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ON THE DEFINABILITY OF LEŚNIEWSKI'S COPULA 'IS' IN SOME ONTOLOGY-LIKE THEORIES

Abstract

We formulate a certain subtheory of Ishimoto's [1] quantifier-free fragment of Leśniewski's ontology, and show that Ishimoto's theory can be reconstructed in it. Using an epimorphism theorem we prove that our theory is complete with respect to a suitable set-theoretic interpretation. Furthermore, we introduce the name constant $\mathsf{1}$ (which corresponds to the universal name 'object') and we prove its adequacy with respect to the set-theoretic interpretation (again using an epimorphism theorem). Ishimoto's theory enriched by the constant $\mathsf{1}$ is also reconstructed in our formalism with into which $\mathsf{1}$ has been introduced. Finally we examine for both our theories their quantifier extensions and their connections with Leśniewski's classical quantified ontology.

Keywords: Leśniewski's ontology, elementary ontology, quantifier-free fragment of ontology, copula 'is', calculus of names, ontology-like theories, subtheories of Leśniewski's ontology.

Introduction

The first part of this paper (sections 1–5) is an introduction to first-order and quantifier-free theories with Leśniewski's copula 'is' ('ε'). Some of these theories also have the name constant $\mathsf{1}$ (which corresponds to the universal name 'object'). We present various connections between these theories and their semantic investigation in the following standard set-theoretic interpretation of 'is' and 'object' (in an arbitrary family $\mathcal{F}$ of sets):

$$X \,\boldsymbol{\varepsilon}_{\mathcal{F}}\, Y \iff X \text{ is a singleton and } X \subseteq Y,$$
$$\mathbf{1}_{\mathcal{F}} = \bigcup \mathcal{F}\,.$$

Notice that quantifier-free theories can be treated as pure (i.e., quantifier-free) calculi of names, in which individual variables are schematic letters for general names and specific symbols are appropriate logical constants.

In Section 6 we formulate a subtheory of the quantifier-free fragment ontology presented by Ishimoto in [1]. Using an epimorphism theorem we show that this subtheory is complete in the following set-theoretic semantics for 'is' (in an arbitrary family $\mathcal{F}$ of sets):

$$X \,\boldsymbol{\varepsilon}^{\star}_{\mathcal{F}}\, Y \iff \emptyset \neq X \subsetneq Y \text{ or both } X \text{ is a singleton and } X = Y.$$

We reconstruct Ishimoto's theory in this subtheory. (Notice that $X \,\boldsymbol{\varepsilon}_{\mathcal{F}}\, Y$ iff $X \,\boldsymbol{\varepsilon}^{\star}_{\mathcal{F}}\, X$ and $X \,\boldsymbol{\varepsilon}^{\star}_{\mathcal{F}}\, Y$.) We also put in place conditions that suffice for us to obtain Leśniewski's elementary ontology on the basis of our subtheory.

In Section 7 we introduce into our formalism the constant '$\mathbf{1}$' and prove its completeness again using an epimorphism theorem. Ishimoto's theory enriched by '$\mathbf{1}$' is also reconstructed in our subtheory with $\mathbf{1}$. We examine the connections both theories have with Leśniewski's first-order ontology.

In Section 8 we study the possibility of defining the predicate designated by our subtheory (i.e., for the relation $\boldsymbol{\varepsilon}^{\star}_{\mathcal{F}}$) in the quantifier-free ontology and the first-order ontology.

1. Open first-order theories vs pure calculi of names

Let L be a first-order language. A formula of L is said to be *open* iff it does not contain any quantifiers (i.e., if it does not contain any bound individual variables). Let L^{o} be the language of open formulas in L (i.e., the alphabet of L^{o} obtained from the alphabet of L by omitting quantifiers and bound individual variables). If F denotes the set of all formulas of L then F^{o} denotes the set of all open formulas in L.

Notice that all open theses of any first-order theory we can treated as universal. Thus, any open thesis $\varphi(x_1, \dots, x_n)$ is equivalent to the closed thesis $\forall x_1 \dots \forall x_n\, \varphi(x_1, \dots, x_n)$.

By a *quantifier-free theory* we understand any theory which for some first-order language L satisfies the following three conditions:

1. it is built from the set F^{o} of open formulas of L,
2. the set of its theses includes the set of formulas from F^{o} which are instances of classical tautologies,
3. the set of its theses is closed under *modus ponens* and the rule of substitution for free individual variables.

Remark 1.1. Quantifier-free theories understood in the above way can be treated as *pure* (i.e., quantifier-free) *calculi of names*, in which individual variables are schematic letters for general names and specific symbols are appropriate logical constants. Of course, when we examine pure calculi of names, we can replace individual variables 'x', 'y', 'z', etc., with appropriate schematic name letters, e.g.: 'S', 'P', 'M', etc. (cf. [3, pp. 11–22] and [4, pp. 5–6]). □

Remark 1.2. Models for pure calculi of names are ordered pairs of the form $\langle U, d\rangle$, where U is any set (a *universe*) and d is a *function of denotation* from Var into 2^U, i.e., for any variable $\boldsymbol{x}$ we assign a subset of U which is treated as a reference of $\boldsymbol{x}$ (cf. [3, pp. 25–27] and [4, pp. 6–7]). □

In both cases where $\boldsymbol{T}$ is a first-order theory or $\boldsymbol{T}$ is a quantifier-free theory, the set of all theses of $\boldsymbol{T}$ will be denoted by $\mathrm{Th}(\boldsymbol{T})$.

Let $\boldsymbol{T}$ be a first-order theory built in a set of formulas F. By a *quantifier-free fragment* of $\boldsymbol{T}$ we understand a quantifier-free theory whose theses are all these and only those open formulas of F^{o} which are theses of $\boldsymbol{T}$. Formally, a quantifier-free theory $\boldsymbol{N}$ is a propositional quantifier-free fragment of a first-order theory $\boldsymbol{T}$ iff $\mathrm{Th}(\boldsymbol{N}) = F^{\mathrm{o}} \cap \mathrm{Th}(\boldsymbol{T})$. Obviously, $\boldsymbol{T}$ may not have a quantifier-free fragment, but if it has a such fragment, it is only one.

A first-order theory $\boldsymbol{T}$ is said to be *open* iff all specific axioms of $\boldsymbol{T}$ are open formulas. In this case, let T^{o} be a quantifier-free theory built in F^{o} and having the same specific axioms as $\boldsymbol{T}$. It is known that (cf., e.g., [6, p. 329]):

THEOREM 1.1. *For any open first-order theory $\boldsymbol{T}$, the quantifier-free theory T^{o} is the quantifier-free fragment of $\boldsymbol{T}$, i.e.,* $\mathrm{Th}(T^{\mathrm{o}}) = F^{\mathrm{o}} \cap \mathrm{Th}(\boldsymbol{T})$.

2. Two elementary Leśniewskian ontologies

Leśniewski's original ontology investigated the copula 'is' represented by the sign 'ε'. This theory is creative in the following sense: it has a creative language and creative definitions (see, e.g., [7, 8, 5]). The only axiom of Leśniewski's ontology is the following formula:

$$x \mathrel{\varepsilon} y \;\leftrightarrow\; \exists z\, z \mathrel{\varepsilon} x \wedge \forall z,u(z \mathrel{\varepsilon} x \wedge u \mathrel{\varepsilon} x \to z \mathrel{\varepsilon} u) \wedge \forall z(z \mathrel{\varepsilon} x \to z \mathrel{\varepsilon} y) \tag{λ}$$

To avoid creativity in ontology, it is studied as a first-order theory (see, e.g., [2, 7, 5]).

2.1. The theory $\mathbf{\Lambda}$

Let L_ε be a first-order language (without equality) with exactly one specific constant – the binary predicate 'ε'. Moreover, let For_ε be the set of all formulas of L_ε and $\mathrm{For}_\varepsilon^{\mathrm{o}}$ be the set of all open formulas from For_ε.

In [10, 5], the first-order theory in the set For_ε based only on axiom (λ) is examined. We denote this theory by '$\mathbf{\Lambda}$'. Directly from the axiom we obtain:

Fact 2.1. *The following formulas are theses of* $\mathbf{\Lambda}$:

$$x \mathrel{\varepsilon} x \;\leftrightarrow\; \exists z\, z \mathrel{\varepsilon} x \;\wedge\; \forall z,u(z \mathrel{\varepsilon} x \wedge u \mathrel{\varepsilon} x \to z \mathrel{\varepsilon} u) \tag{\$}$$

$$x \mathrel{\varepsilon} y \to x \mathrel{\varepsilon} x \tag{ε_1}$$

$$x \mathrel{\varepsilon} y \wedge y \mathrel{\varepsilon} z \to x \mathrel{\varepsilon} z \tag{ε_2}$$

$$x \mathrel{\varepsilon} y \wedge y \mathrel{\varepsilon} y \to y \mathrel{\varepsilon} x \tag{ε_3}$$

$$x \mathrel{\varepsilon} y \wedge y \mathrel{\varepsilon} z \to y \mathrel{\varepsilon} x \tag{ε_4}$$

Fact 2.2. 1. *From* (ε_4) *we obtain* (ε_3). *From* (ε_1) *and* (ε_3) *we obtain* (ε_4).
2. *From* (ε_1)–(ε_3) *we obtain the "$\to$" part of* (λ):

$$x \mathrel{\varepsilon} y \;\to\; \exists z\, z \mathrel{\varepsilon} x \wedge \forall z,u(z \mathrel{\varepsilon} x \wedge u \mathrel{\varepsilon} x \to z \mathrel{\varepsilon} u) \wedge \forall z(z \mathrel{\varepsilon} x \to z \mathrel{\varepsilon} y) \tag{$\to\lambda$}$$

3. *From* (ε_3) *and* (ε_2) *we obtain the "$\to$" part of* (\$):

$$x \mathrel{\varepsilon} x \;\to\; \exists z\, z \mathrel{\varepsilon} x \wedge \forall z,u(z \mathrel{\varepsilon} x \wedge u \mathrel{\varepsilon} x \to z \mathrel{\varepsilon} u) \tag{$\to$\$}$$

4. *The converse implications*:

$$\exists z\, z \mathrel{\varepsilon} x \wedge \forall z,u(z \mathrel{\varepsilon} x \wedge u \mathrel{\varepsilon} x \to z \mathrel{\varepsilon} u) \to x \mathrel{\varepsilon} x \tag{$\leftarrow$\$}$$

$$\exists z\, z \mathrel{\varepsilon} x \wedge \forall z,u(z \mathrel{\varepsilon} x \wedge u \mathrel{\varepsilon} x \to z \mathrel{\varepsilon} u) \wedge \forall z(z \mathrel{\varepsilon} x \to z \mathrel{\varepsilon} y) \to x \mathrel{\varepsilon} y \tag{$\leftarrow\lambda$}$$

we do not obtain from (ε_1)–(ε_4).

Proof: *Ad* 4. In the L_ε-structure $\mathfrak{A} = \langle U_\mathfrak{A}, \varepsilon_\mathfrak{A}\rangle$, where $U_\mathfrak{A} := \{0,1\}$ and $\varepsilon_\mathfrak{A} := \{\langle 0,0\rangle, \langle 0,1\rangle\}$, formulas ($\varepsilon_1$)–($\varepsilon_4$) are true, but ($\leftarrow$\$) is not true. □

It is easy to see that directly from ($\leftarrow\$$) we obtain ($\leftarrow\lambda$). Thus,

FACT 2.3. *The sets* $\{(\varepsilon_1), (\varepsilon_2), (\varepsilon_3), (\leftarrow\$)\}$ *and* $\{(\varepsilon_1), (\varepsilon_2), (\varepsilon_4), (\leftarrow\$)\}$ *create other axiomatizations of* $\mathbf{\Lambda}$. *So we have*:

$$\begin{aligned}\mathrm{Th}(\mathbf{\Lambda}) &= \mathrm{Th}((\varepsilon_1)+(\varepsilon_2)+(\varepsilon_3)+(\leftarrow\$)) \\ &= \mathrm{Th}((\varepsilon_1)+(\varepsilon_2)+(\varepsilon_4)+(\leftarrow\$)).\end{aligned}$$

2.2. The theory EO

In [2] Iwanuś examined the first-order theory which he called the *elementary ontology* and which he denoted by '**EO**'. The theory is based on (λ) and the following two axioms:

$$\forall x \exists y \forall z (z \,\varepsilon\, y \leftrightarrow z \,\varepsilon\, z \wedge \neg\, z \,\varepsilon\, x)$$
$$\forall x, y \exists z \forall u (u \,\varepsilon\, z \leftrightarrow u \,\varepsilon\, x \wedge u \,\varepsilon\, y)$$

Jwanuś proved that these three axioms are enough to obtain a whole elementary ontology, i.e., for any formula φ in which the variable 'y' is not free we obtain the following thesis (see [2, Theorem 1.1a]):

$$\exists y \forall z (z \,\varepsilon\, y \leftrightarrow z \,\varepsilon\, z \wedge \varphi) \tag{$\star$}$$

Moreover, for any variable $\boldsymbol{x}$ which is different from the variable 'y' and any formula φ in which 'y' is not free we obtain the following thesis (see, e.g., [5]):

$$\exists y \forall z (z \,\varepsilon\, y \leftrightarrow z \,\varepsilon\, \boldsymbol{x} \wedge \varphi)$$

So in **EO** we can introduce the definitions of name-forming functors and name constants constructed in the way Leśniewski wanted:

$$\forall z (z \,\varepsilon\, \boldsymbol{f}(x_1, ..., x_n) \leftrightarrow z \,\varepsilon\, \boldsymbol{x} \wedge \varphi_f), \quad \text{for } \boldsymbol{x} \in \{z, x_1, \ldots, x_n\} \tag{$\mathtt{df}\ f$}$$
$$\forall z (z \,\varepsilon\, \boldsymbol{n} \leftrightarrow z \,\varepsilon\, z \wedge \varphi_n) \tag{$\mathtt{df}\ \boldsymbol{n}$}$$

where 'z', 'x_1', . . . , 'x_n' may be the only free variables in φ_f and 'z' may be the only free variable in φ_n (cf. [2, 5]). Formulas φ_f and φ_n may be instances of classical tautologies. Then we can omit them and from ($\mathtt{df}\ \boldsymbol{n}$) in the language $\mathrm{L}_{\varepsilon 1}$ with the constants 'ε' and the name constant '$\mathsf{1}$' we obtain the following definition of '$\mathsf{1}$':

$$x \,\varepsilon\, \mathsf{1} \leftrightarrow x \,\varepsilon\, x \tag{$\mathtt{df}\ \mathsf{1}$}$$

Thus, in the theory **EO** we can define the constant '$\mathsf{1}$', which in Leśniewski's theory represents the universal general name 'object'.

It is known that $\mathbf{\Lambda}$ is a proper subtheory of **EO**, i.e.,

$$\mathrm{Th}(\mathbf{\Lambda}) \subsetneq \mathrm{Th}(\mathbf{EO}).$$

For example, the following thesis of **EO**:

$$\exists y \forall z (z \mathrel{\varepsilon} y \leftrightarrow z \mathrel{\varepsilon} z) \tag{$\star\star$}$$

is not a thesis of $\mathbf{\Lambda}$. So (df 1) cannot be a definition in $\mathbf{\Lambda}$.

Theories **EO** and $\mathbf{\Lambda}$ have, however, the same open theses (see Theorem 4.4), i.e.,

$$\mathrm{For}^{\mathrm{o}}_{\varepsilon} \cap \mathrm{Th}(\mathbf{EO}) = \mathrm{For}^{\mathrm{o}}_{\varepsilon} \cap \mathrm{Th}(\mathbf{\Lambda}).$$

2.3. Set-theoretic interpretations

In this paper we will only consider first-order languages that have one or both of the binary predicates 'ε' and 'ε^*' and a possible name constant '$\mathsf{1}$'. For any first-order language L, any interpretation of L (for short: L-structure) is an relational structure with a universe $U_{\mathfrak{A}}$ in which a binary predicate π is interpreted as a binary relation $\pi_{\mathfrak{A}}$ in $U_{\mathfrak{A}}$ and, optionally, the constant $\mathsf{1}$' is interpreted as a member of $U_{\mathfrak{A}}$. For any L-structure $\mathfrak{A}$, let $\mathrm{Ver}(\mathfrak{A})$ be the set of all formulas of L which are true in $\mathfrak{A}$.

A L-structure $\mathfrak{A}$ is *epimorphic to* a L-structure $\mathfrak{B}$ iff there is a mapping f from $U_{\mathfrak{A}}$ onto $U_{\mathfrak{B}}$ such that for any predicate π of L and arbitrary $a, b \in U_{\mathfrak{A}}$ we have: $\langle a, b\rangle \in \pi_{\mathfrak{A}}$ iff $\langle f(a), f(b)\rangle \in \pi_{\mathfrak{B}}$; and, optionally, $f(\mathsf{1}_{\mathfrak{A}}) = \mathsf{1}_{\mathfrak{B}}$. It is well known that if a L-structure $\mathfrak{A}$ is epimorphic to a L-structure $\mathfrak{B}$ then $\mathrm{Ver}(\mathfrak{A}) = \mathrm{Ver}(\mathfrak{B})$.

Special L-structures are set-theoretic L-structures in which $U_{\mathfrak{A}}$ is any non-empty family $\mathcal{F}$ of sets and for any binary predicate π, the relation $\pi_{\mathfrak{A}}$ is determined in $\mathcal{F}$ by a set-theoretic formula $\Phi_\pi(X, Y)$. This relation will be denoted by $\boldsymbol{\pi}_{\mathcal{F}}$.

