{n \times m}$-algebra, $a \in L$ and $J \subseteq U$. If $card(U) \geq 2$, then the following conditions are equivalent:*

  (i) *J is a support of a,*
 (ii) *$\forall(U \setminus J)a = a$, where $\forall := \sim \circ \exists \circ \sim$,*
(iii) *$\rho \mid_{U \setminus J} = \tau \mid_{U \setminus J}$ implies $S(\rho)a = S(\tau)a$,*
 (iv) *$\rho \mid_{U \setminus J} = 1_{U \setminus J}$ implies $S(\rho)a = a$,*
  (v) *for every $(i, j) \in (n \times m)$, J is a support of $\sigma_{ij}(a)$ in the polyadic Boolean algebra $C(L)$.*

PROOF: It is routine.

In the rest of this section, by polyadic $LM_{n \times m}$-algebra we will mean a locally finite polyadic $LM_{n \times m}$-algebra of infinite degree.

EXAMPLE 3.9. Let $\mathcal{L} = \langle L, \vee, \wedge, \sim, \{\sigma_{ij}\}_{(i,j) \in (n \times m)}, 0_L, 1_L \rangle$ be a complete and completely chrysippian $LM_{n \times m}$-algebra, U an infinite set and $X \neq \emptyset$. The set $L^{(X^U)}$ of all functions from X^U to L has a natural structure of $LM_{n \times m}$-algebra. For every $J \subseteq U$ and $\tau \in U^U$ define two unary operations $\exists(J)$, $S(\tau)$ on $L^{(X^U)}$ by putting:

  - $\exists(J)(p(x)) = \bigvee \{p(y) \mid y \in X^U, y \mid_{U \setminus J} = x \mid_{U \setminus J}\}$,
  - $S(\tau)(p(x)) = p(x \circ \tau)$,

for any $p : X^U \longrightarrow L$, $\tau \in U^U$ and $J \subseteq U$. We can show that $L^{(X^U)}$ is a polyadic $\text{LM}_{n \times m}$-algebra.

DEFINITION 3.10. A polyadic $\text{LM}_{n \times m}$-subalgebra of $L^{(X^U)}$ will be called a *functional polyadic $\text{LM}_{n \times m}$-algebra*. Denote by $\mathcal{F}(X^U, L)$ the functional polyadic $\text{LM}_{n \times m}$-algebra of all elements of $L^{(X^U)}$ having a finite support.

REMARK 3.11. $\mathcal{F}(X^U, L)$ is locally finite.

PROPOSITION 3.12. *Let $(\mathcal{L}, U, S, \exists)$ be a complete and completely chrysippian $\text{LM}_{n \times m}$-algebra. For every $a \in L$, $p \in U^U$ and $J \subseteq U$ the following equality holds:* $S(\tau)\exists(J)a = \bigvee\{S(\rho)a \mid \rho\mid_{U \setminus J} = \tau \mid_{U \setminus J}\}$

PROOF: By [1, Proposition 4.24, pag. 50] we have

$$
\begin{aligned}
\sigma_{ij}S(\tau)\exists(J)a &= \exists(J)S(\tau)\sigma_{ij}a \\
&= \bigvee\{S(\rho)\sigma_{ij}a \mid \rho\mid_{U \setminus J} = \tau\mid_{U \setminus J}\} \\
&= \bigvee\{\sigma_{ij}S(\rho)a \mid \rho\mid_{U \setminus J} = \tau\mid_{U \setminus J}\} \\
&= \sigma_{ij}\left(\bigvee\{S(\rho)a \mid \rho\mid_{U \setminus J} = \tau\mid_{U \setminus J}\}\right)
\end{aligned}
$$

for every $(i, j) \in (n \times m)$. Applying (C5) we get the equality required. $\square$

REMARK 3.13. Let $(\mathcal{L}, U, S, \exists)$ be a polyadic $\text{LM}_{n \times m}$-algebra. Set $E_o(L) = \{a \in L \mid \emptyset$ is support of $a\}$. Then, we can prove that $E_o(L)$ is an $\text{LM}_{n \times m}$-subalgebra of L.

THEOREM 3.14. *Let $(\mathcal{L}, U, S, \exists)$ be a polyadic $\text{LM}_{n \times m}$-algebra and M a proper $n \times m$-filter of $E_o(L)$. Then there exist a non-empty set X and a polyadic $\text{LM}_{n \times m}$-morphism $\Phi : L \longrightarrow \mathcal{F}(X^U, D(\mathbf{2}))$ such that $\Phi(a) = 1$, for each $a \in M$.*

PROOF: Consider the polyadic Boolean algebra $(C(L), U, S, \exists)$ and denote by $E_o(C(L))$ the Boolean algebra of all elements of $C(L)$ having $\emptyset$ as support in $C(L)$, that is, $E_o(C(L)) = \{a \in C(L) : \emptyset$ is support of $a\}$. It is obvious that $E_o(C(L)) = E_o(L) \cap C(L)$ and $M_o = M \cap C(L)$ is a proper filter of the Boolean algebra $E_o(C(L))$. By [1, Theorem 4.28, pag.51] there exists a non-empty set X and a morphism of polyadic Boolean algebras $\Psi : C(L) \longrightarrow \mathcal{F}(X^U, \mathbf{2})$ such that $\Psi(a) = 1$ for each $a \in M_o$.

Define a map $\Phi : L \longrightarrow \mathcal{F}(X^U, D(\mathbf{2}))$ by putting $\Phi(a)(x)(i, j) = \Psi(\sigma_{ij}a)(x)$, for every $a \in L$, $x \in X^U$ and $(i, j) \in (n \times m)$. It is easy

to prove that Φ is a morphism of $\mathrm{LM}_{n\times m}$-algebras. For every $a \in L$, $J \subseteq U$, $\rho \in U^U$, $x \in X^U$ and $(i,j) \in (n \times m)$ we have:

(a) $\Phi(\exists(J)a)(x)(i,j) = \Psi(\sigma_{ij}\exists(J)a)(x)$
$$= \Psi(\exists(J)\sigma_{ij}a)(x)$$
$$= \exists(J)\Psi(\sigma_{ij}a)(x)$$
$$= \bigvee\{\Psi(\sigma_{ij}a)(y) \mid y \mid_{U\setminus J} = x \mid_{U\setminus J}\}$$
$$= \bigvee\{\Phi(a)(y) \mid y \mid_{U\setminus J} = x \mid_{U\setminus J}\}$$
$$= (\exists(J)\Phi(a))(x)(i,j),$$

(b) $\Phi(S(\tau)a)(x)(i,j) = \Psi(\sigma_{ij}S(\tau)a)(x)$
$$= \Psi(S(\tau)\sigma_{ij}a)(x)$$
$$= (S(\tau)\Psi(\sigma_{ij}a))(x)$$
$$= \Psi(\sigma_{ij}a)(x\tau)$$
$$= \Phi(a)(x\tau)(i,j)$$
$$= (S(\tau)\Phi(a))(x)(i,j).$$

By (a) and (b) we obtain that Φ is a polyadic $\mathrm{LM}_{n\times m}$-morphism. If $a \in M$ then $\sigma_{ij}a \in M_o$, therefore $\Psi(\sigma_{ij}a) = 1$ for each $(i,j) \in (n \times m)$. Thus $\Phi(a)(x)(i,j) = \Psi(\sigma_{ij}a)(x) = 1$ for every $x \in X^U$ and $(i,j) \in (n\times m)$. $\qquad\square$

4.   Tense polyadic $\mathrm{LM}_{n\times m}$-algebras

In this section we will introduce the tense polyadic $\mathrm{LM}_{n\times m}$-algebras as a common generalization of tense polyadic Boolean algebras and tense polyadic LM_n-algebras.

DEFINITION 4.1. A *tense polyadic $LM_{n\times m}$-algebra* is a sextuple $(\mathcal{L}, U, S, \exists, G, H)$ such that

(a) $(\mathcal{L}, U, S, \exists)$ is a polyadic $\mathrm{LM}_{n\times m}$-algebra,
(b) $(\mathcal{L}, G, H)$ is a tense $\mathrm{LM}_{n\times m}$-algebra,
(c) $S(\tau)(G(p)) = G(S(\tau))(p))$, for any $\tau \in U^U$ and $p \in L$,
(d) $S(\tau)(H(p)) = H(S(\tau))(p))$, for any $\tau \in U^U$ and $p \in L$.

