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THE INFINITE-VALUED ŁUKASIEWICZ LOGIC AND PROBABILITY

To Grzegorz

Abstract

The paper concerns the algebraic structure of the set of cumulative distribution functions as well as the relationship between the resulting algebra and the infinite-valued Łukasiewicz algebra. The paper also discusses interrelations holding between the logical systems determined by the above algebras.

Keywords: probability, cumulative distribution function, the infinite-valued standard Łukasiewicz algebra, consequence relation.

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One of the interesting and still not satisfactorily resolved problems in metalogic concerns the character and nature of relationship holding between probability theory and the Łuksiewicz infinite-valued logic. It is sometimes argued that Łukasiewicz logic is appropriate for the formulation of the 'fuzzy set theory'. On the other hand, logical values assigned to propositions in accordance with the scheme provided by the infinite Łukasiewicz algebra should not be interpreted as probabilities of these propositions, because the structural algebraic properties of this algebra are incoherent with the standard Kolmogorov's axiom system of probability theory. In the simplest case, the values assigned to compound propositions in the Łukasiewicz infinite-valued algebra do not agree with the values computed according to the rules of probability theory, e.g., for the conjunction of two propositions. But nevertheless, there are some interesting interdependencies holding between probability and infinite-valued logics. More specifically, in this paper we want to shed more light on the relationship between Łukasiewicz logic and the theory of cumulative distribution functions. Some aspects of this problem have been signalled in Czelakowski's [6]. Here the problem is discussed in a more thorough way from the viewpoint of the algebra of cumulative distribution functions.

1. The infinite-valued Łuksiewicz algebra

$\mathbb{R}$ is the set of real numbers. On the unit interval $I := [0,1] := \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ we define the operations $\rightarrow, \wedge, \vee, \otimes, \oplus, \neg$ as follows:

$$\begin{array}{lrcl}
(\rightarrow)_{\text{Ł}} & a \rightarrow b & := & \min(1, 1-a+b), \\
(\wedge)_{\text{Ł}} & a \wedge b & := & \min(a,b), \\
(\vee)_{\text{Ł}} & a \vee b & := & \max(a,b), \\
(\otimes)_{\text{Ł}} & a \otimes b & := & \max(0, a+b-1), \\
(\oplus)_{\text{Ł}} & a \oplus b & := & \min(1, a+b), \\
(\neg)_{\text{Ł}} & \neg a & := & 1-a.
\end{array}$$

They are called the *Łukasiewicz operations*. The operations $\rightarrow, \wedge$ and $\vee$ are succesively called implication, conjunction and disjunction. $\otimes$ is called the strong conjunction and $\oplus$ is the weak disjunction. $\neg$ is the negation operation.

$A_c := \langle I, \rightarrow, \wedge, \vee, \oplus, \otimes, \neg \rangle$ is the infinite-valued *Łukasiewicz algebra*. All the displayed operations are treated here as primitive operations of A_c but they are definable in terms of the operations $\rightarrow$ and $\neg$ in the well-known manner. (One may also take $\oplus$ and $\neg$ as primitive operations, because $a \rightarrow b = \neg a \oplus b, a \vee b = \neg(\neg a \oplus b) \oplus b, a \wedge b = \neg(a \vee \neg b)$, and $a \otimes b = \neg(a \oplus \neg b)$, for all $a, b \in I$.)

The pair $\langle A_c, \{1\} \rangle$ is called the infinite *Łukasiewicz matrix*.

L is the sentential language appropriate for A_c. Thus L is the absolute free algebra freely generated by a countably infinite set Var of propositional variables and endowed with the connectives $\rightarrow, \wedge, \vee, \oplus, \otimes$ and $\neg$. (The connectives of L are marked by the same symbols as the corresponding operations in the algebra A_c; such doubleness in the meaning of the above symbols should not lead to confusion.)