For 'ε' the formula $\Phi_\varepsilon(X, Y)$ has the following form:[1]

$$X \text{ is a singleton and } X \subseteq Y.$$

That is, we put:

$$\boldsymbol{\varepsilon}_{\mathcal{F}} := \left\{\langle X, Y\rangle \in \mathcal{F}^2 : \Phi_\varepsilon(X, Y)\right\}. \tag{df $\boldsymbol{\varepsilon}_{\mathcal{F}}$}$$

Optionally, if L has the constant '$\mathsf{1}$', then for any non-empty family $\mathcal{F}$ of sets we put $\mathbf{1}_{\mathcal{F}} := \bigcup \mathcal{F}$.

[1] For the predicate 'ε^*' the formula $\Phi_{\varepsilon^*}(X, Y)$ will be given on p. 250.

We say that a non-empty family $\mathcal{F}$ of sets is an *s-family* iff $\{p\} \in \mathcal{F}$, for any $p \in \bigcup \mathcal{F}$. We say that a field $\mathcal{F}$ of sets is an *s-field* iff it is a s-family. A special L-structure with a universe $\mathcal{F}$ is *s-special* (resp. *p-special*; *sf-special*) iff $\mathcal{F}$ is an s-family (resp. a power set; an *s-field*).

2.4. Epimorphism theorems for $\mathbf{\Lambda}$ and EO

In [10] the following theorem is proved:[2]

THEOREM 2.4 ([10]). *An* L_ε*-structure is a model of* $\mathbf{\Lambda}$ *iff it is epimorphic to an s-special* L_ε*-structure.*

Thus, we obtain:

THEOREM 2.5 ([10]). $\varphi \in \mathrm{Th}(\mathbf{\Lambda})$ *iff* φ *is true in any s-special* L_ε*-structure.*

PROOF: "$\Rightarrow$" Obvious. "$\Leftarrow$" Let φ be true in any s-special L_ε-structure and let $\mathfrak{A}$ be an arbitrary model of $\mathbf{\Lambda}$. In virtue of Theorem 2.4, $\mathfrak{A}$ is epimorphic to a s-special L_ε-structure $\mathfrak{B}$; so we have $\mathrm{Ver}(\mathfrak{A}) = \mathrm{Ver}(\mathfrak{B})$. But $\varphi \in \mathrm{Ver}(\mathfrak{B})$, by the assumption. Hence $\varphi \in \mathrm{Ver}(\mathfrak{A})$. So φ is true in all models of $\mathbf{\Lambda}$. Thus, $\varphi \in \mathrm{Th}(\mathbf{\Lambda})$, by Gödel's completeness theorem. $\square$

In [2, Theorem 3.II] it is proved that:

THEOREM 2.6 ([2]). $\varphi \in \mathrm{Th}(\mathbf{EO})$ *iff* φ *is true in any p-special* L_ε*-structure.*

Although Theorem 2.6 holds, not every model of **EO** is epimorphic to a p-special L_ε-structure. But in [5] the following theorem is proved:

THEOREM 2.7 ([5]). *An* L_ε*-structure is a model of* **EO** *iff it is epimorphic to an sf-special* L_ε*-structure.*

Thus, we obtain (as Theorem 2.5):

THEOREM 2.8. $\varphi \in \mathrm{Th}(\mathbf{EO})$ *iff* φ *is true in any sf-special* L_ε*-structure.*

Because every sf-special L_ε-structure with set-theoretic operations $\cup$, $\cap$ and $-$ is an atomic Boolean algebra, Theorem 2.8 is a semantical proof the fact that the theory **EO** is definitionally equivalent to the first-order theory of atomic Boolean algebras (see [5, Section 9]). A syntactic proof of this fact has been presented in [2, Theorem 2.I].

[2] For the proof see also the proof of Theorem 3.3 and footnote 3.

3. Theories EO and Λ with the name constant '1'

3.1. The theory EO+(df 1)

We wrote that we can define in the theory **EO** the name constant '1' by (df 1). Obviously, we must extend the language L_{ε} to $\mathrm{L}_{\varepsilon 1}$ and add to **EO** the definition (df 1). Let us denote by **EO**+(df 1) this conservative extension of **EO** in the set $\mathrm{For}_{\varepsilon 1}$ of formulas. Since (df 1) is true in all special structures, from Theorem 2.6 we obtain:

Corollary 3.1. 1. *An* $\mathrm{L}_{\varepsilon 1}$*-structure is a model of* **EO**+(df 1) *iff it is epimorphic to an sf-special* $\mathrm{L}_{\varepsilon 1}$*-structure.*

2. $\varphi \in \mathrm{Th}(\mathbf{EO}+(\mathrm{df}\ 1))$ *iff* φ *is true in any sf-special* $\mathrm{L}_{\varepsilon 1}$*-structure.*

3.2. The theory Λ1

As we mentioned on page 238, the formula (df 1) cannot be a definition in **Λ**. So if we want to consider the constant '1' in **Λ**, we must introduce it with a specific axiom. This axiom can be the following formula:

$$x \mathrel{\varepsilon} x \to x \mathrel{\varepsilon} 1 \tag{$\varepsilon 1_1$}$$

Let **Λ1** be the first-order theory in $\mathrm{For}_{\varepsilon 1}$ having formulas (λ) and ($\varepsilon 1_1$) as specific axioms.

Fact 3.2. *Formula* (df 1) *and the following ones are theses of* **Λ1**:

$$x \mathrel{\varepsilon} y \to x \mathrel{\varepsilon} 1 \tag{$\mathrm{c}\varepsilon 1_1$}$$

$$x \mathrel{\varepsilon} 1 \to x \mathrel{\varepsilon} x \tag{$\mathrm{c}\varepsilon 1_2$}$$

$$x \mathrel{\varepsilon} 1 \ \leftrightarrow\ \exists z\, z \mathrel{\varepsilon} x \wedge \forall z \forall u (z \mathrel{\varepsilon} x \wedge u \mathrel{\varepsilon} x \to z \mathrel{\varepsilon} u) \tag{\$1}$$

$$1 \mathrel{\varepsilon} x \to x \mathrel{\varepsilon} x \tag{$\varepsilon 1_2$}$$

Proof: *For* ($\mathrm{c}\varepsilon 1_1$): We use ($\varepsilon 1_1$) and ($\varepsilon_1$). *For* ($\mathrm{c}\varepsilon 1_2$): We use ($\varepsilon_1$). *For* (df 1): We use ($\varepsilon 1_1$) and ($\mathrm{c}\varepsilon 1_2$). *For* (\$1): We use (\$) and (df 1).

For ($\varepsilon 1_2$): By (λ) and (df 1), we obtain:

$$\begin{aligned} 1 \mathrel{\varepsilon} x \ &\leftrightarrow\ \exists z\, z \mathrel{\varepsilon} 1 \ \wedge\ \forall z, u (z \mathrel{\varepsilon} 1 \wedge u \mathrel{\varepsilon} 1 \to z \mathrel{\varepsilon} u) \ \wedge\ \forall z (z \mathrel{\varepsilon} 1 \to z \mathrel{\varepsilon} x) \\ &\leftrightarrow\ \exists z\, z \mathrel{\varepsilon} 1 \ \wedge\ \forall z, u (z \mathrel{\varepsilon} z \wedge u \mathrel{\varepsilon} u \to z \mathrel{\varepsilon} u) \ \wedge\ \forall z (z \mathrel{\varepsilon} z \to z \mathrel{\varepsilon} x) \end{aligned}$$

But, by (ε_1), we obtain:

$$\forall z, u (z \mathrel{\varepsilon} z \wedge u \mathrel{\varepsilon} u \to z \mathrel{\varepsilon} u) \ \to\ \forall z, u (z \mathrel{\varepsilon} x \wedge u \mathrel{\varepsilon} x \to z \mathrel{\varepsilon} u)$$

Therefore we also obtain the following thesis of $\mathbf{\Lambda 1}$:

$$\mathsf{1}\,\varepsilon\, x \;\rightarrow\; \exists z\; z\,\varepsilon\, x \,\wedge\, \forall z,u(z\,\varepsilon\, x \wedge u\,\varepsilon\, x \rightarrow z\,\varepsilon\, u)$$

Hence, by ($), we obtain ($\varepsilon\mathsf{1}_2$). $\square$

3.3. An epimorphism theorem for $\mathbf{\Lambda 1}$

THEOREM 3.3. *An* $\mathrm{L}_{\varepsilon\mathsf{1}}$*-structure is a model of* $\mathbf{\Lambda 1}$ *iff it is epimorphic to an s-special* $\mathrm{L}_{\varepsilon\mathsf{1}}$*-structure.*

PROOF: "$\Rightarrow$" Let $\mathfrak{A} = \langle U_{\mathfrak{A}}, \varepsilon_{\mathfrak{A}}, \mathsf{1}_{\mathfrak{A}}\rangle$ be a model of $\mathbf{\Lambda 1}$.

We defined the following relation on $U_{\mathfrak{A}}$:

$$a \equiv b \text{ iff either } a = b, \text{ or both } a\,\varepsilon_{\mathfrak{A}}\, b \text{ and } b\,\varepsilon_{\mathfrak{A}}\, a.$$

By (ε_2), $\equiv$ is an equivalence relation and it is a congruence on $\mathfrak{A}$, i.e., if $a_1 \equiv a_2$ and $b_1 \equiv b_2$, then: $a_1\,\varepsilon_{\mathfrak{A}}\, b_1$ iff $a_2\,\varepsilon_{\mathfrak{A}}\, b_2$. We denote the equivalence class of a by $[a]$. Of course, if $a \not\varepsilon_{\mathfrak{A}}\, a$ then $[a] = \{a\}$. We put $U_{\mathfrak{A}}/_{\equiv} := \{[a] : a \in U_{\mathfrak{A}}\}$ and define the following function $f\colon U_{\mathfrak{A}} \to 2^{U_{\mathfrak{A}}/_{\equiv}}$,

$$f(a) := \big\{[c] \in 2^{U_{\mathfrak{A}}/_{\equiv}} : c\,\varepsilon_{\mathfrak{A}}\, a\big\}.$$

Firstly, we prove that for all $a, b \in U_{\mathfrak{A}}$,

$$\text{if } a\,\varepsilon_{\mathfrak{A}}\, b \text{ then } f(a) = \{[a]\}.$$

Suppose that $a\,\varepsilon_{\mathfrak{A}}\, b$. Then, by ($\varepsilon_1$), we have $a\,\varepsilon_{\mathfrak{A}}\, a$; and so $\{[a]\} \subseteq f(a)$. On the other hand, if $[c] \in f(a)$ then $c\,\varepsilon_{\mathfrak{A}}\, a$. So $a\,\varepsilon_{\mathfrak{A}}\, c$, by ($\varepsilon_4$). Therefore, $a \equiv c$ and so $[c] = [a]$. Hence $f(a) \subseteq \{[a]\}$.

We put $\mathcal{F} := \{f(a) : a \in U_{\mathfrak{A}}\}$. Of course, $\langle \mathcal{F}, \boldsymbol{\varepsilon}_{\mathcal{F}}, \mathbf{1}_{\mathcal{F}}\rangle$ is a special $\mathrm{L}_{\varepsilon\mathsf{1}}$-structure. We show that it is an s-special, i.e., $\mathcal{F}$ is an s-family. Assume that $p \in \bigcup \mathcal{F}$, i.e., $p \in f(a)$, for some $a \in U_{\mathfrak{A}}$. Then $p = [c]$ for some $c \in U_{\mathfrak{A}}$ such that $c\,\varepsilon_{\mathfrak{A}}\, a$. Hence $c\,\varepsilon_{\mathfrak{A}}\, c$; and so $f(c) := \{[c]\}$. Therefore $\{p\} \in \mathcal{F}$.

Secondly, we prove that for all $a, b \in U_{\mathfrak{A}}$:

$$a\,\varepsilon_{\mathfrak{A}}\, b \quad \text{iff} \quad f(a)\,\boldsymbol{\varepsilon}_{\mathcal{F}}\, f(b).$$

Suppose that $a\,\varepsilon_{\mathfrak{A}}\, b$. Then $f(a) = \{[a]\} \subseteq f(b)$. So $f(a)\,\boldsymbol{\varepsilon}_{\mathcal{F}}\, f(b)$. Conversely, let $f(a)\,\boldsymbol{\varepsilon}_{\mathcal{F}}\, f(b)$, i.e., $f(a)$ is a singleton and $f(a) \subseteq f(b)$. Then for some $c_0 \in U_{\mathfrak{A}}$ we have $f(a) = \{[c_0]\}$ and $[c_0] \in f(b)$. Since $\mathfrak{A}$ is a model of (λ), for the proof of $a\,\varepsilon_{\mathfrak{A}}\, b$ is suffices to show that: (i) $c\,\varepsilon_{\mathfrak{A}}\, a$, for some $c \in U_{\mathfrak{A}}$;

(ii) for all $c, d \in U_{\mathfrak{A}}$, if $c \,\varepsilon_{\mathfrak{A}}\, a$ and $d \,\varepsilon_{\mathfrak{A}}\, a$, then $c \,\varepsilon_{\mathfrak{A}}\, d$; and (iii) for any $c \in U_{\mathfrak{A}}$, if $c \,\varepsilon_{\mathfrak{A}}\, a$ then $c \,\varepsilon_{\mathfrak{A}}\, b$. For (i): $c_0 \,\varepsilon_{\mathfrak{A}}\, a$, since $f(a) = \{[c_0]\}$. For (ii): Suppose that $c \,\varepsilon_{\mathfrak{A}}\, a$ and $d \,\varepsilon_{\mathfrak{A}}\, a$. Then $[c], [d] \in f(a)$, $c \,\varepsilon_{\mathfrak{A}}\, c$ and $[c] = [d] = [c_0]$. So $c \,\varepsilon_{\mathfrak{A}}\, d$. For (iii): Suppose that $c \,\varepsilon_{\mathfrak{A}}\, a$. Then $[c] \in f(a)$. So $f(a) = \{[c]\}$, since $f(a)$ is a singleton. Hence $[c] \in f(b)$; and so $c \,\varepsilon_{\mathfrak{A}}\, b$. Thus, we obtain, if $f(a) \,\boldsymbol{\varepsilon}_{\mathcal{F}}\, f(b)$ then $a \,\varepsilon_{\mathfrak{A}}\, b$.[3]

Finally, we show that $f(\mathsf{1}_{\mathfrak{A}}) = \bigcup \mathcal{F} =: \mathbf{1}_{\mathcal{F}}$. Indeed, for any $a \in U_{\mathfrak{A}}$ we have $f(a) \subseteq f(\mathsf{1}_{\mathfrak{A}})$. If $[c] \in f(a)$ then $c \,\varepsilon_{\mathfrak{A}}\, a$. Hence $c \,\varepsilon_{\mathfrak{A}}\, c$, by ($\varepsilon_1$). Therefore $c \,\varepsilon_{\mathfrak{A}}\, \mathsf{1}_{\mathfrak{A}}$, by ($\varepsilon\mathsf{1}_1$). So $[c] \in f(\mathsf{1}_{\mathfrak{A}})$. Thus, $f(\mathsf{1}_{\mathfrak{A}}) \subseteq \bigcup \mathcal{F} \subseteq f(\mathsf{1}_{\mathfrak{A}})$.

"$\Leftarrow$" Obvious. □

Thus, we obtain (as Theorem 2.5):

THEOREM 3.4. *For any $\varphi \in \mathrm{For}_{\varepsilon\mathsf{1}}$, φ is a thesis of $\mathbf{\Lambda}\mathsf{1}$ iff φ is true in any s-special $\mathrm{L}_{\varepsilon\mathsf{1}}$-structure.*

4. The quantifier-free fragment of EO

Let us describe the quantifier-free fragment of elementary ontology **EO** in Ishimoto's version from [1].

4.1. The open theory E

Following Ishimoto, we consider an open first-order theory built in L_ε and having (ε_1)–(ε_3) as specific axioms. We denote this theory by '**E**'. Since from (ε_1) and (ε_3) we obtain (ε_4) and from (ε_4) we obtain (ε_3), the formulas (ε_1), (ε_2) and (ε_4) create an another axiomatization of the theory **E**. Notice that ($\rightarrow\lambda$) and ($\rightarrow\$$) are theses of **E**, but ($\leftarrow\lambda$) and ($\leftarrow\$$) are not.

4.2. E versus Λ and EO

By facts 2.1 and 2.2(4), we obtain:

$$\mathrm{Th}(\mathbf{E}) \subsetneq \mathrm{Th}(\mathbf{\Lambda}). \tag{4.1}$$

However, by Fact 2.3, we obtain:

$$\mathrm{Th}(\mathbf{\Lambda}) = \mathrm{Th}(\mathbf{E}+(\leftarrow\$)), \tag{4.2}$$

$$\mathrm{Th}(\mathbf{EO}) = \mathrm{Th}(\mathbf{E}+(\leftarrow\$)+(\star)). \tag{4.3}$$

[3] Note that the part of the above proof which does not apply to the constant '1' is the proof of Theorem 2.4.

4.3. The quantifier-free theory $\mathbf{E}^{\mathrm{o}}$

Let $\mathbf{E}^{\mathrm{o}}$ be the quantifier-free theory built in $\mathrm{For}^{\mathrm{o}}_{\varepsilon}$ and having the same specific axioms as $\mathbf{E}$. Directly from Theorem 1.1 we obtain:

COROLLARY 4.1. *The quantifier-free theory* $\mathbf{E}^{\mathrm{o}}$ *is the quantifier-free fragment of the open theory* $\mathbf{E}$, *i.e.,* $\mathrm{Th}(\mathbf{E}^{\mathrm{o}}) = \mathrm{For}^{\mathrm{o}}_{\varepsilon} \cap \mathrm{Th}(\mathbf{E})$.

