**B**ulletin of the **S**ection of **L**ogic
Volume 49/2 (2020), pp. 181–184
http://dx.doi.org/10.18778/0138-0680.2020.10

C O P E
Member since 2018
JM13714

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# COMPOUNDING OBJECTS

**Abstract**

We prove a characterization theorem for filters, proper filters and ultrafilters which is a kind of converse of Łoś's theorem. It is more natural than the usual intuition of these terms as large sets of coordinates, which is actually unconvincing in the case of ultrafilters. As a bonus, we get a very simple proof of Łoś's theorem.

*Keywords*: Łoś's theorem, converse of Łoś's theorem, filter, proper filter, ultrafilter.

One of the useful methods in formal sciences is the construction of complex structures by compounding objects of simpler structures. For example, by compounding real numbers in triples we construct $(\mathbb{R}^3, +^3, <^3)$ from $(\mathbb{R}, +, <)$. The operation $+^3$ and the relation $<^3$ are defined coordinate-wise e.g. $(2, 3, 1) +^3 (1, -1, 0) = (3, 2, 1)$ and $(2, 3, 1) <^3 (3, 4, 2)$, but we have to be aware that the total order $<$ turns into the partial order $<^3$ (e.g. neither $(2, 3, 1) <^3 (3, 2, 1)$ nor $(3, 2, 1) <^3 (2, 3, 1)$). The interesting question is whether it is possible to construct a compound system with the same 1-order properties as the systems it is compound of. In this way we could construct nonstandard models of standard (intended) structures. For example, by compounding standard $\mathsf{PA}$ structures of natural numbers we could get a nonstandard (non-isomorphic) model of standard $\mathsf{PA}$. For the systems $\mathbf{S}_i = (S_i, \ldots, \circ_i, \ldots, R_i, \ldots)$, $i \in J$, we may always construct the compound system $\Pi\mathbf{S}_i = (\Pi S_i, \ldots, \Pi\circ_i, \ldots, \Pi R_i, \ldots) = (S, \ldots, \circ, \ldots, R, \ldots)$, with sequences $a = (a_1, a_2, a_3, \ldots)$, $a_i \in S_i$, as elements of $S$ and operations $o$ and relations $R$ defined coordinate-wise:

$$a \circ b = (a_1, a_2, a_3, \ldots) \circ (b_1, b_2, b_3, \ldots) = (a_1 \circ_1 b_1, a_2 \circ_2 b_2, a_3 \circ_3 b_3, \ldots)$$

$$aRb \equiv \forall i(a_i R_i b_i) \text{ i.e. } aRb \equiv \{i : a_i R_i b_i\} = J$$

But, as we have already pointed out, the compound system will not share the properties of the components (compare the totality of $<$ and the partiality of $<^3$). It could share them if instead of

$$aRb \equiv (\forall i)(a_i R_i b_i) \equiv \{i : a_i R_i b_i\} = J$$

we define

$$aRb \equiv (\underline{\forall} i)(a_i R_i b_i) \equiv \{i : a_i R_i b_i\} \in B$$

with some appropriate $B$. We may think of B as a family of "big" subsets of $J$ and of $\underline{\forall}$ as meaning "for almost all". It means that something is true $\underline{\forall} i \in J$ if and only if it is true on a big subset of $J$. It was proved by Łoś (in the famous Łoś 's Theorem) that the appropriate "big" families are ultrafilters. Here we want to prove a kind of converse which is the following characterization theorem for filters, proper filters and ultrafilters:

THEOREM 1 (Characterization theorem).

*(i)* *The equality in the compound system, defined by $a = b \equiv \{i : a_i = b_i\} \in B$, is an equivalence relation if and only if $B$ is a filter. Moreover, the equivalence relation is then a congruence i.e. if $a = a^*$ and $b = b^*$ then $a \circ b = a^* \circ b^*$.*

*(ii)* *The equality $a = b \equiv \{i : a_i = b_i\} \in B$ is an equivalence relation and obeys the principle of contradiction i.e. $\neg((a = b) \wedge (a \neq b))$ if and only if $B$ is a proper filter, where $a \neq b$ if $\{i : a_i \neq b_i\} \in B$. Furthermore, compound relations defined by $aRb \equiv \{i : a_i R_i b_i\} \in B$ then obey the principle of contradiction too i.e. $\neg((aRb) \wedge (a\hat{R}b))$, where $a\hat{R}b$ if $\{i : a_i \hat{R}_i b_i\} \in B$.*

*(iii)* *The equality $a = b \equiv \{i : a_i = b_i\} \in B$ is an equivalence relation, satisfies the principle of contradiction and obeys the principle of excluded middle i.e. $(a = b) \vee (a \neq b)$ if and only if $B$ is an ultrafilter. Furthermore, compound relations defined by $aRb \equiv \{i : a_i R_i b_i\} \in B$ then obey the principle of excluded middle too i.e. $(aRb) \vee (a\hat{R}b)$.*

From the characterization theorem it easily follows that $\underline{\forall}$ distributes through every truth-functional connective. Namely, if $X_i$ and $Y_i$ are formulae evaluated in the component $S_i$, we have the following:

Corollary 1.