2. Cumulative distribution functions

In probability theory, one-dimensional cumulative distributions (CDFs, for short) are *defined* as functions $F : \mathbb{R} \longrightarrow [0,1]$ that satisfy the following conditions:

(2a) F is non-decreasing.
(2b) F is right-continuous,
(2c) $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$.

F is therefore a *càdlàg function* which means that for every real r, the left limit $F(r^-)$ exists; and the right limit $F(r^+)$ exists and equals $F(r)$.

Cumulative distribution functions are also called *unary probabilistic attributes*. This name is justified by the fact that each CDF provides a probabilistic characterization of unary properties (attributes) of statistical populations (see [11]). Examples of such attributes are weight, height etc.

One of the main theorems in probability theory states that every CDF F, defined as above, determines a probability measure μ_F on the σ-field $\mathcal{B}(\mathbb{R})$ of Borel subsets of $\mathbb{R}$ such that $\mu_F((a,b]) = F(b) - F(a)$ for any real numbers a, b, where $a < b$. (If $a = b$, then $\mu_F(\{a\}) = F(a) - F(a^-)$, where $F(a^-)$ is the left limit of F at a. If F is continuous at a, then $\mu_F(\{a\}) = 0$.)

Moreover, every probability measure on $B(\mathbb{R})$ is determined by a unique CDF (see e.g. [1]).

If F is a continuous CDF, the measure μ_F takes value zero on one-element subsets of $\mathbb{R}$. Consequently, for any numbers a, b with $a < b$, it is the case that $\mu_F((a,b)) = \mu_F((a,b]) = \mu_F([a,b)) = \mu_F([a,b])$.

3. The Algebra of Cumulative Distribution Functions

Let μ be a probability measure on the σ-field $\mathcal{B}(\mathbb{R})$ of Borel subsets of $\mathbb{R}$. μ^d is the measure *dual* to μ. Thus

(1) $\mu^d(X) := \mu(-X)$,

for any set $X \in \mathcal{B}(\mathbb{R})$, where $-X := \{-x : x \in X\}$. (If X is a Borel set, then so is $-X$.)

μ^d is a probability measure on $\mathcal{B}(\mathbb{R})$. This directly follows from the equivalence that $A \cap B = \emptyset$ if and only if $-A \cap -B = \emptyset$, for any sets

$A, B \subseteq \mathbb{R}$, and the fact that μ is a measure. μ^d agrees with μ on Borel sets A such that $A = -A$. Such sets A are called symmetric.

It is also clear that $(\mu^d)^d = \mu$.

Let $F : \mathbb{R} \longrightarrow [0, 1]$ be a CDF. The cumulative distribution function *dual* to F is the function $F^d : \mathbb{R} \longrightarrow [0, 1]$ defined as follows. Let μ_F be the probability measure on $\mathcal{B}(\mathbb{R})$ corresponding to F. F^d is, by definition, the cumulative distribution function that determines the dual measure μ^d. F^d is unambiguously defined. In fact,

(2) $F^d(x) := \mu^d_F((-\infty, x])$,

for every $x \in \mathbb{R}$. Thus

(3) $F^d(x) := \mu_F([-x, +\infty)$,

for every $x \in \mathbb{R}$. Since $\mu_F([-x, +\infty)) = 1 - \mu_F((-\infty, -x))$, we have that

(4) $F^d(x) := 1 - F((-x)-)$,

where $F((-x)-)$ is the left limit of F at $-x$, for every $x \in \mathbb{R}$.

Lemma 3.1. $(F^d)^d = F$.

Proof. Let μ be the measure corresponding to F and let μ^d be the dual measure. μ^d is the measure corresponding to F^d. Then $(F^d)^d(x) =$ (by (3)) $\mu^d([-x, +\infty)) = \mu(-[-x, +\infty)) = \mu((-\infty, x]) = F(x)$, for all $x \in \mathbb{R}$.