Remark 4.1. In connection with Remark 1.1, the quantifier-free theory $\mathbf{E}^{\mathrm{o}}$ can be treated as a pure calculus of names with one logical constant 'ε' (cf. [3, pp. 26–27 and 96–97] and [4, pp. 6–7 and 24–25]).

Moreover, in connection with Remark 1.2 and (df $\boldsymbol{\varepsilon}_{\mathcal{F}}$), models for the pure calculus of names $\mathbf{E}^{\mathrm{o}}$ are ordered pairs of the form $\langle U, d\rangle$, where U is any set, $d\colon \mathrm{Var} \to 2^U$ and the logical constant 'ε' has the following interpretation:

'$x \,\varepsilon\, y$' is true in $\langle U, d\rangle$ iff $d(x)$ is a singleton and $d(x) \subseteq d(y)$. □

4.4. An epimorphism theorem for E

In [10] and [5] we have, respectively, proofs of "(a) ⇔ (b)" and "(a) ⇔ (c)" parts of the following theorem:

THEOREM 4.2. *For any* L_{ε}*-structure the following conditions are equivalent*:

(a) *it is a model of* $\mathbf{E}$,

(b) *it is epimorphic to a special* L_{ε}*-structure,*

(c) *it is epimorphic to a special* L_{ε}*-structure whose universe is a family of non-empty sets.*[4]

Hence we obtain (as Theorem 2.5):

THEOREM 4.3. *For any* $\varphi \in \mathrm{For}_{\varepsilon}$ *the following conditions are equivalent*:

(a) φ *is a thesis of* $\mathbf{E}$,

(b) φ *is true in any special* L_{ε}*-structure,*

(c) φ *is true in any special* L_{ε}*-structure whose universe is a family of non-empty sets.*

Remark 4.2. In connection with the above theorem, Corollary (4.1) and Remark 4.1, an open formula from $\mathrm{For}^{\mathrm{o}}_{\varepsilon}$ is a thesis of a pure calculus of names $\mathbf{E}^{\mathrm{o}}$ iff it is true in any model $\langle U, d\rangle$, i.e., it is a tautology in the given semantics. Moreover, we also obtain that an open formula φ from $\mathrm{For}^{\mathrm{o}}_{\varepsilon}$ is a thesis of a pure calculus of names $\mathbf{E}^{\mathrm{o}}$ iff φ is true in any model

[4]See the proof of Theorem 5.8 and footnote 7.

$\langle U, d\rangle$ in which we have $d(\boldsymbol{x}) \neq \emptyset$ for any variable $\boldsymbol{x}$, i.e., φ is a traditional tautology in the given semantics.[5] □

4.5. $\mathbf{E}^{\mathrm{o}}$ is the quantifier-free fragment of elementary ontology

From theorems 2.6 and 4.3 we obtain:

THEOREM 4.4. $\mathrm{Th}(\mathbf{E}^{\mathrm{o}}) = \mathrm{For}^{\mathrm{o}}_{\varepsilon} \cap \mathrm{Th}(\mathbf{E}) = \mathrm{For}^{\mathrm{o}}_{\varepsilon} \cap \mathrm{Th}(\mathbf{\Lambda}) = \mathrm{For}^{\mathrm{o}}_{\varepsilon} \cap \mathrm{Th}(\mathbf{EO})$. *So* $\mathbf{E}^{\mathrm{o}}$ *is the quantifier-free fragment of* $\mathbf{E}$, $\mathbf{\Lambda}$ *and* $\mathbf{EO}$.

Moreover, for any first-order theory $\boldsymbol{T}$, *if* $\mathrm{Th}(\mathbf{E}) \subseteq \mathrm{Th}(\boldsymbol{T}) \subseteq \mathrm{Th}(\mathbf{EO})$ *then* $\mathbf{E}^{\mathrm{o}}$ *is the quantifier-free fragment of* $\boldsymbol{T}$.

PROOF: First, $\mathrm{Th}(\mathbf{E}^{\mathrm{o}}) = \mathrm{For}^{\mathrm{o}}_{\varepsilon} \cap \mathrm{Th}(\mathbf{E}) \subseteq \mathrm{For}^{\mathrm{o}}_{\varepsilon} \cap \mathrm{Th}(\mathbf{\Lambda}) \subseteq \mathrm{For}^{\mathrm{o}}_{\varepsilon} \cap \mathrm{Th}(\mathbf{EO})$. Second, let $\varphi \in \mathrm{For}^{\mathrm{o}}_{\varepsilon} \cap \mathrm{Th}(\mathbf{EO})$ and $\langle \mathcal{F}, \boldsymbol{\varepsilon}_{\mathcal{F}}\rangle$ be any special L_{ε}-structure. Notice that $\mathcal{F} \subseteq 2^{\cup\mathcal{F}}$ and for all $X, Y \in \mathcal{F}$ we have: $X \,\boldsymbol{\varepsilon}_{\mathcal{F}}\, Y$ iff $X \,\boldsymbol{\varepsilon}_{2^{\cup\mathcal{F}}}\, Y$. So $\langle \mathcal{F}, \boldsymbol{\varepsilon}_{\mathcal{F}}\rangle$ is a substructure of the p-special L_{ε}-structure $\langle 2^{\mathcal{F}}, \boldsymbol{\varepsilon}_{2^{\cup\mathcal{F}}}\rangle$. By Theorem 2.6, φ is true in $\langle 2^{\mathcal{F}}, \boldsymbol{\varepsilon}_{2^{\cup\mathcal{F}}}\rangle$. Hence φ is true in $\langle \mathcal{F}, \boldsymbol{\varepsilon}_{\mathcal{F}}\rangle$, since φ is open. Therefore $\varphi \in \mathrm{Th}(\mathbf{E})$, by Theorem 4.3. □

5. The theory E with the name constant '1'

5.1. The open theory E1

Since the formula ($\star\star$) is not a thesis of $\mathbf{E}$, if we want to consider the constant $\mathsf{1}$ in $\mathbf{E}$, we must introduce it with specific axioms. These axioms can be the open formulas ($\varepsilon\mathsf{1}_1$) and ($\varepsilon\mathsf{1}_2$). So let $\mathbf{E1}$ be the open first-order theory in $\mathrm{For}_{\varepsilon\mathsf{1}}$ having the formulas (ε_1)–(ε_3), ($\varepsilon\mathsf{1}_1$) and ($\varepsilon\mathsf{1}_2$) as specific axioms.

FACT 5.1. *Axioms* ($\varepsilon\mathsf{1}_1$) *and* ($\varepsilon\mathsf{1}_2$) *are independent in* $\mathbf{E}$.

PROOF: The $\mathrm{L}_{\varepsilon\mathsf{1}}$-structure $\mathfrak{A} = \langle U_{\mathfrak{A}}, \varepsilon_{\mathfrak{A}}, \mathsf{1}_{\mathfrak{A}}\rangle$, where $U_{\mathfrak{A}} := \{1, 2\}$, $\varepsilon_{\mathfrak{A}} := \{\langle 1, 1\rangle, \langle 1, 2\rangle\}$ and $\mathsf{1}_{\mathfrak{A}} := 1$, is a model of $\mathbf{E}$ and ($\varepsilon\mathsf{1}_1$) in which ($\varepsilon\mathsf{1}_2$) is not true. Moreover, the $\mathrm{L}_{\varepsilon\mathsf{1}}$-structure $\mathfrak{A} = \langle U_{\mathfrak{A}}, \varepsilon_{\mathfrak{A}}, \mathsf{1}_{\mathfrak{A}}\rangle$, where $U_{\mathfrak{A}} := \{0, 1\}$, $\varepsilon_{\mathfrak{A}} := \{\langle 0, 0\rangle\}$ and $\mathsf{1}_{\mathfrak{A}} := 1$, is a model of $\mathbf{E}$ and ($\varepsilon\mathsf{1}_2$) in which ($\varepsilon\mathsf{1}_1$) is not true. □

[5] In [3, pp. 96–97] and [4, pp. 24–25] these results were shown using Henkin's method with the maximal consistent sets in $\mathbf{E}^{\mathrm{o}}$.

FACT 5.2. *The formulas* ($\mathtt{c}\varepsilon\mathsf{1}_2$) *and the "$\rightarrow$" part of* ($\$1$)

$$x \mathbin{\varepsilon} \mathsf{1} \;\rightarrow\; \exists z\, z \mathbin{\varepsilon} x \wedge \forall z \forall u (z \mathbin{\varepsilon} x \wedge u \mathbin{\varepsilon} x \rightarrow z \mathbin{\varepsilon} u) \tag{→\$1}$$

are theses of **E** *in the language* $\mathrm{L}_{\varepsilon\mathsf{1}}$.

PROOF: *For* ($\mathtt{c}\varepsilon\mathsf{1}_2$): We use ($\varepsilon_1$) and the substitute $[y/\mathsf{1}]$. *For* (→\$1): Since (→\$) is a thesis of **E**, we use ($\mathtt{c}\varepsilon\mathsf{1}_2$). □

FACT 5.3. *The formulas* ($\mathtt{c}\varepsilon\mathsf{1}_1$), *and* (df 1) *are theses of* **E1**.[6]

PROOF: *For* ($\mathtt{c}\varepsilon\mathsf{1}_1$): We use ($\varepsilon\mathsf{1}_1$) and ($\varepsilon_1$). *For* (df 1): We use ($\varepsilon\mathsf{1}_1$) and ($\mathtt{c}\varepsilon\mathsf{1}_2$), by Fact 5.2. □

5.2. The quantifier-free theory E1°

Let **E1°** be the quantifier-free theory built in $\mathrm{For}^{\mathrm{o}}_{\varepsilon\mathsf{1}}$ and having the same specific axioms as **E1**. Directly from Theorem 1.1 we obtain:

COROLLARY 5.4. **E1°** *is the quantifier-free fragment of* **E1**.

Remark 5.1. The quantifier-free theory **E1°** can be treated as a pure calculus of names with the logical constants 'ε' and '$\mathsf{1}$' (cf. [3, pp. 96–97]).

In connection with remarks 4.1 and 5.1, models for the pure calculus of names **E1°** are ordered pairs of the form $\langle U, d\rangle$, where U is any set and $d\colon \mathrm{Var} \rightarrow 2^U$ such that $d(\mathsf{1}) = U$. The logical constant 'ε' has the same interpretation as in Remark 4.1 (cf. [5, pp. 26–27 and 96–97]). □

5.3. E1 versus Λ1

First, notice that:

FACT 5.5. *The "$\leftarrow$" part of* ($\$1$), *i.e. the following formula*

$$\exists z\, z \mathbin{\varepsilon} x \wedge \forall z \forall u (z \mathbin{\varepsilon} x \wedge u \mathbin{\varepsilon} x \rightarrow z \mathbin{\varepsilon} u) \;\rightarrow\; x \mathbin{\varepsilon} \mathsf{1} \tag{←\$1}$$

as well as the formula (←\$), *are not theses of* **E1**.

PROOF: The $\mathrm{L}_{\varepsilon\mathsf{1}}$-structure $\mathfrak{A} = \langle U_{\mathfrak{A}}, \varepsilon_{\mathfrak{A}}, \mathsf{1}_{\mathfrak{A}}\rangle$, where $U_{\mathfrak{A}} := \{0,1\}$, $\varepsilon_{\mathfrak{A}} := \{\langle 0,0\rangle, \langle 0,1\rangle\}$ and $\mathsf{1}_{\mathfrak{A}} := 1$, is a model of **E1** in which (←\$1) is not true, since $\mathsf{1}_{\mathfrak{A}} \not\varepsilon_{\mathfrak{A}} \mathsf{1}_{\mathfrak{A}}$. □

[6] However, (df 1) is not the definition of '1' in **E**.

Thus, by facts 3.2 and 5.5, we obtain:

FACT 5.6. $\mathbf{\Lambda 1}$ *is a proper extension of* $\mathbf{E1}$, *i.e., we have:*

$$\mathrm{Th}(\mathbf{E1}) \subsetneq \mathrm{Th}(\mathbf{\Lambda 1}).$$

Let $\mathbf{E}+(\leftarrow\$1)$ be the the first-order theory which is built in $\mathrm{For}_{\varepsilon 1}$ and which is a non-conservative extension of $\mathbf{E}$ by one specific axiom $(\leftarrow\$1)$.

THEOREM 5.7. *The three theories* $\mathbf{\Lambda 1}$, $\mathbf{E}+(\leftarrow\$1)$ *and* $\mathbf{E}+(\leftarrow\$)+(\varepsilon 1_1)$ *are equivalent, i.e.,*

$$\mathrm{Th}(\mathbf{E}+(\leftarrow\$1)) = \mathrm{Th}(\mathbf{\Lambda 1}) = \mathrm{Th}(\mathbf{E}+(\leftarrow\$)+(\varepsilon 1_1)).$$

PROOF: Firstly, $\mathrm{Th}(\mathbf{E}+(\leftarrow\$1)) \subseteq \mathrm{Th}(\mathbf{\Lambda 1})$, since $\mathrm{Th}(\mathbf{E1}) \subseteq \mathrm{Th}(\mathbf{\Lambda 1})$ and $(\leftarrow\$1) \in \mathrm{Th}(\mathbf{\Lambda 1})$, by Fact 3.2. Secondly, by (4.2), we have:

$$\mathrm{Th}(\mathbf{\Lambda 1}) := \mathrm{Th}(\mathbf{\Lambda}+(\varepsilon 1_1)) = \mathrm{Th}(\mathbf{E}+(\leftarrow\$)+(\varepsilon 1_1)).$$

Moreover, from $(\rightarrow\$)$ and $(\leftarrow\$1)$ we obtain $(\varepsilon 1_1)$; from $(\mathrm{c}\varepsilon 1_2)$ and $(\leftarrow\$1)$ we obtain $(\leftarrow\$)$. Hence $\mathrm{Th}(\mathbf{E}+(\leftarrow\$)+(\varepsilon 1_1)) \subseteq \mathrm{Th}(\mathbf{E}+(\leftarrow\$1))$. □

5.4. An epimorphism theorem for E1

THEOREM 5.8. *An* $\mathrm{L}_{\varepsilon 1}$*-structure is a model of* $\mathbf{E1}$ *iff it is epimorphic to a special* $\mathrm{L}_{\varepsilon 1}$*-structure.*

PROOF: "⇒" Let $\mathfrak{A} = \langle U_{\mathfrak{A}}, \varepsilon_{\mathfrak{A}}, 1_{\mathfrak{A}}\rangle$ be a model of $\mathbf{E1}$. We consider two cases.

The first case: $1_{\mathfrak{A}}\ \varepsilon_{\mathfrak{A}}\ 1_{\mathfrak{A}}$. We define the function $f\colon U_{\mathfrak{A}} \to \{\emptyset, \{U_{\mathfrak{A}}\}\}$,

$$f(a) := \begin{cases} \emptyset & \text{there is no } c \text{ such that } c\ \varepsilon_{\mathfrak{A}}\ a \\ \{U_{\mathfrak{A}}\} & \text{otherwise} \end{cases}$$

We put $\mathcal{F} := \{f(a) : a \in U_{\mathfrak{A}}\}$ and we show that f is an epimorphism from $\mathfrak{A}$ onto $\langle \mathcal{F}, \boldsymbol{\varepsilon}_{\mathcal{F}}, \mathbf{1}_{\mathcal{F}}\rangle$. In fact, notice that $f(1_{\mathfrak{A}}) = \{U_{\mathfrak{A}}\} = \bigcup \mathcal{F} =: \mathbf{1}_{\mathcal{F}}$.

Moreover, we show that for all $a, b \in U_{\mathfrak{A}}$:

$$a\ \varepsilon_{\mathfrak{A}}\ b \text{ iff } f(a)\ \boldsymbol{\varepsilon}_{\mathcal{F}}\ f(b).$$

Suppose that $a\ \varepsilon_{\mathfrak{A}}\ b$. Then $a\ \varepsilon_{\mathfrak{A}}\ a$, by (ε_1). Hence $f(a) = \{U_{\mathfrak{A}}\} = f(b)$; and so $f(a)\ \boldsymbol{\varepsilon}_{\mathcal{F}}\ f(b)$. Conversely, suppose that $f(a)\ \boldsymbol{\varepsilon}_{\mathcal{F}}\ f(b)$, i.e., $f(a)$ is a singleton and $f(a) \subseteq f(b)$. Then $f(a) = \{U_{\mathfrak{A}}\} = f(b)$. Hence for some c_1, c_2 we have $c_1\ \varepsilon_{\mathfrak{A}}\ a$ and $c_2\ \varepsilon_{\mathfrak{A}}\ b$. For $i = 1, 2$, by (ε_1), $c_i\ \varepsilon_{\mathfrak{A}}\ c_i$; and so

$c_i \mathrel{\varepsilon_{\mathfrak{A}}} 1_{\mathfrak{A}}$, by ($\varepsilon 1_1$). Therefore, $1_{\mathfrak{A}} \mathrel{\varepsilon_{\mathfrak{A}}} c_i$, by ($\varepsilon_3$) and the assumption. Hence $1_{\mathfrak{A}} \mathrel{\varepsilon_{\mathfrak{A}}} a$ and $1_{\mathfrak{A}} \mathrel{\varepsilon_{\mathfrak{A}}} b$, by ($\varepsilon_2$). Hence $a \mathrel{\varepsilon_{\mathfrak{A}}} a$, by ($\varepsilon 1_2$). Hence $a \mathrel{\varepsilon_{\mathfrak{A}}} 1_{\mathfrak{A}}$, by ($\varepsilon 1_1$). Thus, $a \mathrel{\varepsilon_{\mathfrak{A}}} b$, by ($\varepsilon_2$).