DEFINITION 4.2. Let $(\mathcal{L}, U, S, \exists, G, H)$ and $(\mathcal{L}', U, S, \exists, G, H)$ be two tense polyadic $\mathrm{LM}_{n\times m}$-algebras. A function $f : L \longrightarrow L'$ is a *morphism of tense polyadic $LM_{n\times m}$-algebras* if the following properties hold:

(i) f is a morphism of polyadic $LM_{n \times m}$-algebras,

(ii) f is a morphism of tense $LM_{n \times m}$-algebras.

We are going to use the notion of tense system to give an example of tense polyadic $LM_{n \times m}$-algebra.

DEFINITION 4.3. Let $\mathcal{T} = (T, (X_t)_{t \in T}, R, Q, 0)$ be a tense system and L be a complete and completely chrysippian $LM_{n \times m}$-algebra. We denote by:

$$F_{T,L}^{U,n \times m} = \{(f_t)_{t \in T} \mid f_t : X_t^U \longrightarrow L, \text{ for all } t \in T\}.$$

We will denote $F_{T,L}^{U,n \times m}$ by $F_T^{U,n \times m}$ for $L = D(\mathbf{2})$.

On $F_{T,L}^{U,n \times m}$ we will consider the following operations:

- $(f_t)_{t \in T} \wedge (g_t)_{t \in T} = (f_t \wedge g_t)_{t \in T}$, where $(f_t \wedge g_t)(x) = f_t(x) \wedge g_t(x)$, for all $t \in T$ and $x \in X_t^U$,

- $(f_t)_{t \in T} \vee (g_t)_{t \in T} = (f_t \vee g_t)_{t \in T}$, where $(f_t \vee g_t)(x) = f_t(x) \vee g_t(x)$, for all $t \in T$ and $x \in X_t^U$,

- $\sim^{\mathcal{T}} ((f_t)_{t \in T}) = (\sim \circ f_t)_{t \in T}$, where $(\sim \circ f_t)(x) = \sim (f_t(x))$, for all $t \in T$ and $x \in X_t^U$,

- $\sigma_{ij}^{\mathcal{T}}((f_t)_{t \in T}) = (\sigma_{ij} \circ f_t)_{t \in T}$, where $(\sigma_{ij} \circ f_t)(x) = \sigma_{ij}(f_t(x))$, for all $(i,j) \in (n \times m)$, $t \in T$ and $x \in X_t^U$,

- $0^{\mathcal{T}} = (0_t)_{t \in T}$, where $0_t : X_t^U \longrightarrow L$, $0_t(x) = 0_L$, for all $t \in T$ and $x \in X_t^U$,

- $1^{\mathcal{T}} = (1_t)_{t \in T}$, where $1_t : X_t^U \longrightarrow L$, $1_t(x) = 1_L$, for all $t \in T$ and $x \in X_t^U$.

LEMMA 4.4. $\mathcal{F}_{\mathcal{T},\mathcal{L}}^{U,n \times m} = \langle F_{\mathcal{T},\mathcal{L}}^{U,n \times m}, \vee, \wedge, \sim^{\mathcal{T}}, (\sigma_{ij}^{\mathcal{T}})_{(i,j) \in (n \times m)}, 0^{\mathcal{T}}, 1^{\mathcal{T}} \rangle$ *is an* $LM_{n \times m}$*-algebra.*

PROOF: First, we will prove that $\langle F_{\mathcal{T},\mathcal{L}}^{U,n \times m}, \vee, \wedge, \sim^{\mathcal{T}}, 0^{\mathcal{T}}, 1^{\mathcal{T}} \rangle$ is a De Morgan algebra. It is easy to see that $\langle F_{T,L}^{U,n \times m}, \vee, \wedge, 0^{\mathcal{T}}, 1^{\mathcal{T}} \rangle$ is a bounded distributive lattice.

(a) $\sim^{\mathcal{T}} \sim^{\mathcal{T}} ((f_t)_{t \in T}) = \sim^{\mathcal{T}} ((\sim \circ f_t)_{t \in T}) = (\sim \circ \sim \circ f_t)_{t \in T}$, where $(\sim \circ \sim f_t)(x) = \sim (\sim (f_t(x))) = f_t(x)$, for all $t \in T$ and $x \in X_t^U$, so $\sim^{\mathcal{T}} \sim^{\mathcal{T}} ((f_t)_{t \in T}) = (f_t)_{t \in T}$.

(b) $\sim^{\mathcal{T}} ((f_t)_{t \in T} \wedge (g_t)_{t \in T}) = \sim^{\mathcal{T}} ((f_t \wedge g_t)_{t \in T}) = (\sim \circ (f_t \wedge g_t))_{t \in T}$, where $(\sim \circ (f_t \wedge g_t))(x) = \sim ((f_t \wedge g_t)(x)) = \sim (f_t(x) \wedge g_t(x)) = \sim f_t(x) \vee \sim g_t(x)$,

for all $t \in T$ and $x \in X_t^U$, so, $\sim^{\mathcal{T}} ((f_t)_{t \in T} \wedge (g_t)_{t \in T}) = \sim^{\mathcal{T}} (f_t)_{t \in T} \vee \sim^{\mathcal{T}} (g_t)_{t \in T}$.

Now we will prove that $F_{\mathcal{T},\mathcal{L}}^{U,n \times m}$ verify the conditions (C1)-(C5).

(C1) Let $(i,j) \in (n \times m)$ and $(f_t)_{t \in T}, (g_t)_{t \in T} \in F_{T,L}^{U,n \times m}$. Then, $\sigma_{ij}^{\mathcal{T}}((f_t)_{t \in T} \vee (g_t)_{t \in T}) = \sigma_{ij}^{\mathcal{T}}((f_t \vee g_t)_{t \in T}) = (\sigma_{ij} \circ (f_t \vee g_t))_{t \in T}$. Since $\mathcal{L}$ is an $LM_{n \times m}$-algebra we obtain that: $(\sigma_{ij} \circ (f_t \vee g_t))_{t \in T} = ((\sigma_{ij} \circ f_t) \vee (\sigma_{ij} \circ g_t))_{t \in T} = (\sigma_{ij} \circ f_t)_{t \in T} \vee (\sigma_{ij} \circ g_t)_{t \in T} = \sigma_{ij}^{\mathcal{T}}((f_t)_{t \in T}) \vee \sigma_{ij}^{\mathcal{T}}((g_t)_{t \in T})$.

(C2) Let $(i,j) \in (n \times m)$. We will to prove that $\sigma_{ij}^{\mathcal{T}}((f_t)_{t \in T}) \leq \sigma_{(i+1)j}^{T}((f_t)_{t \in T})$, for all $(f_t)_{t \in T} \in F_{T,L}^{U,n \times m}$. Let $(f_t)_{t \in T} \in F_{T,L}^{U,n \times m}$. Then, $\sigma_{ij}^{\mathcal{T}}((f_t)_{t \in T}) = (\sigma_{ij} \circ f_t)_{t \in T}$ and $\sigma_{(i+1)j}^{\mathcal{T}}((f_t)_{t \in T}) = (\sigma_{(i+1)j} \circ f_t)_{t \in T}$. Let $t \in T$ and $x \in X_t^U$. Since $\mathcal{L}$ is an $LM_{n \times m}$-algebra we obtain that: $\sigma_{ij}(f_t(x)) \leq \sigma_{(i+1)j}(f_t(x))$, so, $\sigma_{ij}^{\mathcal{T}}((f_t)_{t \in T}) \leq \sigma_{(i+1)j}^{\mathcal{T}}((f_t)_{t \in T})$. In a similar way we can prove that: $\sigma_{ij}^{\mathcal{T}}((f_t)_{t \in T}) \leq \sigma_{i(j+1)}^{\mathcal{T}}((f_t)_{t \in T})$.

(C4) Now, we will prove that $\sigma_{ij}^{\mathcal{T}} \circ \sigma_{rs}^{\mathcal{T}} = \sigma_{rs}^{\mathcal{T}}$, for all $(i,j), (r,s) \in (n \times m)$. Let $(i,j), (r,s) \in (n \times m)$ and $(f_t)_{t \in T} \in F_{T,L}^{U,n \times m}$. Proving condition $(\sigma_{ij}^{\mathcal{T}} \circ \sigma_{rs}^{\mathcal{T}})((f_t)_{t \in T}) = \sigma_{rs}^{\mathcal{T}}((f_t)_{t \in T})$ is equivalent proving $(\sigma_{ij} \circ \sigma_{rs} \circ f_t)_{t \in T} = (\sigma_{rs} \circ f_t)_{t \in T}$. Let $t \in T$ and $x \in X_t^U$. Then, we have $(\sigma_{ij} \circ \sigma_{rs} \circ f_t)(x) = (\sigma_{ij} \circ \sigma_{rs})(f_t(x)) = \sigma_{rs}(f_t(x)) = (\sigma_{rs} \circ f_t)(x)$, so $(\sigma_{ij} \circ \sigma_{rs} \circ f_t)_{t \in T} = (\sigma_{rs} \circ f_t)_{t \in T}$.