F is continuous at a point a if and only if the left limit of F at a is equal to $F(a)$. It follows from (4) that if F is continuous at $-x$, then $F^d(x) = 1 - F(-x)$. We thus obtain

Corollary 3.2. *If F is a continuous CDF, then F^d is a continuous CDF as well. Moreover*

$$F^d(x) = 1 - F(-x),$$

for every $x \in \mathbb{R}$.

The corollary follows from the above remarks. □

Suppose F is a continuous CDF and it has density, i.e., there exists a measurable non-negative function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that

(5) $F(x) = \int_{-\infty}^{x} f(t)dt,$

for all $x \in \mathbb{R}$. The function $g : \mathbb{R} \longrightarrow \mathbb{R}$ given by $g(x) := f(-x)$, $x \in \mathbb{R}$, is the density function of F^d. The graph of g is obtained by the reflection of the graph of f with respect to the y-axis.

CDF is the set of (unary) cumulative distribution functions. The order relation $\leq$ on CDF is defined pointwise as follows:

(6) $F \leq G \overset{df}{\Longleftrightarrow} F(x) \leq G(x)$ for every real number x.

Thus, in accordance to the meaning attached to cumulative distributions, $F \leq G$ states that for every real number x, the probability that the numerial value of the probabilistic attribute F belongs to the interval $(-\infty, x]$ is smaller or equal to the probability that a numerical value of G belongs to $(-\infty, x]$.

Other operations are also performable in the set CDF. Suppose F and G are cumulative distribution functions, not necessarily continuous. We define further operations on cumulative distribution functions:

$$\begin{aligned} (F \wedge G)(x) &:= \min(F(x), G(x)), \\ (F \vee G)(x) &:= \max(F(x), G(x)), \\ (F \oplus G)(x) &:= \min(1, F(x) + G(x)), \\ (F \otimes G)(x) &:= \max(0, F(x) + G(x) - 1), \end{aligned}$$

for all $x \in \mathbb{R}$.

THEOREM 3.3. *The sets CDF is closed with respect to the operations* $\wedge, \vee, \oplus$ *and* $\otimes$*. Moreover, if* F *and* G *are continuous, then so are* $F \wedge G$*,* $F \vee G$*,* $F \oplus G$ *and* $F \otimes G$*.*

PROOF. See [6].

The operations $\wedge$ and $\vee$ are called the *conjunction* and the *disjunction*, respectively. $\oplus$ is the bounded addition of CDFs. By the analogy to Łukasiewicz algebra Ac, the sum $F \oplus G$ will be called the weak disjunction of cumulative distributions F and G. In turn, $F \otimes G$ is the strong conjunction (also in the analogy to Łukasiewicz algebra).

The set CDF of cumulative distributions equipped with the operations $\wedge$ and $\vee$ forms a distributive lattice. The order relation $\leq$ defined in (6) is thus the order relation of the distributive lattice $\langle \mathrm{CDF}, \wedge, \vee \rangle$. Note that $F \vee G + F \wedge G = F + G$, where $+$ is the addition of real functions.

A slightly less obvious is the fact that the above distributive lattice together with the operation of dualization ^{d} satisfies De Morgan's laws:

Lemma 3.4. *For any cumulative distribution functions F and G,*

$$(F \wedge G)^d = F^d \vee G^d \text{ and } (F \vee G)^d = F^d \wedge G^d.$$

Proof. We shall show the first equality. The proof is restricted here to continuous CDFs. (The proof in the general case is a bit more involved.) Let x be a real number. In view of Corollary 3.2 we have:

$$(F \wedge G)^d(x) = 1 - (F \wedge G)(-x) = 1 - \min(F(-x), G(-x)) =$$
$$\max(1 - F(-x), 1 - G(-x)) = \max(F^d(x), G^d(x)) = (F^d \vee G^d)(x). \quad \square$$

Thus ^{d} is an involution operation satisfying De Morgan's laws.

The algebra

$$\langle CDF, \wedge, \vee, ^d \rangle$$

satisfies the axioms of De Morgan algebras with one exception: it is not bounded as a distributive lattice, that is, it does not possess the bottom and top elements. We shall use, however, the suggestive term *the De Morgan algebra of cumulative distribution functions* as a proper name.