1. $(\underline{\forall} i)(X_i \wedge Y_i) \equiv (\underline{\forall} i)X_i \wedge (\underline{\forall} i)Y_i$
2. $(\underline{\forall} i)(\neg X_i) \equiv \neg(\underline{\forall} i)X_i$

Note that $\forall$ satisfies (1) but does not satisfy (2). Using this corollary and the process of Skolemization, it is easy to prove Łoś's Theorem.

Theorem 2 (Łoś's Theorem). *For every* 1*-order formula* $F, S \models F$ *if and only if* $(\underline{\forall} i)S_i \models F_i$*, where every operation symbol* $\circ$ *and every relation symbol* $R$ *in* $F$ *is replaced by the corresponding operation symbol* $\circ_i$ *and the corresponding relation symbol* $R_i$ *in* $F_i$*.*

Proof of the characterization theorem: In what follows $X = \{i : a_i = b_i\}$, $Y = \{i : b_i = c_i\}$ and $Z = \{i : a_i = c_i\}$.

Proof of (i):

$a = a$ if and only if $\{i : a_i = a_i\} = J \in B$

$a = b \wedge b = c \rightarrow a = c$ if and only if $X \in B \wedge Y \in B \rightarrow X \cap Y \subset Z \in B$ if and only if $(X \in B \wedge Y \in B \rightarrow X \cap Y \in B) \wedge (Z \in B \wedge Z \subset U \rightarrow U \in B)$.

But $J \in B$, $(X \in B \wedge Y \in B \rightarrow X \cap Y \in B)$ and $(Z \in B \wedge Z \subset U \rightarrow U \in B)$ define a filter. Furthermore, if $a = a^* \wedge b = b^*$ then $\{i : a_i = a_i^*\} \in B$ and $\{i : b_i = b_i^*\} \in B$ and it follows that $\{i : a_i \circ b_i = a_i^* \circ b_i^*\} \in B$ because $\{i : a_i = a_i^*\} \cap \{i : b_i = b_i^*\} \subset \{i : a_i \circ b_i = a_i^* \circ b_i^*\}$. □

Proof of (ii):
$\neg((a = b) \wedge (a \neq b))$ if and only if $\neg(X \in B \wedge X^c \in B)$ i.e. $X^c \in B \rightarrow \neg(X \in B)$ i.e. the filter is proper. Furthermore, then $\neg((aRb) \wedge \neg(aRb))$ for every $R$ because $\neg(X \in B \wedge X^c \in B)$ for every $X$. □

Proof of (iii):
$(a = b) \vee (a \neq b)$ if and only if $X \in B \vee X^c \in B$ i.e. $\neg(X \in B) \rightarrow X^c \in B$ i.e. the filter is ultrafilter. Furthermore, then $(aRb) \vee \neg(aRb)$ for every $R$ because $\neg X \in B \rightarrow X^c \in B$ for every $X$. □

Proof of the corollary: (1) is evidently true and (2) follows from $\neg(X \in B) \leftrightarrow X^c \in B$. □

Proof of the Łoś's theorem: For atomic formulae $F$, "$S \models F$ if and only if $(\underline{\forall} i)S_i \models F_i$" is the definition of $\models$. For truth functional $F$ we have to prove that $\forall$ distributes through truth functional connectives and this follows from the corollary. For quantified $F = \exists xG$: $S \models_v \exists xG$ means $(\exists a)S \models_{v(a/x)} G$. By induction $S \models_{v(a/x)} G \leftrightarrow (\underline{\forall} i)S_i \models_{v_i(a_i/x)} G_i$. By skolemization $(\exists a)(\underline{\forall} i)S_i \models_{v_i(a_i/x)} G_i \leftrightarrow (\underline{\forall} i)(\exists a)S_i \models_{v_i(a_i/x)} G_i$. By definition of $\models$ this is equivalent to $(\underline{\forall} i)S_i \models_{v_i} \exists xG_i$. $\square$

## References

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