The second case: $1_{\mathfrak{A}} \not\mathrel{\varepsilon_{\mathfrak{A}}} 1_{\mathfrak{A}}$. We defined the following relation on $U_{\mathfrak{A}}$:

$$a \equiv b \text{ iff either } a = b, \text{ or both } a \mathrel{\varepsilon_{\mathfrak{A}}} b \text{ and } b \mathrel{\varepsilon_{\mathfrak{A}}} a.$$

By (ε_2), $\equiv$ is an equivalence relation and it is a congruence on $\mathfrak{A}$, i.e., if $a_1 \equiv a_2$ and $b_1 \equiv b_2$, then: $a_1 \mathrel{\varepsilon_{\mathfrak{A}}} b_1$ iff $a_2 \mathrel{\varepsilon_{\mathfrak{A}}} b_2$. We denote the equivalence class of a by $[a]$. Of course, if $a \not\mathrel{\varepsilon_{\mathfrak{A}}} a$ then $[a] = \{a\}$. We put $U_{\mathfrak{A}}/_{\equiv} := \{[a] : a \in U_{\mathfrak{A}}\}$ and define the following function $f \colon U_{\mathfrak{A}} \to 2^{U_{\mathfrak{A}}/_{\equiv}}$,

$$f(a) := \begin{cases} \{[a]\} & \text{if } a \mathrel{\varepsilon_{\mathfrak{A}}} a \\ \{[c] : c \mathrel{\varepsilon_{\mathfrak{A}}} a\} \cup \{\{\emptyset\}, \emptyset\} & \text{otherwise} \end{cases}$$

We put $\mathcal{F} := \{f(a) : a \in U_{\mathfrak{A}}\}$ and we show that f is an epimorphism from $\mathfrak{A}$ onto $\langle \mathcal{F}, \boldsymbol{\varepsilon}_{\mathcal{F}}, \mathbf{1}_{\mathcal{F}} \rangle$.

Firstly, we show that for all $a, b \in U_{\mathfrak{A}}$:

$$a \mathrel{\varepsilon_{\mathfrak{A}}} b \quad \text{iff} \quad [a] \in f(b). \tag{$\dagger$}$$

Suppose that $a \mathrel{\varepsilon_{\mathfrak{A}}} b$. If $b \not\mathrel{\varepsilon_{\mathfrak{A}}} b$ then $[a] \in f(b)$. If $b \mathrel{\varepsilon_{\mathfrak{A}}} b$ then $f(b) := \{[b]\}$ and $b \mathrel{\varepsilon_{\mathfrak{A}}} a$, by ($\varepsilon_3$). Hence $a \equiv b$, $[a] = [b]$; and so $[a] \in f(b)$. Conversely, suppose that $[a] \in f(b)$. If $b \not\mathrel{\varepsilon_{\mathfrak{A}}} b$ then $a \mathrel{\varepsilon_{\mathfrak{A}}} b$, since $[a] \notin \{\{\emptyset\}, \emptyset\}$. If $b \mathrel{\varepsilon_{\mathfrak{A}}} b$ then $f(b) = \{[a]\}$; and so $a \equiv b$. Hence $a \mathrel{\varepsilon_{\mathfrak{A}}} b$.

Secondly, we prove that for all $a, b \in U_{\mathfrak{A}}$:

$$a \mathrel{\varepsilon_{\mathfrak{A}}} b \quad \text{iff} \quad f(a) \mathrel{\boldsymbol{\varepsilon}_{\mathcal{F}}} f(b).$$

Suppose that $a \mathrel{\varepsilon_{\mathfrak{A}}} b$. Then $a \mathrel{\varepsilon_{\mathfrak{A}}} a$, by ($\varepsilon_1$). Hence, by ($\dagger$), we have $f(a) := \{[a]\} \subseteq f(b)$, i.e., $f(a) \mathrel{\boldsymbol{\varepsilon}_{\mathcal{F}}} f(b)$. Conversely, let $f(a) \mathrel{\boldsymbol{\varepsilon}_{\mathcal{F}}} f(b)$, i.e., $f(a)$ is a singleton and $f(a) \subseteq f(b)$. Then $a \mathrel{\varepsilon_{\mathfrak{A}}} a$ and so $f(a) = \{[a]\} \subseteq f(b)$. Hence $[a] \in f(b)$; and so $a \mathrel{\varepsilon_{\mathfrak{A}}} b$, by ($\dagger$).[7]

Finally, we show that $f(1_{\mathfrak{A}}) = \bigcup \mathcal{F} =: \mathbf{1}_{\mathcal{F}}$. This is due to the fact that $f(a) \subseteq f(1_{\mathfrak{A}})$, for any $a \in U_{\mathfrak{A}}$. Indeed, if $\{\{\emptyset\}, \emptyset\} \subseteq f(a)$, then $\{\{\emptyset\}, \emptyset\} \subseteq f(1_{\mathfrak{A}})$, since $1_{\mathfrak{A}} \not\mathrel{\varepsilon_{\mathfrak{A}}} 1_{\mathfrak{A}}$. If $[c] \in f(a)$ then $c \mathrel{\varepsilon_{\mathfrak{A}}} a$, by ($\dagger$). Hence $c \mathrel{\varepsilon_{\mathfrak{A}}} c$, by ($\varepsilon_1$). Therefore $c \mathrel{\varepsilon_{\mathfrak{A}}} 1_{\mathfrak{A}}$, by ($\varepsilon 1_1$). So $[c] \in f(1_{\mathfrak{A}})$. Thus, we obtain $f(1_{\mathfrak{A}}) \subseteq \bigcup \mathcal{F} \subseteq f(1_{\mathfrak{A}})$.

"$\Leftarrow$" Obvious. □

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[7] Note that the part of the above proof which does not apply to the constant '1' is the proof of "(a) $\Rightarrow$ (c)" in Theorem 4.2.

Thus, we obtain (as Theorem 4.3):

THEOREM 5.9. *For any* $\varphi \in \mathrm{For}_{\varepsilon 1}$: $\varphi \in \mathrm{Th}(\mathbf{E1})$ *iff* φ *is true in any special* $\mathrm{L}_{\varepsilon 1}$*-structure.*

Remark 5.2. In connection with the above theorem and Remark 5.1, an open formula from $\mathrm{For}^{\mathrm{o}}_{\varepsilon 1}$ is a thesis of a pure calculus of names $\mathbf{E1}^{\mathrm{o}}$ iff it is true in any model $\langle U, d\rangle$, i.e., it is a tautology in the given semantics.[8] □

5.5. $\mathbf{E1}^{\mathrm{o}}$ is the quantifier-free fragment of $\mathbf{\Lambda 1}$ and $\mathbf{EO}+(\mathtt{df}\ 1)$

From theorems 5.9 and 2.6 we obtain:

THEOREM 5.10. $\mathrm{Th}(\mathbf{E1}^{\mathrm{o}}) = \mathrm{For}^{\mathrm{o}}_{\varepsilon 1} \cap \mathrm{Th}(\mathbf{E1}) = \mathrm{For}^{\mathrm{o}}_{\varepsilon 1} \cap \mathrm{Th}(\mathbf{\Lambda 1}) = \mathrm{For}^{\mathrm{o}}_{\varepsilon 1} \cap \mathrm{Th}(\mathbf{EO}+(\mathtt{df}\ 1))$. *So* $\mathbf{E1}^{\mathrm{o}}$ *is the quantifier-free fragment of the first-order theories* $\mathbf{E1}$, $\mathbf{\Lambda 1}$ *and* $\mathbf{EO}+(\mathtt{df}\ 1)$.

PROOF: First, $\mathrm{Th}(\mathbf{E1}^{\mathrm{o}}) = \mathrm{For}^{\mathrm{o}}_{\varepsilon 1} \cap \mathrm{Th}(\mathbf{E1}) \subseteq \mathrm{For}^{\mathrm{o}}_{\varepsilon 1} \cap \mathrm{Th}(\mathbf{\Lambda 1}) \subseteq \mathrm{For}^{\mathrm{o}}_{\varepsilon 1} \cap \mathrm{Th}(\mathbf{EO}+(\mathtt{df}\ 1))$. Second, let $\varphi \in \mathrm{For}^{\mathrm{o}}_{\varepsilon} \cap \mathrm{Th}(\mathbf{EO}+(\mathtt{df}\ 1))$ and $\langle \mathcal{F}, \boldsymbol{\varepsilon}_{\mathcal{F}}, \mathbf{1}_{\mathcal{F}}\rangle$ be any special $\mathrm{L}_{\varepsilon 1}$-structure. Notice that $\bigcup \mathcal{F} = \bigcup 2^{\cup \mathcal{F}}$, $\mathcal{F} \subseteq 2^{\cup \mathcal{F}}$, $\mathbf{1}_{\mathcal{F}} = \mathbf{1}_{2^{\cup \mathcal{F}}}$ and for all $X, Y \in \mathcal{F}$ we have: $X \,\boldsymbol{\varepsilon}_{\mathcal{F}}\, Y$ iff $X \,\boldsymbol{\varepsilon}_{2^{\cup \mathcal{F}}}\, Y$. So $\langle \mathcal{F}, \boldsymbol{\varepsilon}_{\mathcal{F}}, \mathbf{1}_{\mathcal{F}}\rangle$ is a substructure of the p-special $\mathrm{L}_{\varepsilon 1}$-structure $\langle 2^{\mathcal{F}}, \boldsymbol{\varepsilon}_{2^{\cup \mathcal{F}}}, \mathbf{1}_{2^{\cup \mathcal{F}}}\rangle$. By Theorem 2.6, φ is true in $\langle 2^{\mathcal{F}}, \boldsymbol{\varepsilon}_{2^{\cup \mathcal{F}}}, \mathbf{1}_{2^{\cup \mathcal{F}}}\rangle$. Hence φ is also true in $\langle \mathcal{F}, \boldsymbol{\varepsilon}_{\mathcal{F}}, \mathbf{1}_{\mathcal{F}}\rangle$, since φ is open. Therefore $\varphi \in \mathrm{Th}(\mathbf{E1})$, by Theorem 5.9.

Finally, $\mathrm{For}^{\mathrm{o}}_{\varepsilon 1} \cap \mathrm{Th}(\mathbf{EO}+(\mathtt{df}\ 1)) = \mathrm{Th}(\mathbf{E1}^{\mathrm{o}}) = \mathrm{For}^{\mathrm{o}}_{\varepsilon 1} \cap \mathrm{Th}(\mathbf{E1}) \subseteq \mathrm{For}^{\mathrm{o}}_{\varepsilon 1} \cap \mathrm{Th}(\mathbf{\Lambda 1}) \subseteq \mathrm{For}^{\mathrm{o}}_{\varepsilon 1} \cap \mathrm{Th}(\mathbf{EO}+(\mathtt{df}\ 1))$. □

6. A reconstruction of E in one of its subtheories

6.1. The open theory $\mathbf{E}^*$ in the language L_ε

Let $\mathbf{E}^*$ be the open first-order theory in the language L_ε with two specific axioms (ε_2) and (ε_3).

FACT 6.1. $\mathbf{E}^*$ *is a proper subtheory of* $\mathbf{E}$*, i.e.,* $\mathrm{Th}(\mathbf{E}^*) \subsetneq \mathrm{Th}(\mathbf{E})$.

PROOF: First, $\mathrm{Th}(\mathbf{E}^*) \subseteq \mathrm{Th}(\mathbf{E})$. Second, we have $\mathrm{Th}(\mathbf{E}) \nsubseteq \mathrm{Th}(\mathbf{E}^*)$. To show it we take a structure $\langle \mathbb{N}, <\rangle$, where $\mathbb{N}$ is the set of natural numbers and the interpretation of predicate 'ε' is the relation $<$. The formulas (ε_2) and (ε_3) are true in $\langle \mathbb{N}, <\rangle$, but ($\varepsilon_4$) and ($\varepsilon_1$) are not true. □

[8] In [3, pp. 96–97] these results were shown using Henkin's method with the maximal consistent sets in $\mathbf{E1}^{\mathrm{o}}$.

We will prove that in the theory $\mathbf{E}^*$ we can reconstruct the theory $\mathbf{E}$. Between $\mathbf{E}$ and $\mathbf{E}^*$ we define the following transformation $\mathrm{tr}\colon \mathrm{For}_\varepsilon \to \mathrm{For}_\varepsilon$. The function tr fulfils the following conditions for all $\boldsymbol{x}, \boldsymbol{y} \in \mathrm{Var}$ and all $\varphi, \psi \in \mathrm{For}_\varepsilon$:

$$\begin{aligned}
&\mathrm{tr}(\boldsymbol{x}\ \varepsilon\ \boldsymbol{y}) = \ulcorner \boldsymbol{x}\ \varepsilon\ \boldsymbol{y} \wedge \boldsymbol{x}\ \varepsilon\ \boldsymbol{x} \urcorner,\\
&\mathrm{tr}(\neg\, \varphi) = \ulcorner \neg\, \mathrm{tr}(\varphi)\urcorner,\\
&\mathrm{tr}(\varphi \circ \psi) = \ulcorner \mathrm{tr}(\varphi) \circ \mathrm{tr}(\psi)\urcorner, \text{ for } \circ \in \{\wedge, \vee, \to, \leftrightarrow\},\\
&\mathrm{tr}(Q\boldsymbol{x}\, \varphi) = \ulcorner Q\boldsymbol{x}\, \mathrm{tr}(\varphi)\urcorner, \text{ for } Q \in \{\forall, \exists\}.
\end{aligned}$$

We obtain the following:

FACT 6.2. *For any* $\varphi \in \mathrm{For}_\varepsilon$: $\varphi \in \mathrm{Th}(\mathbf{E})$ *iff* $\mathrm{tr}(\varphi) \in \mathrm{Th}(\mathbf{E}^*)$.

PROOF: "$\Rightarrow$" $\mathrm{tr}(\varepsilon_1)$ gives: $x\ \varepsilon\ y \wedge x\ \varepsilon\ x \to x\ \varepsilon\ x \wedge x\ \varepsilon\ x$. So it is an instance of a classical tautology. $\mathrm{tr}(\varepsilon_2)$ gives: $x\,\varepsilon\,y \wedge x\,\varepsilon\,x \wedge y\,\varepsilon\,z \wedge y\,\varepsilon\,y \to x\,\varepsilon\,z \wedge x\,\varepsilon\,x$. So it belongs to $\mathrm{Th}(\mathbf{E}^*)$, by (ε_2) and classical propositional logic. $\mathrm{tr}(\varepsilon_4)$ gives: $x\ \varepsilon\ y \wedge x\ \varepsilon\ x \wedge y\ \varepsilon\ z \wedge y\ \varepsilon\ y \to y\ \varepsilon\ x \wedge y\ \varepsilon\ y$. So it belongs to $\mathrm{Th}(\mathbf{E}^*)$, by (ε_3) and classical propositional logic.

"$\Leftarrow$" By (ε_1) and the rule of substitution for free individual variables, for all variables $\boldsymbol{x}$ and $\boldsymbol{y}$, the equivalence $\ulcorner \boldsymbol{x}\ \varepsilon\ \boldsymbol{y} \leftrightarrow \mathrm{tr}(\boldsymbol{x}\ \varepsilon\ \boldsymbol{y})\urcorner$ is a thesis of $\mathbf{E}$. Hence for any $\varphi \in \mathrm{For}_\varepsilon$: $\varphi \in \mathrm{Th}(\mathbf{E})$ iff $\mathrm{tr}(\varphi) \in \mathrm{Th}(\mathbf{E})$. Thus, since $\mathbf{E}^*$ is a subtheory of $\mathbf{E}$, if $\mathrm{tr}(\varphi) \in \mathrm{Th}(\mathbf{E}^*)$, then $\varphi \in \mathrm{Th}(\mathbf{E})$. □

6.2. The open theory $\mathbf{E}^*$ in the language $\mathrm{L}_{\varepsilon^*}$

For better readability, we will analyse theory $\mathbf{E}^*$ in another language $\mathrm{L}_{\varepsilon^*}$, where we change the predicate 'ε' to 'ε*'. So in place of axioms (ε_2) and (ε_3) we take their $\mathrm{L}_{\varepsilon^*}$-counterparts:

$$x\ \varepsilon^*\ y \wedge y\ \varepsilon^*\ z \to x\ \varepsilon^*\ z \qquad (\varepsilon^*_1)$$

$$x\ \varepsilon^*\ y \wedge y\ \varepsilon^*\ y \to y\ \varepsilon^*\ x \qquad (\varepsilon^*_2)$$

Notice that directly from (ε^*_1) we obtain the following thesis of $\mathbf{E}^*$:

$$x\ \varepsilon^*\ y \wedge y\ \varepsilon^*\ x \to x\ \varepsilon^*\ x \wedge y\ \varepsilon^*\ y \qquad (\mathsf{c}\varepsilon^*_1)$$

Moreover, by (ε^*_2) and (ε^*_1), we obtain the $\mathrm{L}_{\varepsilon^*}$-counterpart of (→\$):

$$x\ \varepsilon^*\ x\ \to\ \exists z\ z\ \varepsilon^*\ x \wedge \forall z, u(z\ \varepsilon^*\ x \wedge u\ \varepsilon^*\ x \to z\ \varepsilon^*\ u) \qquad (\to\$_{\varepsilon^*})$$

6.3. Defining the predicate 'ε' by 'ε*'

We extend the language $\mathrm{L}_{\varepsilon^*}$ to the language $\mathrm{L}_{\varepsilon\varepsilon^*}$ by adding the predicate 'ε'. In $\mathrm{L}_{\varepsilon\varepsilon^*}$ let $\mathbf{E}^*$+(df ε) be a definitional extension of the theory $\mathbf{E}^*$ by adding the following definition:

$$x \mathbin{\varepsilon} y \leftrightarrow x \mathbin{\varepsilon^*} x \wedge x \mathbin{\varepsilon^*} y \tag{df ε}$$

So we obtain:

$$x \mathbin{\varepsilon} x \;\leftrightarrow\; x \mathbin{\varepsilon^*} x \tag{\%}$$

6.4. The quantifier-free theories $\mathbf{E}^{*\mathrm{o}}$ and $(\mathbf{E}^*$+(df ε)$)^{\mathrm{o}}$

Let $\mathbf{E}^{*\mathrm{o}}$ and $(\mathbf{E}^*$+(df ε)$)^{\mathrm{o}}$ be the quantifier-free theories built, respectively, in $\mathrm{For}^{\mathrm{o}}_{\varepsilon^*}$ and $\mathrm{For}^{\mathrm{o}}_{\varepsilon\,\varepsilon^*}$ and having the same specific axioms as $\mathbf{E}^*$ and $\mathbf{E}^*$+(df ε). Directly from Theorem 1.1 we obtain:

Corollary 6.3. *$\mathbf{E}^{*\mathrm{o}}$ and $(\mathbf{E}^*$+(df ε)$)^{\mathrm{o}}$ are quantifier-free fragments of $\mathbf{E}^*$ and $\mathbf{E}^*$+(df ε), respectively.*

Remark 6.1. The quantifier-free theories $\mathbf{E}^{*\mathrm{o}}$ and $(\mathbf{E}^*$+(df ε)$)^{\mathrm{o}}$ can be treated as pure calculi of names with one logical constant 'ε*' and two logical constant 'ε*' and 'ε', respectively (cf. [3, pp. 54–55] and [4, p. 8]). □

6.5. Epimorphism theorems for $\mathbf{E}^*$ and $\mathbf{E}^*$+(df ε)

For 'ε*' the formula $\Phi_{\varepsilon^*}(X, Y)$ (see p. 238) has the following form:

either $\emptyset \neq X \subsetneq Y$ or both X is a singleton and $X = Y$.