The distributive lattice $\langle CDF, \wedge, \vee \rangle$ is not complete because it is lacking the top and bottom elements.

One may extend the universe CDF by augmenting it with two additional elements 0 and 1 so that one obtains a bounded distributive lattice (with zero 0 and unit 1) satisfying all conditions imposed on De Morgan algebras. Since we want 0 and 1 to be functions defined throughout set $\mathbb{R}$, we shall assume that 0 is the constant function with value 0 and, likewise 1 is the constant function taking only value 1. The constants 0 and 1 are not cumulative distribution functions on $\mathbb{R}$. But trivially, $0 \leq F \leq 1$, for every CDF F, i.e., $0(x) \leq F(x) \leq 1(x)$ for all $x \in \mathbb{R}$. We also put: $0^d := 1$ and $1^d := 0$.

We mark $CDF_b := CDF \cup \{0, 1\}$. But despite of the fact that the bounded lattice $\langle CDF_b, \wedge, \vee, ^d, 0, 1 \rangle$ is a De Morgan algebra, it is *not a* complete lattice To show this we define the following sequence $F_n, n = 1, 2, \ldots$ of continuous cumulative distribution functions:

$$F_n(x) := \begin{cases} 0 & x \leq 0 \\ x^{\frac{1}{n}} & 0 < x \leq 1 \\ 1 & 1 < x. \end{cases}$$

The sequence $\{F_n\}$ is pointwise covergent to the function F, where $F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x > 0$. F is not right-continuous at $x = 0$. Hence, F is not a CDF. On the other hand, we have that $\{F_n\}$ is monotone, that is, $F_1 \leq F_2 \leq \ldots$ in the lattice $\langle CDF, \wedge, \vee \rangle$. It is then easy to see that $\sup\{F_n : n \geq 1\}$ does not exist in $\langle CDF_b, \wedge, \vee, {}^d, 0, 1 \rangle$.

CCDF is the set of *continuous* cumulative distribution functions on $\mathbb{R}$. The system

$$\langle CCDF, \wedge, \vee, {}^d \rangle$$

is a subalgebra of $\langle CDF, \wedge, \vee, {}^d \rangle$ called *the De Morgan algebra of continuous cumulative distributions.*

Continuous cumulative distributions F such that $F = F^d$ determine symmetric probability measures with respect to the y-axis. This means that $\mu_F([-r, 0]) = \mu_F([0, r])$ for all real numbers r. (μ_F is the probability measure on the σ-field $\mathcal{B}(\mathbb{R})$ corresponding to F.) If F possesses a density function f , we see that f is an even function, that is $f(x) = f(-x)$ for all x, whenever $F = F^d$.

It is also sensible to talk about the operations $\oplus$ and $\otimes$ in the extended lattice $\langle CDF_b, \wedge, \vee, {}^d, 0, 1 \rangle$. The operations $\oplus$ and $\otimes$ are defined on $CDF_b = CDF \cup \{0, 1\}$ according to the same formulas as above. It is then easy to see that

$$1 \oplus a = a \oplus 1 = 1 \text{ and } 0 \oplus a = a \oplus 0 = a$$

and

$$1 \otimes a = a \otimes 1 = a \text{ and } 0 \otimes a = a \otimes 0 = 0$$

for all $a \in CDF \cup \{0, 1\}$.

Here is a bunch of simple facts concerning the above operations:

LEMMA 3.5. *For any cumulative distribution functions F and G:*

(1) $F \oplus G = G \oplus F$,

(2) $F \oplus (G \oplus H) = (F \oplus G) \oplus H$,

(3) $F \otimes G = (F^d \oplus G^d)^d$ *and* $F \oplus G = (F^d \otimes G^d)^d$,

(4) $F \otimes G \leq F \wedge G \leq F \vee G \leq F \oplus G$,

for any $F, G, H \in CDF$.