(C5) Let $(f_t)_{t \in T}, (g_t)_{t \in T} \in F_{T,L}^{U,n \times m}$ such that $\sigma_{ij}^{\mathcal{T}}((f_t)_{t \in T}) = \sigma_{ij}^{\mathcal{T}}((g_t)_{t \in T})$, for every $(i,j) \in (n \times m)$. Then, $(\sigma_{ij} \circ f_t)_{t \in T} = (\sigma_{ij} \circ g_t)_{t \in T}$, for all $(i,j) \in (n \times m)$. It follows that for every $t \in T$, $\sigma_{ij} \circ f_t = \sigma_{ij} \circ g_t$, that is, $\sigma_{ij}(f_t(x)) = \sigma_{ij}(g_t(x))$, for every $t \in T$ and $x \in X_t^U$. Using (C5) for the $LM_{n \times m}$-algebra $\mathcal{L}$, we obtain that $f_t(x) = g_t(x)$, for every $t \in T$ and $x \in X_t^U$, so $(f_t)_{t \in T} = (g_t)_{t \in T}$.

(C6) $\sigma_{ij}^{\mathcal{T}}((f_t)_{t \in T}) \vee \sim^{\mathcal{T}} (\sigma_{ij}^{\mathcal{T}}((f_t)_{t \in T}) = (\sigma_{ij} \circ f_t)_{t \in T} \vee (\sim \circ \sigma_{ij} \circ f_t)_{t \in T} = ((\sigma_{ij} \circ f_t) \vee (\sim \circ \sigma_{ij} \circ f_t))_{t \in T}$, where $((\sigma_{ij} \circ f_t) \vee (\sim \circ \sigma_{ij} \circ f_t))(x) = \sigma_{ij}(f_t(x)) \vee \sim \sigma_{ij}(f_t(x)) = 1$, for every $t \in T$ and $x \in X_t^U$. So, $\sigma_{ij}^{\mathcal{T}}((f_t)_{t \in T}) \vee \sim^{\mathcal{T}} (\sigma_{ij}^{\mathcal{T}}((f_t)_{t \in T})) = 1^{\mathcal{T}}$.

(C7) $\sigma_{ij}^{\mathcal{T}}(\sim^{\mathcal{T}} (f_t)_{t \in T}) = (\sigma_{ij} \circ \sim \circ f_t)_{t \in T}$, where $(\sigma_{ij} \circ \sim \circ f_t)(x) = \sigma_{ij}(\sim f_t(x)) = \sim \sigma_{n-im-j}(f_t(x)) = (\sim \circ \sigma_{n-im-j} \circ f_t)(x)$, for every $t \in T$ and $x \in X_t^U$. It follows that $\sigma_{ij}^{\mathcal{T}}(\sim^{\mathcal{T}} (f_t)_{t \in T}) = \sim^{\mathcal{T}} (\sigma_{n-im-j}^{T}(f_t)_{t \in T})$. $\qquad\square$

On $F_{T,L}^{U,n \times m}$ we define the operators G and H by

$G((f_t)_{t \in T}) = (g_t)_{t \in T}$, where $g_t : X^U \longrightarrow L$, $g_t(x) = \bigwedge\{f_s(i \circ x) \mid tRs, s \in T\}$,

$H((f_t)_{t \in T}) = (h_t)_{t \in T}$, where $h_t : X^U \longrightarrow L$, $h_t(x) = \bigwedge\{f_s(i \circ x) \mid tQs, s \in T\}$, where $i : X_t \longrightarrow X_s$ is the inclusion map.

LEMMA 4.5. $(\mathcal{F}_{T,L}^{U,n \times m}, G, H)$ *is a tense* $LM_{n \times m}$*-algebra.*

PROOF: By Lemma 4.4, we have that $\mathcal{F}_{T,L}^{U,n \times m}$ is an $LM_{n \times m}$-algebra. Now, we have to prove that G and H are tense operators.

(1) $G(1^{\mathcal{T}}) = G((1_t)_{t \in T}) = (f_t)_{t \in T}$, where $f_t(x) = \bigwedge\{1_s(i \circ x) \mid tRs\} = 1$, for all $t \in T$ and $x \in X_t^U$; hence $(f_t)_{t \in T} = (1_t)_{t \in T}$. It follows that $G(1^{\mathcal{T}}) = 1^{\mathcal{T}}$. Similarly we can prove that $H(1^{\mathcal{T}}) = 1^{\mathcal{T}}$.

(2) Let $(f_t)_{t \in T}, (g_t)_{t \in T} \in F_{T,L}^{U,n \times m}$. Then,

   (a) $G((f_t)_{t \in T}) = (v_t)_{t \in T}$, where $v_t(x) = \bigwedge\{f_s(i \circ x) \mid tRs\}$,

   (b) $G((g_t)_{t \in T}) = (p_t)_{t \in T}$, where $p_t(x) = \bigwedge\{g_s(i \circ x) \mid tRs\}$,

   (c) $G((f_t)_{t \in T} \wedge (g_t)_{t \in T}) = G((f_t \wedge g_t)_{t \in T}) = (u_t)_{t \in T}$, where $u_t(x) = \bigwedge\{(f_s \wedge g_s)(i \circ x) \mid tRs\}$.

     Let $t \in T$ and $x \in X_t^U$. By (a), (b) and (c) we obtain that $u_t(x) = v_t(x) \wedge p_t(x)$, hence $(u_t)_{t \in T} = (v_t)_{t \in T} \wedge (p_t)_{t \in T}$, so $G((f_t)_{t \in T} \wedge (g_t)_{t \in T}) = G((f_t)_{t \in T}) \wedge G((g_t)_{t \in T})$. Similarly we can prove that $H((f_t)_{t \in T} \wedge (g_t)_{t \in T}) = H((f_t)_{t \in T}) \wedge H((g_t)_{t \in T})$.

(3) Let $(f_t)_{t \in T} \in F_{T,L}^{U,n \times m}$. Then,

   (a) $G(\sigma_{ij}^{\mathcal{T}}(f_t)_{t \in T}) = G((\sigma_{ij} \circ f_t)_{t \in T}) = (g_t)_{t \in T}$, where $g_t(x) = \bigwedge\{(\sigma_{ij} \circ f_s)(i \circ x) \mid tRs\}$.

   (b) $\sigma_{ij}^{\mathcal{T}}(G((f_t)_{t \in T})) = \sigma_{ij}^{\mathcal{T}}((p_t)_{t \in T})$, where $p_t(x) = \bigwedge\{f_s(i \circ x) \mid tRs\}$.

     Let $t \in T$ and $x \in X_t^U$. By (a), (b) and the fact that $\mathcal{L}$ is completely chrysippian, we obtain that $g_t(x) = \sigma_{ij}(p_t(x))$, hence $(g_t)_{t \in T} = \sigma_{ij}^{\mathcal{T}}(p_t)_{t \in T}$. So, $G \circ \sigma_{ij}^{\mathcal{T}} = \sigma_{ij}^{\mathcal{T}} \circ G$. In a similar way we can prove that H commutes with σ_{ij}. $\qquad\square$

     For any $\tau \in U^U$, we define the function $S(\tau) : F_{T,L}^{U,n \times m} \longrightarrow F_{T,L}^{U,n \times m}$ by

- $S(\tau)((f_t)_{t \in T}) = (g_t)_{t \in T}$, where $g_t : X_t^U \longrightarrow L$ is defined by: $g_t(x) = f_t(x \circ \tau)$, for every $t \in T$ and $x \in X_t^U$.

     For any $J \subseteq U$, we define the function $\exists(J) : F_{T,L}^{U,n \times m} \longrightarrow F_{T,L}^{U,n \times m}$ by

- $\exists(J)((f_t)_{t \in T}) = (g_t)_{t \in T}$, where $g_t : X_t^U \longrightarrow L$ is defined by: $g_t(x) = \bigvee\{f_t(y) \mid y \in X_t^U, y\mid_{U \setminus J} = x\mid_{U \setminus J}\}$, for every $t \in T$ and $x \in X_t^U$.