That is, we put:

$$\boldsymbol{\varepsilon}^{\star}_{\mathcal{F}} := \big\{\langle X, Y\rangle \in \mathcal{F}^2 : \Phi_{\varepsilon^*}(X, Y)\big\}. \tag{df $\boldsymbol{\varepsilon}^{\star}_{\mathcal{F}}$}$$

Fact 6.4. *In any special $\mathrm{L}_{\varepsilon^*}$-structure, the predicate 'ε' defined by* (df ε) *is interpreted by the relation $\boldsymbol{\varepsilon}_{\mathcal{F}}$ defined by* (df $\boldsymbol{\varepsilon}_{\mathcal{F}}$)*. So* (df ε) *is true in any special $\mathrm{L}_{\varepsilon\varepsilon^*}$-structure $\langle \mathcal{F}, \boldsymbol{\varepsilon}_{\mathcal{F}}, \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}}\rangle$.*

Proof: Suppose that $\mathcal{F}$ is a non-empty family of sets and $\mathcal{R} \subseteq \mathcal{F}^2$ is an interpretation of the predicate 'ε' defined by (df ε). We show that $\mathcal{R} = \boldsymbol{\varepsilon}_{\mathcal{F}}$. For all $X, Y \in \mathcal{F}$ we obtain: $X \mathrel{\mathcal{R}} Y$ iff $X \mathrel{\boldsymbol{\varepsilon}^{\star}_{\mathcal{F}}} Y$ and $X \mathrel{\boldsymbol{\varepsilon}^{\star}_{\mathcal{F}}} X$ iff both either $\emptyset \neq X \subsetneq Y$ or there is a $p \in \bigcup\mathcal{F}$ such that $X = \{p\} = Y$, and there is a $q \in \bigcup\mathcal{F}$ such that $X = \{q\}$ iff either both $\emptyset \neq X \subsetneq Y$ and there is a $q \in \bigcup\mathcal{F}$ such that $X = \{q\}$, or both there is a $p \in \bigcup\mathcal{F}$ such that

$X = \{p\} = Y$ and there is a $q \in \bigcup\mathcal{F}$ such that $X = \{q\}$ iff either there is a $p \in \bigcup\mathcal{F}$ such that $X = \{p\} \subsetneq Y$ or there is a $p \in \bigcup\mathcal{F}$ such that $X = \{p\} = Y$ iff there is a $p \in \bigcup\mathcal{F}$ such that $X = \{p\} \subseteq Y$ iff $X \boldsymbol{\varepsilon}_{\mathcal{F}} Y$. $\square$

Remark 6.2. In connection with remarks 1.2, 4.1 and 6.1, models for the pure calculi of names $\mathbf{E}^{*\mathrm{o}}$ and $(\mathbf{E}^{*}+(\texttt{df}\ \varepsilon))^{\mathrm{o}}$ are ordered pairs of the form $\langle U, d\rangle$, where U is any set and $d\colon \mathrm{Var} \to 2^{U}$. The logical constant '$\varepsilon^{*}$' has the following interpretation:

'$x\ \varepsilon^{*}\ y$' is true in $\langle U, d\rangle$ iff either $\emptyset \neq d(x) \subsetneq d(y)$
or both $d(x) = d(y)$ and $d(x)$ is a singleton.

The logical constant 'ε' is interpreted as in Remark 4.1. $\square$

Theorem 6.5. *For any* $\mathrm{L}_{\varepsilon^{*}}$*-structure (resp.* $\mathrm{L}_{\varepsilon\varepsilon^{*}}$*-structure) the following conditions are equivalent:*

(a) *it is a model of* $\mathbf{E}^{*}$ *(resp.* $\mathbf{E}^{*}+(\texttt{df}\ \varepsilon)$*),*

(b) *it is epimorphic to a special* $\mathrm{L}_{\varepsilon^{*}}$*-structure (resp.* $\mathrm{L}_{\varepsilon\varepsilon^{*}}$*-structure),*

(c) *it is epimorphic to a special* $\mathrm{L}_{\varepsilon^{*}}$*-structure (resp.* $\mathrm{L}_{\varepsilon\varepsilon^{*}}$*-structure) whose universe is a family of non-empty sets.*

Proof: "(c) $\Rightarrow$ (b)" Obvious.

"(b) $\Rightarrow$ (a)" Let $\langle \mathcal{F}, \boldsymbol{\varepsilon}_{\mathcal{F}}, \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}}\rangle$ be an arbitrary special $\mathrm{L}_{\varepsilon\varepsilon^{*}}$-structure. Then, by Fact 6.4, $(\texttt{df}\ \varepsilon)$ is true in $\langle \mathcal{F}, \boldsymbol{\varepsilon}_{\mathcal{F}}, \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}}\rangle$. We show that both axioms of $\mathbf{E}^{*}$ are true in $\langle \mathcal{F}, \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}}\rangle$. Consequently, in virtue of Fact 6.4, all axioms of $\mathbf{E}^{*}+(\texttt{df}\ \varepsilon)$ will be true in every epimorphic structure with $\langle \mathcal{F}, \boldsymbol{\varepsilon}_{\mathcal{F}}, \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}}\rangle$.

For (ε^{*}_{1}): We take an arbitrary valuation v such that $v(x) = X$, $v(y) = Y$ and $v(z) = Z$. Assume that $X \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}} Y$ and $Y \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}} Z$. Then both either $\emptyset \neq X \subsetneq Y$ or there is a $p \in \bigcup\mathcal{F}$ such that $X = \{p\} = Y$, and either $\emptyset \neq Y \subsetneq Z$ or there is a $q \in \bigcup\mathcal{F}$ such that $Y = \{q\} = Z$. So we have the following cases:

(i) $\emptyset \neq X \subsetneq Y$ and $\emptyset \neq Y \subsetneq Z$; so $\emptyset \neq X \subsetneq Z$;

(ii) $\emptyset \neq Y \subsetneq Z$ and there is a $p \in \bigcup\mathcal{F}$ such that $X = \{p\} = Y$; so there is a $p \in \bigcup\mathcal{F}$ such that $X = \{p\} \subsetneq Z$;

(iii) there is a $p \in \bigcup\mathcal{F}$ such that $X = \{a\} = Y$ and there is a $q \in \bigcup\mathcal{F}$ such that $Y = \{q\} = Z$; so there is a $p \in \bigcup\mathcal{F}$ such that $X = \{p\} = Z$.

Thus, $X \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}} Z$. (The following case cannot obtain: $\emptyset \neq X \subsetneq Y$ and there is a $q \in \bigcup\mathcal{F}$ such that $Y = \{q\} = Z$.)

For (ε^{*}_{2}): We take an arbitrary valuation v such that $v(x) = X$ and $v(y) = Y$. Assume that $X \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}} Y$ and $Y \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}} Y$. Then both either $\emptyset \neq X \subsetneq Y$

or there is a $p \in \bigcup \mathcal{F}$ such that $X = \{p\} = Y$, and there is a $q \in \bigcup \mathcal{F}$ such that $Y = \{q\}$. So we have: there is a $p \in \bigcup \mathcal{F}$ such that $X = \{p\} = Y$ and there is a $q \in \bigcup \mathcal{F}$ such that $Y = \{q\}$. So there is a $p \in \bigcup \mathcal{F}$ such that $X = \{p\} = Y$. Thus, $Y \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}} X$. (The following case cannot obtain: $\emptyset \neq X \subsetneq Y$ and there is a $b \in \bigcup \mathcal{F}$ such that $Y = \{b\}$.)

"(a) $\Rightarrow$ (c)" *For the theory* $\mathbf{E}^*$. Let $\mathfrak{A} = \langle U_{\mathfrak{A}}, \varepsilon_{\mathfrak{A}} \rangle$ be a model of $\mathbf{E}^*$. We define the following relation on $U_{\mathfrak{A}}$:

$$a \sim b \text{ iff either } a = b, \text{ or both } a \; \varepsilon^*_{\mathfrak{A}} \; b \text{ and } b \; \varepsilon^*_{\mathfrak{A}} \; a.$$

By (ε^*_1), $\sim$ is an equivalence relation and it is a congruence on $\mathfrak{A}$, i.e., if $a_1 \sim a_2$ and $b_1 \sim b_2$, then: $a_1 \; \varepsilon^*_{\mathfrak{A}} \; b_1$ iff $a_2 \; \varepsilon^*_{\mathfrak{A}} \; b_2$. We denote the equivalence class of a by $[a]$. Notice that, by (ε^*_1), for any $a \in U_{\mathfrak{A}}$ we have:

$$\text{if } a \not\varepsilon^*_{\mathfrak{A}} \; a \text{ then } [a] = \{a\}. \tag{$\dagger$}$$

Let $U_{\mathfrak{A}}/_{\sim} := \{[a] : a \in U_{\mathfrak{A}}\}$ and we define the function $f \colon U_{\mathfrak{A}} \to 2^{U_{\mathfrak{A}}/_{\sim}}$,

$$f(a) := \begin{cases} \{[a]\} & \text{if } a \; \varepsilon^*_{\mathfrak{A}} \; a \\ \{[c] : c \; \varepsilon^*_{\mathfrak{A}} \; a\} \cup \{[a], \emptyset\} & \text{otherwise} \end{cases}$$

We put $\mathcal{F} := \{f(a) : a \in U_{\mathfrak{A}}\}$. We show that f is an epimorphism from $\mathfrak{A}$ onto $\langle \mathcal{F}, \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}} \rangle$.

We prove that for all $a, b \in U_{\mathfrak{A}}$:

$$a \; \varepsilon^*_{\mathfrak{A}} \; b \quad \text{iff} \quad f(a) \; \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}} \; f(b). \tag{$\ddagger$}$$

Suppose that $a \; \varepsilon^*_{\mathfrak{A}} \; b$. We consider three possibilities.

1) $b \; \varepsilon^*_{\mathfrak{A}} \; b$. Then $b \; \varepsilon^*_{\mathfrak{A}} \; a$, by ($\varepsilon^*_2$). So $a \sim b$ and $[a] = [b]$. Moreover, $a \; \varepsilon^*_{\mathfrak{A}} \; a$, by ($\varepsilon^*_1$). So $f(a) = \{[a]\} = \{[b]\} = f(b)$. Thus, $f(a) \; \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}} \; f(b)$.

2) $a \; \varepsilon^*_{\mathfrak{A}} \; a$ and $b \not\varepsilon^*_{\mathfrak{A}} \; b$. Then $[b] \in f(b)$, $f(a) = \{[a]\}$ and $[a] \in f(b)$. Moreover, $[b] \notin f(a)$, since $a \nsim b$ by (ε^*_1). Thus, $\emptyset \neq f(a) \subsetneq f(b)$; and so $f(a) \; \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}} \; f(b)$.

3) $a \not\varepsilon^*_{\mathfrak{A}} a$ and $b \not\varepsilon^*_{\mathfrak{A}} b$. Then $a \neq b$, $[a] \in f(a)$, $[a] \in f(b)$ and $[b] \in f(b)$. By ($\dagger$), we have $[a] = \{a\} \neq \{b\} = [b]$. Moreover, $b \not\varepsilon^*_{\mathfrak{A}} \; a$, by ($\varepsilon^*_1$); and so $a \nsim b$. Therefore, $[b] \notin f(a)$. If $[c] \in f(a)$, then either $c \; \varepsilon^*_{\mathfrak{A}} \; a$ or $c = a$. So $c \; \varepsilon^*_{\mathfrak{A}} \; b$, by ($\varepsilon^*_1$) and the assumption. Hence $[c] \in f(b)$. Thus, $\emptyset \neq f(a) \subsetneq f(b)$; and so $f(a) \; \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}} \; f(b)$.

Conversely, let $f(a) \; \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}} \; f(b)$, i.e., either (i) $\emptyset \neq f(a) \subsetneq f(b)$ or (ii) both $f(a)$ is a singleton and $f(a) = f(b)$. In the case (i) we have: $a \neq b$,

$[a] \in f(a)$ and $b \not\varepsilon^*_{\mathfrak{A}} b$. So $a \nsim b$, by ($\mathtt{c}\varepsilon^*_1$) and the assumption. Moreover, $[a] \in f(b)$ and so $a\ \varepsilon^*_{\mathfrak{A}}\ b$, since $[a] \neq [b]$. In the case (ii) we have: $a\ \varepsilon^*_{\mathfrak{A}}\ a$, $b\ \varepsilon^*_{\mathfrak{A}}\ b$ and $\{[a]\} = f(a) = f(b) = \{[b]\}$. So $[a] = [b]$, i.e., $a \sim b$. Hence either $a = b$, or both $a\ \varepsilon^*_{\mathfrak{A}}\ b$ and $b\ \varepsilon^*_{\mathfrak{A}}\ a$. In both cases, we get: $a\ \varepsilon^*_{\mathfrak{A}}\ b$.

For the theory $\mathbf{E}^*$+($\mathtt{df}\ \varepsilon$). Let $\mathfrak{A} = \langle U_{\mathfrak{A}}, \varepsilon_{\mathfrak{A}}, \varepsilon^*_{\mathfrak{A}}\rangle$ be a model of $\mathbf{E}^*$+ ($\mathtt{df}\ \varepsilon$). As for $\mathbf{E}^*$ we construct the family $\mathcal{F}$ and the epimorphism f. Then for all $a, b \in U_{\mathfrak{A}}$ we have: $a\ \varepsilon_{\mathfrak{A}}\ b$ iff $a\ \varepsilon^*_{\mathfrak{A}}\ a$ and $a\ \varepsilon^*_{\mathfrak{A}}\ b$ iff $f(a)\ \boldsymbol{\varepsilon}^\star_{\mathcal{F}}\ f(a)$ and $f(a)\ \boldsymbol{\varepsilon}^\star_{\mathcal{F}}\ f(b)$ iff $f(a)\ \boldsymbol{\varepsilon}_{\mathcal{F}}\ f(b)$ (by Fact 6.4). □

Thus, we obtain (as Theorem 2.5):

THEOREM 6.6. *For any* $\varphi \in \mathrm{For}_{\varepsilon^*}$ (*resp.* $\varphi \in \mathrm{For}_{\varepsilon\,\varepsilon^*}$) *the following conditions are equivalent*:

(a) φ *is a thesis of* $\mathbf{E}^*$ (*resp.* $\mathbf{E}^*$+($\mathtt{df}\ \varepsilon$)),

(b) φ *is true in any special* $\mathrm{L}_{\varepsilon^*}$*-structure* (*resp.* $\mathrm{L}_{\varepsilon\varepsilon^*}$*-structure*),

(c) φ *is true in any special* $\mathrm{L}_{\varepsilon^*}$*-structure* (*resp.* $\mathrm{L}_{\varepsilon\varepsilon^*}$*-structure*) *whose universe is a family of non-empty sets.*

Remark 6.3. In connection with the above theorem, remarks 6.1 and 6.2, an open formula from $\mathrm{For}^{\mathrm{o}}_{\varepsilon^*}$ (resp. $\mathrm{For}^{\mathrm{o}}_{\varepsilon\,\varepsilon^*}$) is a thesis of a pure calculus of names $\mathbf{E}^{*\mathrm{o}}$ (resp. $(\mathbf{E}^*$+($\mathtt{df}\ \varepsilon$)$)^{\mathrm{o}}$) iff it is true in any model $\langle U, d\rangle$, i.e., it is a tautology in the given semantics. □

6.6. A reconstruction of E in E*

It is easy to see that (ε_1)–(ε_4) are theses of $\mathbf{E}^*$+($\mathtt{df}\ \varepsilon$). Thus, we obtain that $\mathbf{E}^*$+($\mathtt{df}\ \varepsilon$) is a proper extension of $\mathbf{E}$, i.e.,

$$\mathrm{Th}(\mathbf{E}) \subsetneq \mathrm{Th}(\mathbf{E}^*+(\mathtt{df}\ \varepsilon)). \tag{6.1}$$

However, in the light of theorems 4.3 and 6.6, the theories $\mathbf{E}$ and $\mathbf{E}^*$+($\mathtt{df}\ \varepsilon$) have the same theses from the language L_ε, i.e., we obtain:

$$\mathrm{Th}(\mathbf{E}) = \mathrm{For}_\varepsilon \cap \mathrm{Th}(\mathbf{E}^*+(\mathtt{df}\ \varepsilon)). \tag{6.2}$$

6.7. Reconstructions of Λ and EO in some extensions of E*

If we use the language $\mathrm{L}_{\varepsilon\varepsilon^*}$ then we can extend theories in $\mathrm{L}_{\varepsilon^*}$ using formulas from L_ε. Let us recall that the formula ($\mathtt{\leftarrow\$}$) is not a thesis of $\mathbf{E}$. So, by (6.2), it is not a thesis of $\mathbf{E}^*$+($\mathtt{df}\ \varepsilon$).