PROOF. Straightforward. For details – see [6]. □

The above lemma continues to hold for the bounded lattice $\langle CDF_b, \wedge, \vee, ^d, 0, 1\rangle$ augmented with $\otimes$ and $\oplus$.

The algebraic structure of the set CDF of cumulative distribution funtions is much richer. This set is also endowed with the operation of *convolution*

$$(7) \quad (F * G)(x) := \int_{-\infty}^{+\infty} F(t)G(x - t)dt,$$

for all $x \in \mathbb{R}$. The convolution of cumulative distributions is a CDF. The convolution operation preserves continuity of CDFs. As it is known, the operation $*$ is associative and commutative. It is also distributive: $(F + G) * H = (F * H) + (G * H)$. (But the sum $F + G$ is not a CDF.) If F and G are cumulative distributions corresponding to independent random variables X and Y defined on a probabilistic space, then the convolution $F * G$ is the cumulative distribution of the sum $X + Y$ of these random variables.

The list of operations which are performable on the set CDF is longer. We mention here convex combinations of finite sequences of cumulative distributions as well as translations along the x-axis.These operations have not been included to the list of primitive operations of $\langle CDF, \wedge, \vee, ^d, \oplus, \otimes\rangle$.

Let F be a cumulative distribution. For each real number r we define the function F_r by the condition:

$$F_r(x) := F(x + r), \text{ for all } x \in \mathbb{R}.$$

F_r is a cumulative distribution. If $r > 0$, the graph of F_r is obtained from the graph of F by means of the translation r units to the left. Obviously, $F_0 = F$. We have:

LEMMA 3.6. *For any* $a, b \in \mathbb{R}$, $a \leq b$ *if and only if* $F_a \leq F_b$.

PROOF. Straightforward. □

It follows from the lemma that the family of cumulative distributions $\{F_r : r \in \mathbb{R}\}$ forms a chain in the poset $\langle CDF, \leq\rangle$ and the order type of this chain is equal to λ, the order type of the set $\mathbb{R}$. One may then say that the poset $\langle CDF, \leq\rangle$ has a rather complicated order structure, also due to the fact that $\mathbb{R}$ is unbounded. For example, the poset $\langle CDF, \leq\rangle$ contains neither maximal nor minimal elements.

It is easy to see that $F \leq G$ implies that $F * H \leq G * H$ for any cumulative distribution functions F, G and H.

The next lemma is rather special. It shows that the convolution partially distributes over the weak disjunction.

LEMMA 3.7. *$(F \oplus G) * H \leq (F * H) \oplus (G * H)$, for any continuous cumulative distribution functions F, G and any $H \in CDF$.*

PROOF. Let a be an arbitrary but fixed real number such that $F(a) + G)(a) = 1$. Such an a exists because $F + G$ is continuous and has the Darboux property. Moreover $F(t) + G)(t) \geq 1$ for all $t \geq a$. We compute:

$$(*)\ ((F \oplus G) * H)(x) = \int_{-\infty}^{+\infty} (F \oplus G)(t)H(x-t)dt = \int_{-\infty}^{+\infty} \min(1, F(t) + G(t))H(x-t)dt = \int_{-\infty}^{a} (F(t) + G(t))H(x-t)dt + \int_{a}^{+\infty} H(x-t)dt.$$

We then obtain:

$$((F * H) \oplus (G * H))(x) = \min(1, (F * H)(x) + (G * H)(x)) =$$
$$\min(1, \int_{-\infty}^{+\infty} F(t)H(x-t)dt + \int_{-\infty}^{+\infty} G(t)H(x-t)dt =$$
$$\min(1, \int_{-\infty}^{+\infty} (F(t) + G(t))H(x-t)dt =$$
$$\min(1, \int_{-\infty}^{a} (F(t) + G(t))H(x-t)dt + \int_{a}^{+\infty} (F(t) + G(t))H(x-t)dt) \geq$$
$$\min(1, \int_{-\infty}^{a} (F(t) + G(t))H(x-t)dt + \int_{a}^{+\infty} H(x-t)dt) \overset{(\text{by } (*))}{=}$$
$$\min(1, ((F \oplus G) * H)(x)) = ((F \oplus G) * H)(x).$$