PROPOSITION 4.6. *For any $J \subseteq U$, $(\mathcal{F}_{T,L}^{U,n \times m}, \exists(J))$ is a monadic $LM_{n \times m}$-algebra.*

PROOF: Let $J \subseteq U$. We will prove that $\exists(J)$ is an existential quantifier on $F_{T,L}^{U,n \times m}$.

(E1) $\exists(J)(0^{\mathcal{T}}) = \exists(J)((0_t)_{t \in T}) = (g_t)_{t \in T}$, where $g_t(x) = \bigvee\{0_t(y) \mid y \in X_t^U, y \mid_{U \setminus J} = x \mid_{U \setminus J}\} = \bigvee\{0\} = 0$, for every $t \in T$ and $x \in X_t^U$. We obtain that $(g_t)_{t \in T} = 0^{\mathcal{T}}$, hence $\exists(J)(0^{\mathcal{T}}) = 0^{\mathcal{T}}$.

(E2) Let $(f_t)_{t \in T} \in F_{T,L}^{U,n \times m}$. We will prove that $(f_t)_{t \in T} \leq \exists(J)((f_t)_{t \in T})$. We have: $\exists(J)((f_t)_{t \in T}) = (g_t)_{t \in T}$, where $g_t(x) = \bigvee\{f_t(y) \mid y \in X_t^U, y \mid_{U \setminus J} = x \mid_{U \setminus J}\}$, for every $t \in T$ and $x \in X_t^U$. We obtain that $f_t(x) \leq g_t(x)$, for every $t \in T$ and $x \in X_t^U$, hence $(f_t)_{t \in T} \leq (g_t)_{t \in T}$.

(E3) Let $(f_t)_{t \in T}, (g_t)_{t \in T} \in F_{T,L}^{U,n \times m}$. We have:

(a) $\exists(J)((f_t)_{t \in T} \wedge \exists(J)(g_t)_{t \in T}) = \exists(J)((f_t)_{t \in T} \wedge (h_t)_{t \in T}) = \exists(J)((f_t \wedge h_t)_{t \in T}) = (u_t)_{t \in T}$,

(b) $\exists(J)((f_t)_{t \in T}) \wedge \exists(J)((g_t)_{t \in T}) = (p_t)_{t \in T} \wedge (v_t)_{t \in T} = (p_t \wedge v_t)_{t \in T}$, where, for every $t \in T$ and $x \in X_t^U$,
$h_t(x) = \bigvee\{g_t(y) \mid y \in X_t^U, y \mid_{U \setminus J} = x \mid_{U \setminus J}\}$,
$u_t(x) = \bigvee\{(f_t(z) \wedge h_t(z)) \mid z \in X_t^U, z \mid_{U \setminus J} = x \mid_{U \setminus J}\} = \bigvee\{f_t(z) \wedge g_t(y) \mid z, y \in X_t^U, z \mid_{U \setminus J} = x \mid_{U \setminus J} = y \mid_{U \setminus J}\}$,
$p_t(x) = \bigvee\{f_t(z) \mid z \in X_t^U, z \mid_{U \setminus J} = x \mid_{U \setminus J}\}$,
$v_t(x) = \bigvee\{g_t(y) \mid y \in X_t^U, y \mid_{U \setminus J} = x \mid_{U \setminus J}\}$.

It follows that, for every $t \in T$ and $x \in X_t^U$, $p_t(x) \wedge v_t(x) = \bigvee\{f_t(z) \wedge g_t(y) \mid z, y \in X_t^U, z \mid_{U \setminus J} = x \mid_{U \setminus J} = y \mid_{U \setminus J}\} = u_t(x)$. Hence, $\exists(J)((f_t)_{t \in T} \wedge \exists(J)((g_t)_{t \in T}) = \exists(J)((f_t)_{t \in T}) \wedge \exists(J)((g_t)_{t \in T})$.

(E4) Let $(i, j) \in (n \times m)$ and $(f_t)_{t \in T} \in F_{T,L}^{U,n \times m}$. Then, we have

(a) $\exists(J)(\sigma_{ij}^{\mathcal{T}})((f_t)_{t \in T}) = \exists(J)((\sigma_{ij} \circ f_t)_{t \in T}) = (g_t)_{t \in T}$, where
$g_t(x) = \bigvee\{\sigma_{ij}(f_t(y)) \mid y \in X_t^U, y \mid_{U \setminus J} = x \mid_{U \setminus J}\}$, for all $t \in T$ and $x \in X_t^U$.

(b) $\sigma_{ij}^{\mathcal{T}}(\exists(J)((f_t)_{t \in T}) = \sigma_{ij}^{\mathcal{T}}((h_t)_{t \in T}) = (\sigma_{ij} \circ h_t)_{t \in T}$ with $h_t(x) = \bigvee\{f_t(y) \mid y \in X_t^U, y \mid_{U \setminus J} = x \mid_{U \setminus J}\}$, for every $t \in T$ and $x \in X_t^U$.

Using the fact that $\mathcal{L}$ is completely chrysippian we deduce that $\sigma_{ij}(h_t(x)) = g_t(x)$, for every $t \in T$ and $x \in X_t^U$, hence $\exists(J)(\sigma_{ij}^{\mathcal{T}}((f_t)_{t \in T})) = \sigma_{ij}^{\mathcal{T}}(\exists(J)((f_t)_{t \in T}))$. $\square$

The following proposition provides the main example of tense polyadic $LM_{n \times m}$-algebra.

PROPOSITION 4.7. $(\mathcal{F}_{T,L}^{U,n\times m}, U, S, \exists, G, H)$ *is a tense polyadic $LM_{n\times m}$-algebra.*

PROOF: We will verify the conditions of Definition 4.1.

(a): We have to prove that the conditions of Definition 3.3 are satisfied.

(i): Let $(f_t)_{t\in T} \in F_{T,L}^{U,n\times m}$, $t \in T$ and $x \in X_t^U$. By applying the definition of S, we obtain: $S(1_U)((f_t)_{t\in T}) = (g_t)_{t\in T}$, where $g_t(x) = f_t(x \circ 1_U) = f_t(x)$, so $S(1_U)((f_t)_{t\in T}) = (f_t)_{t\in T}$, hence $S(1_U) = 1_{\mathcal{F}_{T,L}^{U,n\times m}}$.

(ii): Let $\rho, \tau \in U^U$, $(f_t)_{t\in T} \in F_{T,L}^{U,n\times m}$, $t \in T$ and $x \in X_t^U$.
$S(\rho \circ \tau)((f_t)_{t\in T}) = (g_t)_{t\in T}$ with $g_t(x) = f_t(x \circ \rho \circ \tau)$.
$(S(\rho) \circ S(\tau))((f_t)_{t\in T}) = S(\rho)(S(\tau)((f_t)_{t\in T})) = S(\rho)((h_t)_{t\in T}) = (p_t)_{t\in T}$,
where $h_t(x) = f_t(x \circ \tau)$ and $p_t(x) = h_t(x \circ \rho) = f_t(x \circ \rho \circ \tau)$.
It follows that $S(\rho \circ \tau)((f_t)_{t\in T}) = (S(\rho) \circ S(\tau))((f_t)_{t\in T})$, hence $S(\rho \circ \tau) = S(\rho) \circ S(\tau)$.

(iii): Let $(f_t)_{t\in T} \in F_{T,L}^{U,n\times m}$, $t \in T$ and $x \in X_t^U$. We have: $\exists(\emptyset)((f_t)_{t\in T}) = (g_t)_{t\in T}$, where
$g_t(x) = \bigvee\{f_t(y) \mid y \in X_t^U, y \mid_U = x \mid_U\} = \bigvee\{f_t(x)\} = f_t(x)$,
so $\exists(\emptyset)((f_t)_{t\in T}) = (f_t)_{t\in T}$, i.e. $\exists(\emptyset) = 1_{F_{T,L}^{U,n\times m}}$.

(iv): Let $J, J' \subseteq U$ and $(f_t)_{t\in T} \in F_{T,L}^{U,n\times m}$. Then,

(1) $\exists(J \cup J')((f_t)_{t\in T}) = (g_t)_{t\in T}$ with
$g_t(x) = \bigvee\{f_t(y) \mid y \in X_t^U, y \mid_{U\setminus(J\cup J')} = x \mid_{U\setminus(J\cup J')}\}$,
for every $t \in T$ and $x \in X_t^U$.