In virtue of (4.2) and (6.1), we obtain that $\mathbf{E}^*$+(df ε) is a proper extension of $\mathbf{\Lambda}$. Moreover, in virtue of (4.3) and (6.1), we obtain that $\mathbf{E}^*$+(df ε)+($\star$) is a proper extension of **EO**. That is,

$$\mathrm{Th}(\mathbf{\Lambda}) \subsetneq \mathrm{Th}(\mathbf{E}^*+(\mathtt{df}\ \varepsilon)+(\leftarrow\$)), \tag{6.3}$$

$$\mathrm{Th}(\mathbf{EO}) \subsetneq \mathrm{Th}(\mathbf{E}^*+(\mathtt{df}\ \varepsilon)+(\leftarrow\$)+(\star)). \tag{6.4}$$

However, in the light of theorems 2.5 and 6.6, the theories $\mathbf{\Lambda}$ and $\mathbf{E}^*$+(df ε)+(←\$) have the same theses from the language L_ε, i.e., we obtain:

$$\mathrm{Th}(\mathbf{\Lambda}) = \mathrm{For}_\varepsilon \cap \mathrm{Th}(\mathbf{E}^*+(\mathtt{df}\ \varepsilon)+(\leftarrow\$)).$$

In fact, by theorems 2.5 and 6.6, all theses of $\mathbf{E}^*$+(df ε)+(←\$) are true in all s-special $\mathrm{L}_{\varepsilon\varepsilon^*}$-structures. So if φ belongs to $\mathrm{For}_\varepsilon \cap \mathrm{Th}(\mathbf{E}^*+(\mathtt{df}\ \varepsilon)+(\leftarrow\$))$, then it is true in all s-special L_ε-structures. Hence, by Theorem 2.5, φ is a thesis of $\mathbf{\Lambda}$.

Moreover, in the light of theorems 2.6 (or 2.8) and 6.6, the theories **EO** and $\mathbf{E}^*$+(df ε)+(←\$)+($\star$) have the same theses from the language L_ε, i.e.:

$$\mathrm{Th}(\mathbf{EO}) = \mathrm{For}_\varepsilon \cap \mathrm{Th}(\mathbf{E}^*+(\mathtt{df}\ \varepsilon)+(\leftarrow\$)+(\star)).$$

In fact, by theorems 2.6 and 6.6, all theses of $\mathbf{E}^*$+(df ε)+(←\$) are true in all p-special $\mathrm{L}_{\varepsilon\varepsilon^*}$-structures. So if φ belongs to $\mathrm{For}_\varepsilon \cap \mathrm{Th}(\mathbf{E}^*+(\mathtt{df}\ \varepsilon)+(\leftarrow\$))$, then it is true in all p-special L_ε-structures. Hence, by Theorem 2.6, φ is a thesis of **EO**.

7. The theory $\mathbf{E}^*$ with the name constant ‘1’

7.1. The theory $\mathbf{E}^*\mathsf{1}$

Let $\mathbf{E}^*\mathsf{1}$ be a non-conservative extension of the theory $\mathbf{E}^*$ which is an open first-order theory built in $\mathrm{For}_{\varepsilon\mathsf{1}}$ and has the following specific axioms:

$$x\ \varepsilon^*\ y \rightarrow x\ \varepsilon^*\ \mathsf{1} \tag{$\varepsilon^*\mathsf{1}_1$}$$

$$\mathsf{1}\ \varepsilon^*\ x \rightarrow x\ \varepsilon^*\ x \tag{$\varepsilon^*\mathsf{1}_2$}$$

$$y\ \varepsilon^*\ x \wedge z\ \varepsilon^*\ \mathsf{1} \wedge \neg\, z\ \varepsilon^*\ x \ \rightarrow\ x\ \varepsilon^*\ \mathsf{1} \tag{$\varepsilon^*\mathsf{1}_3$}$$

Notice that ($\varepsilon^*\mathsf{1}_3$) is logically equivalent to:

$$\exists u\, u\ \varepsilon^*\ x \wedge \exists u(u\ \varepsilon^*\ \mathsf{1} \wedge \neg\, u\ \varepsilon^*\ x) \ \rightarrow\ x\ \varepsilon^*\ \mathsf{1}$$

From ($\varepsilon^*\mathsf{1}_1$) we obtain the $\mathrm{L}_{\varepsilon^*\mathsf{1}}$-counterpart of ($\varepsilon\mathsf{1}_1$), i.e.,

$$x\ \varepsilon^*\ x \rightarrow x\ \varepsilon^*\ \mathsf{1}$$

But the implication ‘$x\ \varepsilon^*\ \mathsf{1} \rightarrow x\ \varepsilon^*\ x$’, and so the $\mathrm{L}_{\varepsilon^*\mathsf{1}}$-counterpart of (df 1), i.e., ‘$x\ \varepsilon^*\ \mathsf{1} \leftrightarrow x\ \varepsilon^*\ x$’, are not theses of $\mathbf{E}^*\mathsf{1}$. In fact, the $\mathrm{L}_{\varepsilon^*\mathsf{1}}$-structure

$\mathfrak{A} = \langle U_{\mathfrak{A}}, \varepsilon^*_{\mathfrak{A}}, 1_{\mathfrak{A}}\rangle$, where $U_{\mathfrak{A}} := \{0,1\}$, $\varepsilon^*_{\mathfrak{A}} := \{\langle 0,1\rangle\}$ and $1_{\mathfrak{A}} := 1$, is a model of $\mathbf{E}^*1$ in which '$x \ \varepsilon^* \ 1 \rightarrow x \ \varepsilon^* \ x$' is not true.

Axiom (ε^*1_1) is the $\mathrm{L}_{\varepsilon^*1}$-counterpart of '$x \ \varepsilon \ y \rightarrow x \ \varepsilon \ 1$' belonging to $\mathrm{Th}(\mathbf{E}1)$. Axiom (ε^*1_2) is the $\mathrm{L}_{\varepsilon^*1}$-counterpart of axiom ($\varepsilon 1_2$) of $\mathbf{E}1$. However, we show that the $\mathrm{L}_{\varepsilon 1}$-counterpart of ($\varepsilon^*1_3$) is not a thesis of $\mathbf{E}1$.

Fact 7.1. *The axioms of* $\mathbf{E}^*1$ *are independent.*

Proof: Firstly note that both $\mathrm{L}_{\varepsilon^*1}$-structures from Fact 5.1 are models of (ε^*1_3). So (ε^*1_2) does not follow form (ε^*1_1) and (ε^*1_3); and (ε^*1_1) does not follow from (ε^*1_2) and (ε^*1_3). Secondly, the $\mathrm{L}_{\varepsilon^*1}$-structure $\mathfrak{A} = \langle U_{\mathfrak{A}}, \varepsilon^*_{\mathfrak{A}}, 1_{\mathfrak{A}}\rangle$, where $U_{\mathfrak{A}} := \{0,1,2,3\}$, $\varepsilon^*_{\mathfrak{A}} := \{\langle 0,3\rangle, \langle 1,3\rangle, \langle 1,2\rangle\}$ and $1_{\mathfrak{A}} := 3$ is a model of $\mathbf{E}^*$ and the formulas (ε^*1_1) and (ε^*1_2). But any valuation v for which $v(x) = 2$, $v(y) = 1$ and $v(z) = 0$ does not satisfy (ε^*1_3). □

We will get similarly:

Corollary 7.2. *The* $\mathrm{L}_{\varepsilon 1}$*-counterpart of* ($\varepsilon^*1_3$) *is not a thesis of* $\mathbf{E}1$.

Proof: The $\mathrm{L}_{\varepsilon^*1}$-structure $\mathfrak{A} = \langle U_{\mathfrak{A}}, \varepsilon_{\mathfrak{A}}, 1_{\mathfrak{A}}\rangle$, where $U_{\mathfrak{A}} := \{0,1,2,3\}$, $\varepsilon_{\mathfrak{A}} := \{\langle 0,0\rangle, \langle 1,1\rangle, \langle 0,3\rangle, \langle 1,3\rangle, \langle 1,2\rangle\}$ and $1_{\mathfrak{A}} := 3$ is a model of $\mathbf{E}1$. But any valuation v for which $v(x) = 2$, $v(y) = 1$ and $v(z) = 0$ does not satisfy the L_{ε}-counterpart of (ε^*1_3). □ Now, notice that:

Fact 7.3. *All axioms of* $\mathbf{E}^*1$ *are true in all special* $\mathrm{L}_{\varepsilon^*1}$ *structures.*

Proof: Let $\mathcal{F}$ be any non-empty family of sets.

For (ε^*1_1): We take an arbitrary valuation v such that $v(x) = X$ and $v(y) = Y$. Assume that $X \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}} Y$. Then either $\emptyset \neq X \subsetneq Y$ or there is $p \in \bigcup \mathcal{F}$ such that $X = \{p\} = Y$. Of course, $X, Y \subseteq \bigcup \mathcal{F} =: \mathbf{1}_{\mathcal{F}}$. If $\bigcup \mathcal{F} = \{p\}$ then $X = \{p\} = \mathbf{1}_{\mathcal{F}}$. If $\bigcup \mathcal{F}$ is not a singleton then $X \subsetneq \mathbf{1}_{\mathcal{F}}$. So in both cases we have $X \ \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}} \ \mathbf{1}_{\mathcal{F}}$.

For (ε^*1_2): We take an arbitrary valuation v such that $v(x) = X$. Assume that $\mathbf{1}_{\mathcal{F}} \ \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}} \ X$. Then there is a $p \in \bigcup \mathcal{F}$ such that $\mathbf{1}_{\mathcal{F}} = \{p\} = X$. So we have $X \ \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}} \ X$.

For (ε^*1_3): We take an arbitrary valuation v such that $v(x) = X$. Assume that for some $Y_0, Z_0 \in \mathcal{F}$ we have $Y_0 \ \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}} \ X$, $Z_0 \ \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}} \ \mathbf{1}_{\mathcal{F}}$ and $Z_0 \not\boldsymbol{\varepsilon}^{\star}_{\mathcal{F}} X$. We consider three cases.

(a) $\mathbf{1}_{\mathcal{F}}$ is a singleton. Then $X = \mathbf{1}_{\mathcal{F}}$, since $\emptyset \neq X \subseteq \mathbf{1}_{\mathcal{F}}$. So $X \ \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}} \ \mathbf{1}_{\mathcal{F}}$.

(b) X is a singleton and $\mathbf{1}_{\mathcal{F}}$ is not. Then $\emptyset \neq X \subsetneq \mathbf{1}_{\mathcal{F}}$. So $X \ \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}} \ \mathbf{1}_{\mathcal{F}}$.

(c) X is not a singleton. Then $\mathbf{1}_{\mathcal{F}}$ is not a singleton and $\emptyset \neq Y_0 \subsetneq X$. Moreover, $\emptyset \neq Z_0 \subsetneq \mathbf{1}_{\mathcal{F}}$ and either $Z_0 = X$ or $Z_0 \nsubseteq X$. So either $\emptyset \neq Z_0 =$

$X \subsetneq \mathbf{1}_{\mathcal{F}}$ or both $\emptyset \neq Z_0 \subsetneq \mathbf{1}_{\mathcal{F}}$ and $Z_0 \nsubseteq X$. So in both cases $X \neq \mathbf{1}_{\mathcal{F}}$. Thus, $\emptyset \neq X \subsetneq \mathbf{1}_{\mathcal{F}}$, i.e., $X \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}} \mathbf{1}_{\mathcal{F}}$. □

7.2. The quantifier-free theories $\mathbf{E}^*\mathsf{1}^{\mathrm{o}}$ and $(\mathbf{E}^*\mathsf{1}+(\mathtt{df}\ \varepsilon))^{\mathrm{o}}$

Let $\mathbf{E}^*\mathsf{1}^{\mathrm{o}}$ and $(\mathbf{E}^*\mathsf{1}+(\mathtt{df}\ \varepsilon))^{\mathrm{o}}$ be quantifier-free theories in $\mathrm{For}^{\mathrm{o}}_{\varepsilon 1}$ and $\mathrm{For}^{\mathrm{o}}_{\varepsilon\varepsilon^*1}$ having the same specific axioms as $\mathbf{E}^*\mathsf{1}$ and $\mathbf{E}^*\mathsf{1}^{\mathrm{o}}+(\mathtt{df}\ \varepsilon)$, respectively. Directly from Theorem 1.1 we obtain:

Corollary 7.4. $\mathbf{E}^*\mathsf{1}^{\mathrm{o}}$ *is the quantifier-free fragment of* $\mathbf{E}^*\mathsf{1}$*. Moreover,* $(\mathbf{E}^*\mathsf{1}+(\mathtt{df}\ \varepsilon))^{\mathrm{o}}$ *is the quantifier-free fragment of* $\mathbf{E}^*\mathsf{1}+(\mathtt{df}\ \varepsilon)$*.*

Remark 7.1. The quantifier-free theory $\mathbf{E}^*\mathsf{1}^{\mathrm{o}}$ can be treated as a pure calculus of names with logical constants 'ε^*' and '$\mathsf{1}$'.

The quantifier-free theory $(\mathbf{E}^*\mathsf{1}+(\mathtt{df}\ \varepsilon))^{\mathrm{o}}$ can be treated as a pure calculus of names with logical constants 'ε', 'ε^*' and '$\mathsf{1}$'. □

7.3. Epimorphism theorems for $\mathbf{E}^*\mathsf{1}$ and $\mathbf{E}^*\mathsf{1}+(\mathtt{df}\ \varepsilon)$

Theorem 7.5. *An* $\mathrm{L}_{\varepsilon^*1}$*-structure is a model of* $\mathbf{E}^*\mathsf{1}$ *(resp.* $\mathbf{E}^*\mathsf{1}+(\mathtt{df}\ \varepsilon)$*) iff it is epimorphic to a special* $\mathrm{L}_{\varepsilon^*1}$*-structure (resp.* $\mathrm{L}_{\varepsilon\varepsilon^*1}$*-structure).*

Proof: "⇒" *For the theory* $\mathbf{E}^*\mathsf{1}$. Let $\mathfrak{A} = \langle U_{\mathfrak{A}}, \varepsilon^*_{\mathfrak{A}}, \mathsf{1}_{\mathfrak{A}}\rangle$ be a model of $\mathbf{E}^*\mathsf{1}$. We consider three cases.

The first case: there is no c such that $c\ \varepsilon^*_{\mathfrak{A}}\ \mathsf{1}_{\mathfrak{A}}$. Then, by $(\varepsilon^*\mathsf{1}_1)$, for all $a, b \in U_{\mathfrak{A}}$ we have $a \not\varepsilon^*_{\mathfrak{A}} b$. We define the function $f \colon U_{\mathfrak{A}} \to \{\emptyset\}$ by $f(a) := \emptyset$, for any $a \in U_{\mathfrak{A}}$. Moreover, we put $\mathcal{F} := \{f(a) : a \in U_{\mathfrak{A}}\} = \{\emptyset\}$. Of course, $f(\mathsf{1}_{\mathfrak{A}}) = \{\emptyset\} = \bigcup \mathcal{F} =: \mathbf{1}_{\mathcal{F}}$ and for all $a, b \in U_{\mathfrak{A}}$ we have: $a\ \varepsilon^*_{\mathfrak{A}}\ b$ iff $f(a)\ \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}}\ f(b)$. So f is an epimorphism from $\mathfrak{A}$ onto $\langle \mathcal{F}, \boldsymbol{\varepsilon}_{\mathcal{F}}, \mathbf{1}_{\mathcal{F}}\rangle$.

The second case: $\mathsf{1}_{\mathfrak{A}}\ \varepsilon^*_{\mathfrak{A}}\ \mathsf{1}_{\mathfrak{A}}$. We define the function $f \colon U_{\mathfrak{A}} \to \{\emptyset, \{\emptyset\}\}$,

$$f(a) := \begin{cases} \{\emptyset\} & \text{if } a\ \varepsilon^*_{\mathfrak{A}}\ a \\ \emptyset & \text{otherwise} \end{cases}$$

and we put $\mathcal{F} := \{f(a) : a \in U_{\mathfrak{A}}\} \subseteq \{\emptyset, \{\emptyset\}\}$. Of course, $f(\mathsf{1}_{\mathfrak{A}}) = \{\emptyset\} = \bigcup \mathcal{F} =: \mathbf{1}_{\mathcal{F}}$. Moreover, we show that for all $a, b \in U_{\mathfrak{A}}$ we have:

$$a\ \varepsilon^*_{\mathfrak{A}}\ b \text{ iff } f(a)\ \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}}\ f(b).$$

Suppose that $a\ \varepsilon^*_{\mathfrak{A}}\ b$. Then $a\ \varepsilon^*_{\mathfrak{A}}\ \mathsf{1}$, by $(\varepsilon^*\mathsf{1}_1)$. Hence $\mathsf{1}\ \varepsilon^*_{\mathfrak{A}}\ a$, by (ε^*_2), since $\mathsf{1}_{\mathfrak{A}}\ \varepsilon^*_{\mathfrak{A}}\ \mathsf{1}_{\mathfrak{A}}$. Hence $\mathsf{1}\ \varepsilon^*_{\mathfrak{A}}\ b$, by (ε^*_1). Therefore $a\ \varepsilon^*_{\mathfrak{A}}\ a$ and $b\ \varepsilon^*_{\mathfrak{A}}\ b$, by $(\varepsilon^*\mathsf{1}_2)$. Therefore $f(a) = \{\emptyset\} = f(b)$; and so $f(a)\ \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}}\ f(b)$. Conversely, suppose that

$f(a)\,\boldsymbol{\varepsilon}^\star_{\mathcal{F}}\,f(b)$. Then $f(a)=\{\emptyset\}=f(b)$. Hence $a\,\varepsilon^*_{\mathfrak{A}}\,a$ and $b\,\varepsilon^*_{\mathfrak{A}}\,b$. So $a\,\varepsilon^*_{\mathfrak{A}}\,\mathsf{1}$ and $b\,\varepsilon^*_{\mathfrak{A}}\,\mathsf{1}$, by ($\varepsilon^*_1$). Hence $\mathsf{1}\,\varepsilon^*_{\mathfrak{A}}\,b$, by ($\varepsilon^*_2$), since $\mathsf{1}_{\mathfrak{A}}\,\varepsilon^*_{\mathfrak{A}}\,\mathsf{1}_{\mathfrak{A}}$. Hence $a\,\varepsilon^*_{\mathfrak{A}}\,b$, by ($\varepsilon^*_1$).

Thus, in this case f is an epimorphism from $\mathfrak{A}$ onto $\langle\mathcal{F},\boldsymbol{\varepsilon}_{\mathcal{F}},\mathbf{1}_{\mathcal{F}}\rangle$.

The third case: there is a c such that $c\,\varepsilon^*_{\mathfrak{A}}\,\mathsf{1}_{\mathfrak{A}}$ and $\mathsf{1}_{\mathfrak{A}}\not\varepsilon_{\mathfrak{A}}\,\mathsf{1}_{\mathfrak{A}}$. As in the "(a) $\Rightarrow$ (c)" part of the proof of Theorem 6.5, we defined the congruence $\sim$. Moreover, we define the function $f\colon U_{\mathfrak{A}}\to 2^{U_{\mathfrak{A}}/\sim}$,

$$f(a):=\begin{cases}\emptyset & \text{if } a\not\varepsilon^*_{\mathfrak{A}}\,\mathsf{1}_{\mathfrak{A}} \text{ and there is no } c \text{ such that } c\,\varepsilon^*_{\mathfrak{A}}\,a\\ \{[c]:c\,\varepsilon^*_{\mathfrak{A}}\,a\}\cup\{\emptyset\} & a\not\varepsilon^*_{\mathfrak{A}}\,\mathsf{1}_{\mathfrak{A}} \text{ and for some } c \text{ we have } c\,\varepsilon^*_{\mathfrak{A}}\,a\\ \{[a]\} & \text{if } a\,\varepsilon^*_{\mathfrak{A}}\,a \text{ (and } a\,\varepsilon^*_{\mathfrak{A}}\,\mathsf{1}_{\mathfrak{A}})\\ \{[c]:c\,\varepsilon^*_{\mathfrak{A}}\,a\}\cup\{[a],\emptyset\} & \text{if } a\not\varepsilon^*_{\mathfrak{A}}\,a \text{ and } a\,\varepsilon^*_{\mathfrak{A}}\,\mathsf{1}_{\mathfrak{A}}\end{cases}$$

So $f(\mathsf{1}_{\mathfrak{A}}):=\{[c]:c\,\varepsilon^*_{\mathfrak{A}}\,\mathsf{1}_{\mathfrak{A}}\}\cup\{\emptyset\}$. We put $\mathcal{F}:=\{f(a):a\in U_{\mathfrak{A}}\}$.

We must show that $f(\mathsf{1}_{\mathfrak{A}})=\bigcup\mathcal{F}=:\mathbf{1}_{\mathcal{F}}$. This is due to the fact that $f(a)\subseteq f(\mathsf{1}_{\mathfrak{A}})$, for any $a\in U_{\mathfrak{A}}$. Firstly, $\emptyset\in f(\mathsf{1}_{\mathfrak{A}})$. Secondly, if $a\,\varepsilon^*_{\mathfrak{A}}\,a$, then $f(a):=\{[a]\}$ and $a\,\varepsilon^*_{\mathfrak{A}}\,\mathsf{1}_{\mathfrak{A}}$, by ($\varepsilon^*\mathsf{1}_1$). Hence $[a]\in f(\mathsf{1}_{\mathfrak{A}})$; and so $f(a)\subseteq f(\mathsf{1}_{\mathfrak{A}})$. Thirdly, if $a\not\varepsilon^*_{\mathfrak{A}}\,a$ and $[c]\in f(a)$, then either $c\,\varepsilon^*_{\mathfrak{A}}\,a$ or both $c=a$ and $a\,\varepsilon^*_{\mathfrak{A}}\,\mathsf{1}_{\mathfrak{A}}$. In both cases $c\,\varepsilon^*_{\mathfrak{A}}\,\mathsf{1}_{\mathfrak{A}}$. Thus, $f(a)\subseteq f(\mathsf{1}_{\mathfrak{A}})$.

Therefore, we obtain $f(\mathsf{1}_{\mathfrak{A}})\subseteq\bigcup\mathcal{F}\subseteq f(\mathsf{1}_{\mathfrak{A}})$.

Now we show that for all $a,b\in U_{\mathfrak{A}}$:

$$a\,\varepsilon^*_{\mathfrak{A}}\,b \quad\text{iff}\quad f(a)\,\boldsymbol{\varepsilon}^\star_{\mathcal{F}}\,f(b).$$

Suppose that $a\,\varepsilon^*_{\mathfrak{A}}\,b$. Then $a\,\varepsilon^*_{\mathfrak{A}}\,\mathsf{1}_{\mathfrak{A}}$, by ($\varepsilon^*\mathsf{1}_1$). Hence $f(a)\neq\emptyset$ and $a\neq\mathsf{1}_{\mathfrak{A}}$, by the assumption. We consider five possibilities.

(1) $b\,\varepsilon^*_{\mathfrak{A}}\,b$. Then $b\,\varepsilon^*_{\mathfrak{A}}\,a$, by ($\varepsilon^*_2$). Moreover, $a\,\varepsilon^*_{\mathfrak{A}}\,a$, by ($\varepsilon^*_1$). Hence, by ($\varepsilon^*_1$), $a\sim b$; so $[a]=[b]$ and $f(a)=\{[a]\}=\{[b]\}=f(b)$.

(2) $a\,\varepsilon^*_{\mathfrak{A}}\,a$ and $b\not\varepsilon^*_{\mathfrak{A}}\,\mathsf{1}_{\mathfrak{A}}$ (and so $b\not\varepsilon^*_{\mathfrak{A}}\,b$). Then $f(a)=\{[a]\}$, $[a]\neq[b]$; and so $\emptyset\neq f(a)\subsetneq f(b):=\{[c]:c\,\varepsilon^*_{\mathfrak{A}}\,b\}\cup\{\emptyset\}$.

(3) $b\not\varepsilon^*_{\mathfrak{A}}\,b$, $a\,\varepsilon^*_{\mathfrak{A}}\,a$ and $b\,\varepsilon^*_{\mathfrak{A}}\,\mathsf{1}_{\mathfrak{A}}$. Then $f(a)=\{[a]\}$, $[a]\neq[b]$, $[b]\notin f(a)$; and so $\emptyset\neq f(a)\subsetneq f(b):=\{[c]:c\,\varepsilon^*_{\mathfrak{A}}\,b\}\cup\{[b],\emptyset\}$.

(4) $a\not\varepsilon^*_{\mathfrak{A}}\,a$ and $b\not\varepsilon^*_{\mathfrak{A}}\,\mathsf{1}_{\mathfrak{A}}$ (and so $b\not\varepsilon^*_{\mathfrak{A}}\,b$). Then $b\not\varepsilon^*_{\mathfrak{A}}\,a$, by ($\varepsilon^*_1$) and the assumption. So $[a]\neq[b]$. Moreover, $f(a):=\{[c]:c\,\varepsilon^*_{\mathfrak{A}}\,a\}\cup\{[a],\emptyset\}$ and $f(b):=\{[c]:c\,\varepsilon^*_{\mathfrak{A}}\,b\}\cup\{\emptyset\}$. Therefore, $\emptyset\neq f(a)\subsetneq f(b)$.

(5) $a\not\varepsilon^*_{\mathfrak{A}}\,a$ and $b\,\varepsilon^*_{\mathfrak{A}}\,\mathsf{1}_{\mathfrak{A}}$ and $b\not\varepsilon^*_{\mathfrak{A}}\,b$. Then $b\not\varepsilon^*_{\mathfrak{A}}\,a$, by ($\varepsilon^*_1$) and the assumption. So $[a]\neq[b]$. Moreover, $f(a):=\{[c]:c\,\varepsilon^*_{\mathfrak{A}}\,a\}\cup\{[a],\emptyset\}$ and $f(b):=\{[c]:c\,\varepsilon^*_{\mathfrak{A}}\,b\}\cup\{[b],\emptyset\}$. Hence $[b]\notin f(a)$. Therefore, $\emptyset\neq f(a)\subsetneq f(b)$.

Thus, in all five cases we have $f(a)\,\boldsymbol{\varepsilon}^\star_{\mathcal{F}}\,f(b)$.

Conversely, let $f(a)\, \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}}\, f(b)$, i.e., either (1) both $f(a)$ is a singleton and $f(a) = f(b)$, or (2) $\emptyset \neq f(a) \subsetneq f(b)$. Then, in both cases, for some c_0 we have $c_0\ \varepsilon^*_{\mathfrak{A}}\ a$. Hence $c_0\ \varepsilon^*_{\mathfrak{A}}\ 1_{\mathfrak{A}}$.

(1) Then $f(a) = \{[a]\} = \{[b]\} = f(b)$. Hence $a\ \varepsilon^*_{\mathfrak{A}}\ b$.

(2) We consider the following cases.

(2a) $a\ \varepsilon^*_{\mathfrak{A}}\ a$. Then $\emptyset \neq f(a) := \{[a]\} \subsetneq f(b) \neq \{[b]\}$. Hence $b\ \not\varepsilon^*_{\mathfrak{A}}\ b$; and so $a \nsim b$, i.e., $[a] \neq [b]$. Hence $a\ \varepsilon^*_{\mathfrak{A}}\ b$.

(2b) $a\ \not\varepsilon^*_{\mathfrak{A}}\ a$. We show that $a\ \varepsilon^*_{\mathfrak{A}}\ 1_{\mathfrak{A}}$. (It is also when $a\ \varepsilon^*_{\mathfrak{A}}\ a$.)

Indeed, suppose that $a\ \not\varepsilon^*_{\mathfrak{A}}\ 1_{\mathfrak{A}}$, i.e., $\emptyset \neq f(a) := \{[c] : c\ \varepsilon^*_{\mathfrak{A}}\ a\} \cup \{\emptyset\}$. But we have $\emptyset \neq f(a) \subsetneq f(b) \subseteq f(1_{\mathfrak{A}})$. Hence $\emptyset \neq f(a) \subsetneq f(1_{\mathfrak{A}})$. So $\emptyset \neq \{[c] : c\ \varepsilon^*_{\mathfrak{A}}\ a\} \cup \{\emptyset\} \subsetneq \{[c] : c\ \varepsilon^*_{\mathfrak{A}}\ 1_{\mathfrak{A}}\} \cup \{\emptyset\}$. So there is $c_1 \in U_{\mathfrak{A}}$ such that $c_1\ \varepsilon^*_{\mathfrak{A}}\ 1_{\mathfrak{A}}$ and $c_1\ \not\varepsilon^*_{\mathfrak{A}}\ a$. Moreover, since we have $c_0\ \varepsilon^*_{\mathfrak{A}}\ a$, by ($\varepsilon^*1_3$), we obtain a contradiction: $a\ \varepsilon^*_{\mathfrak{A}}\ 1_{\mathfrak{A}}$.

Therefore $\emptyset \neq \{[c] : c\ \varepsilon^*_{\mathfrak{A}}\ a\} \cup \{[a], \emptyset\} \subsetneq f(b)$. Hence $[a] \in f(b)$. In the case where $b\ \not\varepsilon^*_{\mathfrak{A}}\ 1_{\mathfrak{A}}$ we have $a\ \varepsilon^*_{\mathfrak{A}}\ b$. In the case where $b\ \varepsilon^*_{\mathfrak{A}}\ 1_{\mathfrak{A}}$ either $a\ \varepsilon^*_{\mathfrak{A}}\ b$ or $[a] = [b]$; and so also $a\ \varepsilon^*_{\mathfrak{A}}\ b$.

For the theory $\mathbf{E}^*1$+(df ε). As for the theory $\mathbf{E}^*$+(df ε) in the proof of Theorem 6.5.

"⇐" By Theorem 6.5 and Fact 7.3. □

Thus, we obtain (as Theorem 2.5):

THEOREM 7.6. *For any* $\varphi \in \mathrm{For}_{\varepsilon^*1}$ (*resp.* $\varphi \in \mathrm{For}_{\varepsilon\varepsilon^*1}$): φ *is a thesis of* $\mathbf{E}^*1$ (*resp.* $\mathbf{E}^*1$+(df ε)) *iff* φ *is true in any special* $\mathrm{L}_{\varepsilon^*1}$*-structure* (*resp.* $\mathrm{L}_{\varepsilon\varepsilon^*1}$*-structure*).

Remark 7.2. In connection with the above theorem, an open formula from $\mathrm{For}^{\mathrm{o}}_{\varepsilon^*1}$ (resp. $\mathrm{For}^{\mathrm{o}}_{\varepsilon\,\varepsilon^*1}$) is a thesis of a pure calculus of names $\mathbf{E}^*1^{\mathrm{o}}$ (resp. $(\mathbf{E}^*1$+(df ε)$)^{\mathrm{o}}$) iff it is true in any model $\langle U, d\rangle$. □

7.4. A reconstruction of E1 in E*1

In $\mathrm{L}_{\varepsilon\varepsilon^*1}$ we can build definitional extensions of two theories $\mathbf{E}^*$+(ε^*1_1)+(ε^*1_2) and $\mathbf{E}^*1$ by adding the definition (df ε). Notice that

FACT 7.7. $\mathrm{Th}(\mathbf{E}1) \subsetneq \mathrm{Th}(\mathbf{E}^*$+($\varepsilon^*1_1$)+($\varepsilon^*1_2$)+(df ε)$) \subsetneq \mathrm{Th}(\mathbf{E}^*1$+(df ε)$)$. *So the theory* $\mathbf{E}^*1$+(df ε) *is a proper extension of* $\mathbf{E}1$.

PROOF: Firstly, (ε_1)–(ε_3) are theses of $\mathbf{E}^*$+(df ε). Secondly, from (ε^*1_1) and (df ε) we obtain ($\varepsilon 1_1$), and from (ε^*1_2) and (df ε) we obtain ($\varepsilon 1_2$). Thirdly, by Fact 7.1, the formula (ε^*1_3) is not a thesis of $\mathbf{E}^*$+(ε^*1_1)+(ε^*1_2). So it is not a thesis of $\mathbf{E}^*$+(ε^*1_1)+(ε^*1_2)+(df ε). □

However, in the light of theorems 5.9 and 7.6, the theories $\mathbf{E1}$ and $\mathbf{E}^*1+(\mathtt{df}\ \varepsilon)$ have the same theses from the language $\mathrm{L}_{\varepsilon 1}$, i.e., we obtain:

$$\mathrm{For}_{\varepsilon 1} \cap \mathrm{Th}(\mathbf{E}^*1+(\mathtt{df}\ \varepsilon)) = \mathrm{Th}(\mathbf{E}1). \tag{7.1}$$

7.5. A reconstruction of $\mathbf{\Lambda}1$ in some extension of $\mathbf{E}^*$

If we use the language $\mathrm{L}_{\varepsilon\varepsilon^*1}$ then we can extend theories in $\mathrm{L}_{\varepsilon^*1}$ using formulas from $\mathrm{L}_{\varepsilon 1}$. Let us remind that the formula $(\leftarrow\$1)$ is not a thesis of $\mathbf{E}1$. So, by (7.1), it is not a thesis of $\mathbf{E}^*1+(\mathtt{df}\ \varepsilon)$. Moreover, notice that:

Fact 7.8. *All of* (ε^*1_1)–(ε^*1_3) *do not belong to* $\mathrm{Th}(\mathbf{E}^*+(\mathtt{df}\ \varepsilon)+(\leftarrow\$1))$.