(2) $(\exists(J) \circ \exists(J'))((f_t)_{t\in T}) = \exists(J)(\exists(J')(f_t)_{t\in T}) = \exists(J)((h_t)_{t\in T}) = (p_t)_{t\in T}$,
where $h_t(x) = \bigvee\{f_t(y) \mid y \in X_t^U, y \mid_{U\setminus J'} = x \mid_{U\setminus J'}\}$ and
$p_t(x) = \bigvee\{h_t(y) \mid y \in X_t^U, y \mid_{U\setminus J} = x \mid_{U\setminus J}\}$, for every $t \in T$ and $x \in X_t^U$.
We obtain that

$p_t(x) = \bigvee\{f_t(z) \mid z \in X_t^U, \text{ exists } y \in X_t^U : z \mid_{U\setminus J'} = y \mid_{U\setminus J'}, x \mid_{U\setminus J} = y \mid_{U\setminus J}\}$.

We will prove that the sets
$A = \{f_t(y) \mid y \in X_t^U, y \mid_{U\setminus(J\cup J')} = x \mid_{U\setminus(J\cup J')}\}$
and $B = \{f_t(z) \mid z \in X_t^U, \text{ exists } y \in X_t^U \text{ such that } z \mid_{U\setminus J'} = y \mid_{U\setminus J'}, x \mid_{U\setminus J} = y \mid_{U\setminus J}\}$ are equal.

Let $z \in X_t^U$ such that $z \mid_{U \setminus (J \cup J')} = x \mid_{U \setminus (J \cup J')}$. We consider $y \in X_t^U$, defined by

$$y(a) = \begin{cases} z(a), & \text{if } a \in U \setminus J', \\ x(a), & \text{if } a \in J', \end{cases}.$$

It follows that $y \mid_{U \setminus J'} = z \mid_{U \setminus J'}$. If $a \in U \setminus J$, we have two cases:

(I) If $a \in J'$ then, $y(a) = x(a)$.

(II) If $a \notin J'$ it results that $a \in U \setminus (J \cup J')$, so $y(a) = z(a) = x(a)$.

By (I) and (II), we get that $z \mid_{U \setminus J'} = y \mid_{U \setminus J'}$ and $x \mid_{U \setminus J} = y \mid_{U \setminus J}$, so $A \subseteq B$. Conversely, let $z \in X_t^U$ such that, exists $y \in X_t^U$ with $z \mid_{U \setminus J'} = y \mid_{U \setminus J'}$ and $x \mid_{U \setminus J} = y \mid_{U \setminus J}$. It follows that

$z \mid_{(U \setminus J') \cap (U \setminus J)} = y \mid_{(U \setminus J') \cap (U \setminus J)}$ and $x \mid_{(U \setminus J) \cap (U \setminus J')} = y \mid_{(U \setminus J) \cap (U \setminus J')}$, hence $z \mid_{U \setminus (J \cup J')} = x \mid_{U \setminus (J \cup J')}$.

We obtain that $B \subseteq A$, hence $A = B$. We get that $g_t(x) = p_t(x)$ for every $t \in T$ and $x \in X_t^U$, so $\exists (J \cup J') = \exists (J) \circ \exists (J')$.

(v): Let $J \subseteq U$, $\rho, \tau \in U^U$ and $(f_t)_{t \in T} \in F_{T,L}^{U,n \times m}$, such that $\rho \mid_{U \setminus J} = \tau \mid_{U \setminus J}$.

We obtain:

(1) $(S(\rho) \circ \exists (J))((f_t)_{t \in T}) = S(\rho)(\exists (J)((f_t)_{t \in T})) = (g_t)_{t \in T}$, where $g_t(x) = \bigvee \{ f_t(y) \mid y \in X_t^U, y \mid_{U \setminus J} = (x \circ \rho) \mid_{U \setminus J} \}$, for every $t \in T$ and $x \in X_t^U$.

(2) $(S(\tau) \circ \exists (J))((f_t)_{t \in T}) = S(\tau)(\exists (J))((f_t)_{t \in T})) = (h_t)_{t \in T}$, where $h_t(x) = \bigvee \{ f_t(y) \mid y \in X_t^U, y \mid_{U \setminus J} = (x \circ \tau)_{U \setminus J} \}$, for every $t \in T$ and $x \in X_t^U$. By $\rho \mid_{U \setminus J} = \tau \mid_{U \setminus J}$ it follows that $(x \circ \rho) \mid_{U \setminus J} = (x \circ \tau) \mid_{U \setminus J}$, for every $x \in X_t^U$, hence $g_t(x) = h_t(x)$, for every $t \in T$ and $x \in X_t^U$. It results that $S(\rho) \circ \exists (J) = S(\tau) \circ \exists (J)$.

(vi): Let $J \subseteq U$, $(f_t)_{t \in T} \in F_{T,L}^{U,n \times m}$ and $\rho \in U^U$ such that $\rho \mid_{\rho^{-1}(J)}$ is injective.

We have:

(1) $(\exists (J) \circ S(\rho))((f_t)_{t \in T})) = (g_t)_{t \in T}$, where $g_t(x) = \bigvee \{ f_t(y \circ \rho) \mid y \in X_t^U, y \mid_{U \setminus J} = x \mid_{U \setminus J} \}$, for every $t \in T$ and $x \in X_t^U$.

(2) $(S(\rho) \circ \exists (\rho^{-1}(J)))((f_t)_{t \in T})) = (h_t)_{t \in T}$, where $h_t(x) = \bigvee \{ f_t(y) \mid y \in X_t^U, y \mid_{U \setminus \rho^{-1}(J)} = (x \circ \rho) \mid_{U \setminus \rho^{-1}(J)} \}$, for every $t \in T$ and $x \in X_t^U$.

We must prove that A and B are equal, where
$A = \{f_t(y \circ \rho) \mid y \in X_t^U, y \mid_{U \setminus J} = x \mid_{U \setminus J}\}$ y
$B = \{f_t(y) \mid y \in X_t^U, y \mid_{U \setminus \rho^{-1}(J)} = (x \circ \rho) \mid_{U \setminus \rho^{-1}(J)}\}$.
Let $y \in X_t^U$ such that $y \mid_{U \setminus J} = x \mid_{U \setminus J}$. We consider $z = y \circ \rho$.
Let $a \in U \setminus \rho^{-1}(J)$. Then, $z(a) = y(\rho(a)) = x(\rho(a)) = (x \circ \rho)(a)$,
so $z \mid_{U \setminus \rho^{-1}(J)} = (x \circ \rho) \mid_{U \setminus \rho^{-1}(J)}$. We get that $A \subseteq B$.
Conversely, let $y \in X_t^U$ such that $y \mid_{U \setminus \rho^{-1}(J)} = (x \circ \rho) \mid_{U \setminus \rho^{-1}(J)}$.
Since $\rho \mid_{\rho^{-1}(J)}$ is injective, we can consider the bijective function
$\rho' : \rho^{-1}(J) \longrightarrow J$, defined by $\rho'(a) = \rho(a)$ for all $a \in \rho^{-1}(J)$.
Let us consider $z \in X_t^U$, defined by:

$$z(a) = \begin{cases} y(\rho'^{-1}(a)), & \text{if } a \in J, \\ x(a), & \text{if } a \in U \setminus J, \end{cases}$$

We see that $z \mid_{U \setminus J} = x \mid_{U \setminus J}$. By calculus we get that $(z \circ \rho)(a) = y(a)$, for every $a \in U$, so $z \circ \rho = y$. It follows that $B \subseteq A$, so $A = B$.

(vii): It follows by Proposition 4.6.

(b): It follows by Lemma 4.5.

(c): Let $\tau \in U^U$, $(f_t)_{t \in T} \in F_{T,L}^{U, n \times m}$, $t \in T$ and $x \in X_t^U$. It follows that:

(1) $S(\tau)(G((f_t)_{t \in T})) = S(\tau)((g_t)_{t \in T}) = (h_t)_{t \in T}$, where
$g_t(x) = \bigwedge \{f_s(i \circ x) \mid tRs, s \in T\}$ and
$h_t(x) = g_t(x \circ \tau) = \bigwedge \{f_s(i \circ x \circ \tau) \mid tRs\}$.