Proof: The $\mathrm{L}_{\varepsilon\varepsilon^*1}$-structure $\mathfrak{A} = \langle U_{\mathfrak{A}}, \varepsilon_{\mathfrak{A}}, \varepsilon^*_{\mathfrak{A}}, 1_{\mathfrak{A}}\rangle$, where $U_{\mathfrak{A}} := \{1, 2\}$, $\varepsilon^*_{\mathfrak{A}} := \emptyset$, $\varepsilon^*_{\mathfrak{A}} := \{\langle 1, 2\rangle\}$ and $1_{\mathfrak{A}} := 1$, is a model of $\mathbf{E}^*+(\mathtt{df}\ \varepsilon^*)+(\leftarrow\$1)$ in which (ε^*1_1) and (ε^*1_2) are not true.

Moreover, $\mathrm{L}_{\varepsilon\varepsilon^*1}$-structure $\mathfrak{A} = \langle U_{\mathfrak{A}}, \varepsilon_{\mathfrak{A}}, \varepsilon^*_{\mathfrak{A}}, 1_{\mathfrak{A}}\rangle$, where $U_{\mathfrak{A}} := \{0, 1, 2, 3\}$, $\varepsilon^*_{\mathfrak{A}} := \emptyset$, $\varepsilon^*_{\mathfrak{A}} := \{\langle 0, 1\rangle, \langle 1, 2\rangle\}$ and $1_{\mathfrak{A}} := 1$, is a model of $\mathbf{E}^*+(\mathtt{df}\ \varepsilon^*)+(\leftarrow\$1)$ in which (ε^*1_3) is not true. □

In virtue of (6.1) and Theorem 5.7, we obtain that $\mathbf{E}^*+(\mathtt{df}\ \varepsilon)+(\leftarrow\$1)$ is a proper extension of $\mathbf{\Lambda}1$, i.e.,

$$\mathrm{Th}(\mathbf{\Lambda}1) \subsetneq \mathrm{Th}(\mathbf{E}^*+(\mathtt{df}\ \varepsilon)+(\leftarrow\$1)). \tag{7.2}$$

However, in the light of theorems 3.4 and 7.6, the theories $\mathbf{\Lambda}1$ and $\mathbf{E}^*+(\mathtt{df}\ \varepsilon)+(\leftarrow\$1)$ have the same theses from the language $\mathrm{L}_{\varepsilon 1}$, i.e., we obtain:

$$\mathrm{Th}(\mathbf{\Lambda}1) = \mathrm{For}_{\varepsilon 1} \cap \mathrm{Th}(\mathbf{E}^*+(\mathtt{df}\ \varepsilon)+(\leftarrow\$1)). \tag{7.3}$$

In fact, by theorems 3.4 and 7.6, all theses of $\mathbf{E}^*+(\mathtt{df}\ \varepsilon)+(\leftarrow\$1)$ are true in all s-special $\mathrm{L}_{\varepsilon\varepsilon^*1}$-structures. So if φ belongs to $\mathrm{For}_{\varepsilon 1} \cap \mathrm{Th}(\mathbf{E}^*+(\mathtt{df}\ \varepsilon)+(\leftarrow\$))$, then it is true in all s-special $\mathrm{L}_{\varepsilon 1}$-structures. Hence, by Theorem 3.4, φ is a thesis of $\mathbf{\Lambda}$.

8. Defining the predicate 'ε^*' by 'ε'

As the definition of 'ε^*' by 'ε' we adopt the following non-open formula:

$$\begin{aligned} x\,\varepsilon^*\,y \ \leftrightarrow\ & (x\,\varepsilon\,y \wedge y\,\varepsilon\,x) \vee \\ & (\exists u\, u\,\varepsilon\,x \wedge \forall u(u\,\varepsilon\,x \to u\,\varepsilon\,y) \wedge \neg\forall u(u\,\varepsilon\,y \to u\,\varepsilon\,x)) \end{aligned} \tag{$\mathtt{df}\ \varepsilon^*$}$$

8.1. The definition (df ε^*) in the theory E

Let $\mathbf{E}$+(df ε^*) be a definitional extension of $\mathbf{E}$ by adding (df ε^*). We prove:

FACT 8.1. *The theory* $\mathbf{E}$+(df ε^*) *is a proper extension of* $\mathbf{E}^*$.

PROOF: For (ε^*_1): Directly by (df ε^*) we obtain:

$$\begin{aligned} x \,\varepsilon^*\, y \wedge y \,\varepsilon^*\, z \leftrightarrow {}& ((\exists u\, u \,\varepsilon\, x \wedge \forall u(u \,\varepsilon\, x \rightarrow u \,\varepsilon\, y) \wedge \exists u(u \,\varepsilon\, y \wedge \neg\, u \,\varepsilon\, x)) \vee \\ & (x \,\varepsilon\, y \wedge y \,\varepsilon\, x)) \wedge ((\exists u\, u \,\varepsilon\, y \wedge \forall u(u \,\varepsilon\, y \rightarrow u \,\varepsilon\, z) \wedge \\ & \exists u(u \,\varepsilon\, z \wedge \neg\, u \,\varepsilon\, y)) \vee (y \,\varepsilon\, z \wedge z \,\varepsilon\, y)) \end{aligned}$$

Hence, by (ε_2), we have:

$$x \,\varepsilon^*\, y \wedge y \,\varepsilon^*\, z \rightarrow (x \,\varepsilon\, z \wedge z \,\varepsilon\, x) \vee (\exists u\, u \,\varepsilon\, x \wedge \forall u(u \,\varepsilon\, x \rightarrow u \,\varepsilon\, z) \wedge \exists u(u \,\varepsilon\, z \wedge \neg\, u \,\varepsilon\, x))$$

For (ε^*_2): Directly by (df ε^*) we obtain:

$$\begin{aligned} x \,\varepsilon^*\, y \wedge y \,\varepsilon^*\, y \;\leftrightarrow\;\; & ((\exists u\, u \,\varepsilon\, x \wedge \forall u(u \,\varepsilon\, x \rightarrow u \,\varepsilon\, y) \wedge \neg\forall u(u \,\varepsilon\, y \rightarrow u \,\varepsilon\, x)) \vee \\ & (x \,\varepsilon\, y \wedge y \,\varepsilon\, x)) \wedge y \,\varepsilon\, y \\ \leftrightarrow\;\; & (\exists u\, u \,\varepsilon\, x \wedge \forall u(u \,\varepsilon\, x \rightarrow u \,\varepsilon\, y) \wedge \\ & \neg\forall u(u \,\varepsilon\, y \rightarrow u \,\varepsilon\, x) \wedge y \,\varepsilon\, y) \vee (x \,\varepsilon\, y \wedge y \,\varepsilon\, x \wedge y \,\varepsilon\, y) \end{aligned}$$

However, the first component of the above disjunction is contradictory. In fact, if $y \,\varepsilon\, y$ and for some u_1 we have $u_1 \,\varepsilon\, x$, then also $u_1 \,\varepsilon\, y$. So, by (ε_3), $y \,\varepsilon\, u_1$. So, by (ε_2), we obtain: $\forall u(u \,\varepsilon\, y \rightarrow u \,\varepsilon\, x)$. Thus, we obtain the following (the first one by (ε_1); the second one by (df ε^*)):

$$x \,\varepsilon^*\, y \wedge y \,\varepsilon^*\, y \;\leftrightarrow\; y \,\varepsilon\, x \wedge x \,\varepsilon\, y \wedge y \,\varepsilon\, y \;\leftrightarrow\; y \,\varepsilon\, x \wedge x \,\varepsilon\, y$$
$$x \,\varepsilon\, y \wedge y \,\varepsilon\, x \;\rightarrow\; y \,\varepsilon^*\, x$$

So we also have '$x \,\varepsilon^*\, y \wedge y \,\varepsilon^*\, y \;\rightarrow\; y \,\varepsilon^*\, x$'. □

Notice that directly from (df ε^*) we obtain the formula (%). However,

FACT 8.2. *The implication '$x \,\varepsilon\, y \rightarrow x \,\varepsilon^*\, y$' is not a thesis of* $\mathbf{E}$+(df ε^*).[9] *Hence we obtain*:

$$(\mathtt{df}\ \varepsilon) \notin \mathrm{Th}(\mathbf{E}+(\mathtt{df}\ \varepsilon^*))$$
$$\mathrm{Th}(\mathbf{E}^*+(\mathtt{df}\ \varepsilon)) \not\subseteq \mathrm{Th}(\mathbf{E}+(\mathtt{df}\ \varepsilon^*)).$$

PROOF: The $\mathrm{L}_{\varepsilon\varepsilon^*}$-structure $\mathfrak{A} = \langle U_{\mathfrak{A}}, \varepsilon_{\mathfrak{A}}, \varepsilon^*_{\mathfrak{A}}\rangle$, where $U_{\mathfrak{A}} := \{0,1\}$, $\varepsilon_{\mathfrak{A}} := \{\langle 0,0\rangle, \langle 0,1\rangle\}$ and $\varepsilon^*_{\mathfrak{A}} := \{\langle 0,0\rangle\}$ is a model of $\mathbf{E}$+(df ε^*) in which '$x \,\varepsilon\, y \rightarrow x \,\varepsilon^*\, y$' is not true. So also (df ε) is not true in the model. □

[9] But the implications '$x \,\varepsilon\, x \rightarrow x \,\varepsilon^*\, x$' and '$x \,\varepsilon^*\, x \wedge x \,\varepsilon^*\, y \rightarrow x \,\varepsilon\, y$' are theses of $\mathbf{E}$+(df ε^*).

We obtain:

FACT 8.3. (df ε^*) $\notin$ Th(**EO**+(df ε)+(ε^*_1)+(ε^*_2)). *So* (df ε^*) $\notin$ Th($\mathbf{E}^*$+(df ε)+(←$1)).

PROOF: The $\mathrm{L}_{\varepsilon\varepsilon^*}$-structure $\mathfrak{A} = \langle U_{\mathfrak{A}}, \varepsilon_{\mathfrak{A}}, \varepsilon^*_{\mathfrak{A}}\rangle$, where $U_{\mathfrak{A}} := \{0,1\}$ and $\varepsilon^*_{\mathfrak{A}} := \{\langle 0,1\rangle\}$ and $\varepsilon^*_{\mathfrak{A}} = \emptyset$, is a model of **EO** and formulas (df ε), (ε^*_1) and (ε^*_2). We have $0\ \varepsilon^*_{\mathfrak{A}}\ 1$, but the substitution $[x/0, y/1]$ does not satisfy the right-side of the equivalence (df ε^*). □

Thus, although Th(**E**) $\subsetneq$ Th($\mathbf{E}^*$+(df ε)), we have:

$$\mathrm{Th}(\mathbf{E}+(\mathtt{df}\ \varepsilon^*)) \nsubseteq \mathrm{Th}(\mathbf{E}^*+(\mathtt{df}\ \varepsilon)).$$

8.2. The definition (df ε^*) in the theories $\mathbf{\Lambda}$ and $\mathbf{\Lambda1}$

First we notice:

FACT 8.4. *In any s-special* L_{ε}*-structure, the predicate '*ε^**' defined by* (df ε^*) *is interpreted by the relation* $\boldsymbol{\varepsilon}^{\star}_{\mathcal{F}}$ *defined by* (df $\boldsymbol{\varepsilon}^{\star}_{\mathcal{F}}$). *So* (df ε^*) *is true in any s-special* $\mathrm{L}_{\varepsilon\varepsilon^*}$*-structure* $\langle \mathcal{F}, \boldsymbol{\varepsilon}_{\mathcal{F}}, \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}}\rangle$.

PROOF: Suppose that $\mathcal{F}$ is a non-empty s-family of sets and $\mathcal{R} \subseteq \mathcal{F}^2$ is an interpretation of the predicate 'ε^*' defined by (df ε^*). We show that $\mathcal{R} = \boldsymbol{\varepsilon}^{\star}_{\mathcal{F}}$. For all $X, Y \in \mathcal{F}$ we obtain: $X\,\mathcal{R}\,Y$ iff either (i) both $X\,\boldsymbol{\varepsilon}_{\mathcal{F}}\,Y$ and $Y\,\boldsymbol{\varepsilon}_{\mathcal{F}}\,X$, or (ii) both for some $X_1 \in \mathcal{F}$ we have $X_1\,\boldsymbol{\varepsilon}_{\mathcal{F}}\,X$ and for all $Z \in \mathcal{F}$: if $Z\,\boldsymbol{\varepsilon}_{\mathcal{F}}\,X$ then $Z\,\boldsymbol{\varepsilon}_{\mathcal{F}}\,Y$, and for some $X_2 \in \mathcal{F}$ we have $X_2\,\boldsymbol{\varepsilon}_{\mathcal{F}}\,Y$ and $X_2\,\not\boldsymbol{\varepsilon}_{\mathcal{F}}\,X$.

In the case (i): X is a singleton and $X = Y$. So we have $X\,\boldsymbol{\varepsilon}^{\star}_{\mathcal{F}}\,Y$.

In the case (ii): (a) for some singleton $X_1 \in \mathcal{F}$ we have $X_1 \subseteq X$; (b) for any singleton $Z \in \mathcal{F}$ such that $Z \subseteq X$ we have $Z \subseteq Y$; (c) for some singleton $X_2 \in \mathcal{F}$ we have $X_2 \subseteq Y$ and $X_2 \nsubseteq X$. By (a), $X \neq \emptyset$. By (b) $X \subseteq Y$, since $\mathcal{F}$ is an s-family of sets. By (c), $X \nsubseteq Y$. So we have $X\,\boldsymbol{\varepsilon}^{\star}_{\mathcal{F}}\,Y$.

Conversely, if $X\,\boldsymbol{\varepsilon}^{\star}_{\mathcal{F}}\,Y$ then either case (i) or case (ii) holds. □

In virtue of Theorem 2.4 and Fact 8.4, for the theory $\mathbf{\Lambda}$+(df ε^*) we get:

THEOREM 8.5. *An* $\mathrm{L}_{\varepsilon\varepsilon^*}$*-structure is a model of* $\mathbf{\Lambda}$+(df ε^*) *iff it is epimorphic to an s-special* $\mathrm{L}_{\varepsilon\varepsilon^*}$*-structure.*

Hence we can prove (as Theorem 2.5):

THEOREM 8.6. φ *belongs to* Th($\mathbf{\Lambda}$+(df ε^*)) *iff* φ *is true in any s-special* $\mathrm{L}_{\varepsilon\varepsilon^*}$*-structure.*

Thus, in virtue of Fact 8.3 and theorems 6.6 and 8.6 we get:

FACT 8.7. $\mathrm{Th}(\mathbf{E^*}+(\mathtt{df}\ \varepsilon)) \subsetneq \mathrm{Th}(\mathbf{\Lambda}+(\mathtt{df}\ \varepsilon^*))$ *and* $\mathrm{Th}(\mathbf{E^*}+(\mathtt{df}\ \varepsilon)+(\mathtt{\leftarrow\$1})) \subsetneq \mathrm{Th}(\mathbf{\Lambda1}+(\mathtt{df}\ \varepsilon^*))$

PROOF: Suppose that φ is a thesis of $\mathbf{E^*}+(\mathtt{df}\ \varepsilon)$. Then, in virtue of Theorem 6.6, φ is true in all special $\mathrm{L}_{\varepsilon\varepsilon^*}$-structures. So φ is true in all s-special $\mathrm{L}_{\varepsilon\varepsilon^*}$-structures. Hence $\varphi \in \mathrm{Th}(\mathbf{\Lambda}+(\mathtt{df}\ \varepsilon^*))$, by Theorem 8.6.[10]

Moreover, we use Fact 3.2. □

Finally, we prove that:

FACT 8.8. $\mathrm{Th}(\mathbf{E^*}+(\mathtt{df}\ \varepsilon)+(\mathtt{df}\ \varepsilon^*)+(\mathtt{\leftarrow\$1})) = \mathrm{Th}(\mathbf{\Lambda1}+(\mathtt{df}\ \varepsilon^*))$.

PROOF: Firstly, by (7.2), we have $\mathrm{Th}(\mathbf{\Lambda1}+(\mathtt{df}\ \varepsilon^*)) \subseteq \mathrm{Th}(\mathbf{E^*}+(\mathtt{df}\ \varepsilon)+(\mathtt{\leftarrow\$1})+(\mathtt{df}\ \varepsilon^*))$. Secondly, by Fact 8.7, we have $\mathrm{Th}(\mathbf{E^*}+(\mathtt{df}\ \varepsilon)+(\mathtt{\leftarrow\$1})) \subseteq \mathrm{Th}(\mathbf{\Lambda1}+(\mathtt{df}\ \varepsilon^*))$. □

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[10] We also have another proof of this fact. In [1] Ishimoto proved that in any model $\langle U_{\mathfrak{A}}, \varepsilon_{\mathfrak{A}} \rangle$ of theory $\mathbf{\Lambda}$, if $a\ \varepsilon_{\mathfrak{A}}\ b$ then there is c such that $c\ \varepsilon_{\mathfrak{A}}\ b$, $c \not\varepsilon_{\mathfrak{A}} a$ and $a \not\varepsilon_{\mathfrak{A}} c$. So the implication '$x\ \varepsilon\ y \to x\ \varepsilon^*\ y$' from Fact 8.2 is a true in all such models. Hence it is a thesis of $\mathbf{\Lambda}+(\mathtt{df}\ \varepsilon^*)$. Moreover, the formulas (%) and '$x\ \varepsilon^*\ x \wedge x\ \varepsilon^*\ y \to x\ \varepsilon\ y$' are theses of $\mathbf{E}+(\mathtt{df}\ \varepsilon^*)$ (see footnote 9). Hence $(\mathtt{df}\ \varepsilon)$ is a thesis of $\mathbf{\Lambda}+(\mathtt{df}\ \varepsilon^*)$. Finally, we use facts 8.1 and 8.3.

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