(2) $G(S(\tau)((f_t)_{t \in T})) = G((p_t)_{t \in T}) = (u_t)_{t \in T}$, where $p_t(x) = f_t(x \circ \tau)$ and
$u_t(x) = \bigwedge \{p_s(i \circ x) \mid tRs, s \in T\}$.
By (1) and (2) we obtain that $h_t(x) = u_t(x)$, for all $t \in T$ and $x \in X_t^U$, so $(h_t)_{t \in T} = (u_t)_{t \in T}$, i.e. $S(\tau)(G((f_t)_{t \in T})) = G(S(\tau))((f_t)_{t \in T}))$.

(d): Similar with (c). □

REMARK 4.8. Proposition 4.7 is an extension of Lemma 2.13, in the sense that if we take $B = C(L)$, we obtain Lemma 2.13.

DEFINITION 4.9. Let $(\mathcal{L}, U, S, \exists, G, H)$ be a tense polyadic $\text{LM}_{n \times m}$-algebra. A subset J of U is a *support* of $p \in L$ if $\exists(U \setminus J)p = p$. The intersection of the supports of an element $p \in L$ will be denoted by J_p. A tense polyadic $\text{LM}_{n \times m}$-algebra is *locally finite* if every element has a finite support.

REMARK 4.10. We consider the tense polyadic $LM_{n\times m}$-algebra $(\mathcal{F}_{T,L}^{U,n\times m}, U,$ $S, \exists, G, H)$. By applying Definition 4.9, $M \subseteq U$ is a support of $(f_t)_{t\in T} \in$ $F_{T,L}^{U,n\times m}$ if $\exists(U \setminus M)((f_t)_{t\in T}) = (f_t)_{t\in T}$. By using the definition of $\exists$, we obtain that $\bigvee\{f_t(y) \mid y \in X_t^U, y \mid_M = x \mid_M\} = f_t(x)$, for all $t \in T$ and $x \in X_t^U$.

LEMMA 4.11. *Let us consider the tense polyadic $LM_{n\times m}$-algebra $(\mathcal{F}_T^{U,n\times m},$* $U, S, \exists, G, H)$, *where $F_T^{U,n\times m} = \{(f_t)_{t\in T} \mid f_t : X_t^U \longrightarrow D(\mathbf{2})$ for all $t \in T\}$,* $(f_t)_{t\in T} \in F_T^{U,n\times m}$ *y $Q \subseteq U$. Then the following conditions are equivalent:*

(a) *Q is a support of $(f_t)_{t\in T}$,*
(b) *for every $(x_t)_{t\in T}, (y_t)_{t\in T}, x_t, y_t \in X_t^U$, for all $t \in T$ we have:*
  $x_t \mid_Q = y_t \mid_Q, t \in T \Rightarrow f_t(x_t) = f_t(y_t), t \in T.$

PROOF: (a)$\Rightarrow$ (b): We assume that Q is a support of $(f_t)_{t\in T}$. By applying Definition 4.9 and definition of $\exists$, it follows that $\bigvee\{f_t(y) \mid y \in X_t^U, y \mid_Q = x \mid_Q\} = f_t(x)$, for all $t \in T$ and $x \in X_t^U$. Let $t \in T$, $x_t, y_t \in X_t^U$ such that $x_t \mid_Q = y_t \mid_Q$. We have:

$f_t(x_t) = \bigvee\{f_t(y) \mid y \in X_t^U, y \mid_Q = x_t \mid_Q\} \geq f_t(y_t)$
$f_t(y_t) = \bigvee\{f_t(x) \mid x \in X_t^U, x \mid_Q = y_t \mid_Q\} \geq f_t(x_t)$

So, $f_t(x_t) = f_t(y_t)$.

(b)$\Rightarrow$ (a): Using definition of $\exists$ we obtain that $\exists(U \setminus Q)(f_t)_{t\in T} =$ $(g_t)_{t\in T}$, where $g_t : X_t^U \longrightarrow D(\mathbf{2})$, $g_t(x) = \bigvee\{f_t(y) \mid y \in X_t^U, y \mid_Q = x \mid_Q\}$, for every $t \in T$ and $x \in X_t^U$. Let $t \in T$ and $x \in X_t^U$. By (b) it follows that $g_t(x) = \bigvee\{f_t(x) \mid y \in X_t^U, y \mid_Q = x \mid_Q\} = f_t(x)$. We obtain that $(g_t)_{t\in T} = (f_t)_{t\in T}$, so $\exists(U \setminus Q)(f_t)_{t\in T} = (f_t)_{t\in T}$, i.e. Q is a support of $(f_t)_{t\in T}$.   □

LEMMA 4.12. *Let $f : L \longrightarrow L'$ be a morphism of tense polyadic $LM_{n\times m}$-algebras, $p \in L$, $Q \subseteq U$. If Q is a support of p, then Q is a support of $f(p)$.*

PROOF: Because Q is a support of p, it follows that $\exists(U \setminus Q)p = p$. By applying the definition of morphism of tense polyadic $LM_{n\times m}$-algebras we obtain that $f(\exists(U \setminus Q)p) = \exists(U \setminus Q)f(p) = f(p)$, hence Q is a support of $f(p)$.   □

LEMMA 4.13. *Let $(\mathcal{L}, U, S, \exists, G, H)$ be a tense polyadic $LM_{n \times m}$-algebra. Then,*

(i) *$(C(L), U, S, \exists, C(G), C(H))$ is a tense polyadic Boolean algebra.*
(ii) *If L is locally finite, then $C(L)$ is locally finite.*

PROOF: We only prove (i). By applying [1, p. 453, Remark 4.2], we obtain that $C(L)$ can be endowed with a canonical structure of polyadic Boolean algebra. By [18, Remark 1.15], we have that $(C(L), C(G), C(H))$ is a tense Boolean algebra. The conditions (iii) and (iv) of Definition 2.9 are met for the elements of $C(L)$ as well, hence $C(L)$ is a tense polyadic Boolean algebra. □

Let $(\mathcal{B}, U, S, \exists, G, H)$ be a tense polyadic Boolean algebra. We consider on $D(B)$ the following operations, for every $\tau \in U^U$, $f \in D(B)$ and $J \subseteq U$:

- $(D(S)(\tau))(f) = S(\tau) \circ f$,
- $(D(\exists)(J))(f) = \exists(J) \circ f$,
- $(D(G))(f) = G \circ f$,
- $(D(H))(f) = H \circ f$.

LEMMA 4.14.
(i) *$(D(B), U, D(S), D(\exists), D(G), D(H))$ is a tense polyadic $LM_{n \times m}$-algebra.*
(ii) *If B is locally finite, then $D(B)$ is locally finite.*

The assignments $B \mapsto C(B)$, $B \mapsto D(B)$ establish the adjoint functors C and D between the category of tense polyadic Boolean algebras and the category of tense polyadic $LM_{n \times m}$-algebras.

DEFINITION 4.15. Let $(\mathcal{L}, U, S, \exists, G, H)$ be a tense polyadic $LM_{n \times m}$-algebra. We consider the function $\omega_L : L \longrightarrow D(C(L))$, defined by: for all $x \in L$ and $(i, j) \in (n \times m)$, $\omega_L(x)(i, j) = \sigma_{ij}(x)$.

LEMMA 4.16. *ω_L is an injective morphism of tense polyadic $LM_{n \times m}$-algebras.*

PROOF: By [18, Lemma 2.6], ω_L is an injective morphism of tense $LM_{n \times m}$-algebras. We have to prove that ω_L commutes with S and $\exists$.

Let $J \subseteq U$, $\tau \in U^U$, $x \in L$ and $(i, j) \in (n \times m)$.
(a) We have: $\omega_L(S(\tau)(x))(i, j) = \sigma_{ij}(S(\tau))(x) = S(\tau)(\sigma_{ij}(x))$.
$D(S)(\tau)(\omega_L(x))(i, j) = S(\tau)(\omega_L(x)(i, j)) = S(\tau)(\sigma_{ij}(x))$.

Hence $\omega_L \circ S(\tau) = D(S)(\tau) \circ \omega_L$.

(b) We have: $\omega_L(\exists(J)(x))(i,j) = \sigma_{ij}(\exists(J)(x))$.

$D(\exists)(J)(\omega_L(x))(i,j) = \exists(J)(\omega_L(x)(i,j)) = \exists(J)(\sigma_{ij}(x))$.

As $\exists(J)$ commutes with σ_{ij}, we obtain that $D(\exists)(J)\circ\omega_L = \omega_L\circ\exists(J)$.

$\square$

LEMMA 4.17. *Let* $\mathcal{T} = (T, (X_t)_{t\in T}, R, Q, 0)$ *be a tense system. Then* $C(F_T^{U,n\times m}) \simeq F_T^U$.

PROOF: By [19, Lemma 4.5.], we have that $\mathbf{2} \simeq C(D(\mathbf{2}))$. Let us consider an isomorphism $u : \mathbf{2} \longrightarrow C(D(\mathbf{2})) \subseteq D(\mathbf{2})$. We will define the function $\Phi : F_T^U \longrightarrow C(F_T^{U,n\times m})$, by: $\Phi((f_t)_{t\in T}) = (g_t)_{t\in T}$ with $f_t : X_t^U \longrightarrow \mathbf{2}$, $g_t : X_t^U \longrightarrow D(\mathbf{2})$, $g_t = u \circ f_t$, for every $t \in T$. It is easy to prove that Φ is an injective morphism of tense polyadic Boolean algebras. Let $(h_t)_{t\in T} \in C(F_T^{U,n\times m})$. Then $\sigma_{ij}^{\mathcal{T}}((h_t)_{t\in T}) = (h_t)_{t\in T}$, for every $(i,j) \in (n \times m)$ iff $\sigma_{ij} \circ h_t = h_t$, for every $(i,j) \in (n \times m)$ and $t \in T$ iff $\sigma_{ij}(h_t(x)) = h_t(x)$, for every $(i,j) \in (n \times m)$, $t \in T$ and $x \in X_t^U$ iff $h_t(x) \in C(D(\mathbf{2})) \simeq \mathbf{2}$, for every $t \in T$ and $x \in X_t^U$, hence Φ is surjective. $\square$

5.   Representation theorem

This section contains the main result of this paper: the representation theorem for tense polyadic $\mathrm{LM}_{n\times m}$-algebras (see Theorem 5.2). It extends the representation of tense polyadic Boolean algebras ([21]), as well as the representation of tense $\mathrm{LM}_{n\times m}$-algebras ([18]). In order to obtain a proof of this representation theorem we need some preliminary results.

PROPOSITION 5.1. *Let* $\mathcal{T} = (T, (X_t)_{t\in T}, R, Q, 0)$ *be a tense system. Then there exists an injective morphism of tense polyadic* $\mathrm{LM}_{n\times m}$-*algebras* $\lambda : D(F_T^U) \longrightarrow F_T^{U,n\times m}$.

PROOF: We have that $D(F_T^U) = \{\nu : (n \times m) \longrightarrow F_T^U \mid r \leq s$ implies $\nu(i,r) \leq \nu(i,s), \nu(r,j) \leq \nu(s,j)\}$. Let $\nu \in D(F_T^U)$. For every $(i,j) \in (n \times m)$ we will denote $\nu(i,j) = (g_t^{ij})_{t\in T}$, where $g_t^{ij} : X_t^U \longrightarrow \mathbf{2}$, such that, for all $r \leq s$ and $t \in T$, $g_t^{ir} \leq g_t^{is}$, $g_t^{rj} \leq g_t^{sj}$. We will define $\lambda : D(F_T^U) \longrightarrow F_T^{U,n\times m}$, $\lambda(\nu) = (f_t)_{t\in T}$, where for every $t \in T$, $x \in X_t^U$ and $(i,j) \in (n \times m)$, $f_t : X_t^U \longrightarrow D(\mathbf{2})$ is defined by: $f_t(x)(i,j) = g_t^{ij}(x)$. As

g_t^{ij} are increasing it follows that $f_t(x)$ are increasing, so $f_t(x) \in D(\mathbf{2})$. We must prove that λ is an morphism of tense polyadic $\text{LM}_{n \times m}$-algebras, i.e. λ is an morphism of tense $\text{LM}_{n \times m}$-algebras and it commutes with operations S and $\exists$.

Let $\nu_1, \nu_2 \in D(F_T^U)$ with $\nu_1(i,j) = (g_t^{ij})_{t \in T}$ and $\nu_2(i,j) = (u_t^{ij})_{t \in T}$, where $g_t^{ij}, u_t^{ij} : X_t^U \longrightarrow \mathbf{2}$.

We want to prove that $\lambda(0_{D(F_T^U)}) = 0_{F_{TU,n \times m}}$. We have:

(1) $0_{D(F_T^U)} = 0 : (n \times m) \longrightarrow F_T^U$, $0(i,j) = (0_t^{ij})_{t \in T}$ with $0_t^{ij} : X_t^U \longrightarrow \mathbf{2}$, $0_t^{ij}(x) = 0$, for all $t \in T$ and $x \in X_t^U$.

(2) $0_{F_{TU,n \times m}} = (0_t)_{t \in T}$ with $0_t : X_t^U \longrightarrow D(2)$ is defined by $0_t(x)(i,j) = 0$, for all $x \in X_t^U$ and $(i,j) \in (n \times m)$.

By (1) and (2) we obtain that $0_t(x)(i,j) = 0_t^{ij}(x)$, for all $t \in T, x \in X_t^U$ and $(i,j) \in (n \times m)$, so $\lambda(0_{D(F_{TU})}) = 0_{F_{TU,n \times m}}$. In a similar way we can prove that $\lambda(1_{D(F_T^U)}) = 1_{F_{TU,n \times m}}$.

- We will prove that $\lambda(\nu_1 \vee \nu_2) = \lambda(\nu_1) \vee \lambda(\nu_2)$.

  By the definition of λ, we have: $\lambda(\nu_1 \vee \nu_2) = (p_t)_{t \in T}$, $\lambda(\nu_1) = (f_t)_{t \in T}$, $\lambda(\nu_2) = (h_t)_{t \in T}$, where $p_t, f_t, h_t : X_t^U \longrightarrow D(\mathbf{2})$, $(p_t(x))(i,j)(i,j) = (g_t^{ij} \vee u_t^{ij})(x)$, $(f_t(x))(i,j) = g_t^{ij}(x)$, $(h_t(x))(i,j) = u_t^{ij}(x)$, for al $t \in T, x \in X_t^U$ and $(i,j) \in (n \times m)$.

  Let $t \in T$ and $x \in X_t^U$. The relation $(g_t^{ij} \vee u_t^{ij})(x) = g_t^{ij}(x) \vee u_t^{ij}(x)$ is true, so it follows that $(p_t(x))(i,j) = (f_t(x))(i,j) \vee (h_t(x))(i,j)$, for all $(i,j) \in (n \times m)$. Hence $\lambda(\nu_1 \vee \nu_2) = \lambda(\nu_1) \vee \lambda(\nu_2)$.

In the same way we can prove that $\lambda(\nu_1 \wedge \nu_2) = \lambda(\nu_1) \wedge \lambda(\nu_2)$.

- We must prove that $\lambda \circ \sigma_{ij} = \sigma_{ij} \circ \lambda$.

  Let $(i,j) \in (n \times m)$. We have: $(\sigma_{ij}(\nu_1))(i,j) = \sigma_{ij}(\nu_1(i,j)) = \sigma_{ij}((g_t^{ij})_{t \in T})$ $= (\sigma_{ij} \circ g_{t \in T}^{ij})$, hence $\lambda(\sigma_{ij}(\nu_1)) = (f_t)_{t \in T}$ with $f_t(x)(i,j) = (\sigma_{ij} \circ g_t^{ij})(x)$, for all $t \in T, x \in X_t^U$ and $(i,j) \in (n \times m)$.
  $\sigma_{ij}(\lambda(\nu_1)) = \sigma_{ij}((h_t)_{t \in T}) = (\sigma_{ij} \circ h_t)_{t \in T}$, where $h_t(x)(i,j) = g_t^{ij}(x)$. Let $x \in X_t^U$ and $t \in T$. It results that $f_t(x)(i,j) = \sigma_{ij}(h_t(x)(i,j))$, for all $(i,j) \in (n \times m)$, so $\lambda(\sigma_{ij}(\nu_1)) = \sigma_{ij}(\lambda(\nu_1))$.

- We will to prove that $\lambda \circ G = G \circ \lambda$ and $\lambda \circ H = H \circ \lambda$.

  Let $(i,j) \in (n \times m)$. Then $D(G)(\nu_1)(i,j) = G(\nu_1(i,j)) = G((g_t^{ij})_{t \in T}) = (h_t^{ij})_{t \in T}$, where $h_t^{ij}(x) = \bigwedge \{g_s^{ij}(i \circ x) \mid tRs, s \in T\}$, for every $t \in T$

and $x \in X_t^U$. It follows that $\lambda(D(G)(\nu_1)) = (f_t)_{t \in T}$ with $f_t(x)(i,j) = h_t^{ij}(x)$, for every $t \in T$ and $x \in X_t^U$.

$G(\lambda(\nu_1)) = G((g_t^{ij})_{t \in T}) = (u_t^{ij})$ with $u_t^{ij} = \bigwedge\{g_s^{ij}(i \circ x) \mid tRs\}$, for every $t \in T$ and $x \in X_t^U$. We can se that $f_t(x)(i,j) = u_t^{ij}(x)$ for all $t \in T$, $x \in X_t^U$ and $(i,j) \in (n \times m)$, hence $\lambda \circ G = G \circ \lambda$. In a similar way we can prove that $\lambda \circ H = H \circ \lambda$.

- We will to prove that λ commute with S.

  Let $\tau \in U^U$ and $(i,j) \in (n \times m)$. Then $D(S)(\tau)(\nu_1)(i,j) = S(\tau)(\nu_1(i,j)) = S(\tau)((g_t^{ij})_{t \in T}) = (h_t^{ij})_{t \in T}$ with $h_t^{ij}(x) = g_t^{ij}(x \circ \tau)$. It follows that $(\lambda \circ D(S)(\tau))(\nu_1) = \lambda(D(S)(\tau)(\nu_1)) = (f_t)_{t \in T}$, where $f_t(x)(i,j) = h_t^{ij}(x)$.
  $(S(\tau) \circ \lambda)(\nu_1) = S(\tau)(\lambda(\nu_1)) = (p_t)_{t \in T}$, where $p_t(x)(i,j) = g_t^{ij}(x \circ \tau)$. It follows: $f_t(x)(i,j) = p_t(x)(i,j)$, for all $t \in T$, $x \in X_t^U$ and $(i,j) \in (n \times m)$, so $\lambda \circ D(S)(\tau) = S(\tau) \circ \lambda$.

- We will to prove that λ commute with $\exists$.

  Let $J \subseteq U$ and $(i,j) \in (n \times m)$. We have:
  $D(\exists)(J)(\nu_1)(i,j) = \exists(J)(\nu_1(i,j)) = \exists(J)((g_t^{ij})_{t \in T}) = (h_t^{ij})_{t \in T}$, where $h_t^{ij}(x) = \bigvee\{g_t^{ij}(y) \mid y \in X_t^U, y \mid_{U \setminus J} = x \mid_{U \setminus J}\}$, for all $t \in T$ and $x \in X_t^U$. It follows: $(\lambda \circ D(\exists)(J))(\nu_1) = \lambda(D(\exists)(J)(\nu_1))) = (f_t)_{t \in T}$ with $f_t(x)(i,j) = h_t^{ij}(x)$, for every $t \in T$ and $x \in X_t^U$.
  $(\exists(J) \circ \lambda)(\nu_1) = \exists(J)(\lambda(\nu_1)) = \exists(J)((p_t)_{t \in T}) = (v_t)_{t \in T}$, where $p_t(x)(i,j) = g_t^{ij}(x)$ and $v_t(x)(i,j) = \bigvee\{p_t(y)(i,j) \mid y \in X_t^U, y \mid_{U \setminus J} = x \mid_{U \setminus J}\}$. It results that $v_t(x)(i,j) = h_t^{ij}(x)$ for every $t \in T$, $x \in X_t^U$ and $(i,j) \in (n \times m)$ so $(v_t)_{t \in T} = (h_t^{ij})_{t \in T}$, i.e. $\lambda \circ D(\exists)(J) = \exists(J) \circ \lambda$.

- We will to prove that λ is injective.
  Let $\nu_1, \nu_2 \in D(F_T^U)$, $\nu_1(i,j) = (g_t^{ij})_{t \in T}$ and $\nu_2(i,j) = (p_t^{ij})_{t \in T}$, for all $(i,j) \in (n \times m)$ such that $\lambda(\nu_1) = \lambda(\nu_2)$. Using the definition of λ, we obtain that $g_t^{ij}(x) = p_t^{ij}(x)$, for all $t \in T, x \in X_t^U$ and $(i,j) \in (n \times m)$. It follows that $\nu_1(i,j) = \nu_2(i,j)$, for all $(i,j) \in (n \times m)$, hence $\nu_1 = \nu_2$. The injectivity of λ was proved.   □

The following theorem shows that any tense polyadic $\mathrm{LM}_{n \times m}$-algebra can be represented by means of the tense polyadic $\mathrm{LM}_{n \times m}$-algebra $F_T^{U, n \times m}$ associated with a certain tense system $\mathcal{T}$.

THEOREM 5.2. (**Representation theorem**) *Let* $(\mathcal{L}, U, S, \exists, G, H)$ *be a tense polyadic* $LM_{n \times m}$*-algebra, locally finite, of infinite degree and* Γ *be a proper filter of* L *with* $J_p = \emptyset$ *for all* $p \in \Gamma$*. Then there exist a tense system* $\mathcal{T} = (T, (X_t)_{t \in T}, R, Q, 0)$ *and a morphism of tense polyadic* $LM_{n \times m}$*-algebras* $\Phi : L \longrightarrow F_T^{U, n \times m}$ *such that, for all* $p \in \Gamma$*, the following property holds:*

(P) $\Phi(p) = (f_t)_{t \in T} \Rightarrow (f_0(x))(i, j) = 1$*, for all* $x \in X_t^U$ *and* $(i, j) \in (n \times m)$*.*

PROOF: Let $(\mathcal{L}, U, S, \exists, G, H)$ be a tense polyadic $LM_{n \times m}$-algebra and Γ be a proper filter of L. By Lemma 4.13, we have that $(C(L), U, S, \exists, C(G), C(H))$ is a tense polyadic Boolean algebra and $\Gamma_0 = \Gamma \cap C(L)$ is a proper filter of $C(L)$. Applying the representation theorem for tense polyadic Boolean algebras, it follows that there exist a tense system $\mathcal{T} = (T, (X_t)_{t \in T}, R, Q, 0)$ and a morphism of tense polyadic Boolean algebras $\mu : C(L) \longrightarrow F_T^U$, such that for all $p \in \Gamma_o$ the following property holds: $\mu(p) = (g_t)_{t \in T} \Rightarrow g_0(x) = 1$, for all $x \in X_t^U$. Let $D(\mu) : D(C(L)) \longrightarrow D(F_T^U)$ be the corresponding morphism of μ by the functor D. By using Lemma 4.16, we have an injective morphism of tense polyadic $LM_{n \times m}$-algebras $\omega_L : L \longrightarrow D(C(L))$ and by using Proposition 5.1, we have an injective morphism of tense polyadic $LM_{n \times m}$-algebra $\lambda : D(F_T^U) \longrightarrow F_T^{U, n \times m}$. We consider the following morphisms of tense polyadic $LM_{n \times m}$-algebras:

$$L \xrightarrow{\omega_L} D(C(L)) \xrightarrow{D(\mu)} D(F_F^U) \xrightarrow{\lambda} F_T^{U, n \times m}$$

It follows that $\lambda \circ D(\mu) \circ \omega_L$ is an morphism of tense polyadic $LM_{n \times m}$-algebras.

Now, we will verify the condition (P) of the theorem. Let $p \in \Gamma$ and $(i, j) \in (n \times m)$. We know that $\omega_L(p)(i, j) = \sigma_{ij}(p)$ and $\sigma_{ij}(p) \in \Gamma_0$. Then $D(\mu)(\omega_L(p)) = \mu \circ \omega_L(p)$, hence $(\mu \circ \omega_L(p))(i, j) = \mu(\omega_L(p)(i, j)) = \mu(\sigma_{ij}(p))$. We assume that $\mu(\sigma_{ij}p) = (g_t^{ij})_{t \in T}$, where $g_t^{ij} : X_t^U \longrightarrow \mathbf{2}$. As $\sigma_{ij}p \in \Gamma_0$, we obtain that $g_0^{ij}(x) = 1$, for every $x \in X_t^U$. It results that: $\Phi(p) = \lambda(D(\mu)(\omega_L(p))) = \lambda(D(\mu)(\sigma_{ij}p)) = \lambda(\mu(\sigma_{ij}p))$. It follows that $\Phi(p)(i, j) = (f_t)_{t \in T}$, where, applying the proof of Proposition 5.1, we have that $f_t(x)(i, j) = g_t^{ij}(x)$, for every $t \in T$ and $x \in X_t^U$. Then, $f_0(x)(i, j) = g_0^{ij}(x) = 1$. □

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