(The last equality follows from the fact that every cumulative distributive function is bounded by 1.) $\square$

It is unclear how to meaningfully combine the constant function 1 with the convolution operation $*$. In the technical sense, one may declare that 1 is the both-sided unit (that is, the neutral element) for the convolution operation $*$ and 0 is the zero for $*$, i.e.,

$$1 * a = a * 1 = a \text{ and } 0 * a = a * 0 = 0$$

hold for all $a \in CCDF \cup \{0, 1\}$. But then formula (7) does not apply to the above extension of $*$, because the definite integral of the function 1 throughout $\mathbb{R}$ does not exist. Therefore $1 * 1$ is not definable by means of formula (7).

In what follows we shall diregard the convolution operation $*$ on CDF.

We introduce some notation for the algebras we have defined:

$$CDF := \langle CDF, \wedge, \vee, {}^{d}, \oplus, \otimes \rangle.$$

CDF is called *the algebra* of cumulative distribution functions.

$$CDF_b := \langle CDF_b, \wedge, \vee, {}^{d}, \oplus, \otimes, 0, 1 \rangle.$$

CDF is called *the extended algebra* of cumulative distribution functions. In a fully analogous way one defines

$$CCDF := \langle CCDF, \wedge, \vee, {}^{d}, \oplus, \otimes \rangle,$$

the algebra of continuous cumulative distribution functions, and

$$CCDF_b := \langle CCDF_b, \wedge, \vee, {}^{d}, \oplus, \otimes, 0, 1 \rangle,$$

the extended algebra of continuous cumulative distribution functions.

The algebra CDF is also endowed with the operation $\rightarrow$, where

$$(F \rightarrow G) := F^d \oplus G,$$

for all $F, G \in CDF$. It is clear that $F \rightarrow G$ is a cumulative distribution, that is, $F \rightarrow G \in CDF$ whenever $F, G \in CDF$. Moreover $F \rightarrow G$ is a continuous cumulative distribution whenever $F, G \in CCDF$.

We obviously have that

$$(F \rightarrow G)(x) = \min(1, F^d(x) + G(x)) = \min(1, 1 - F(-x) + G(x)),$$

for any real number x. Moreover

(1) $F \to G = G^d \to F^d$

and

(2) $G \leq F \to G$

for all $F, G \in CDF$.

For the proof of (1) and (2) – see [6].

PROPOSITION 3.8. *Suppose $F, G \in CDF$ and $x \in \mathbb{R}$. Then the conditions $F(-x) = 1$ and $(F \to G)(x) = 1$ imply $G(x) = 1$.*

PROOF. We assume that $F(-x) = 1$ and $(F \to G)(x) = 1$. We have: $1 = (F \to G)(x) = min(1, 1 - F(-x) + G(x))$, which gives that $1 - F(-x) + G(x) \geq 1$. Hence $F(-x) \leq G(x)$. As $F(-x) = 1$, we infer that $G(x) = 1$. $\square$

The property of the operation $\to$ expressed in Proposition 3.8 may be regarded as the validity of a version of the detachment rule.

Although the implication symbol is used to denote the above operation, it would be rather unnatural to attach the name 'implication' to the above function. The reason is in the fact that the operation $\to$ fails to satisfy the law of identity, relevant in metalogical consequences, that is, it is *not* the case that $(F \to F)(x) = 1$ for all $x \in \mathbb{R}$. But we have:

$$(F \to F)(x) = \begin{cases} 1 & \text{if } 0 \leq x \\ 1 - F(-x) + F(x) & \text{otherwise,} \end{cases}$$

as one can easily check.

It is an open question whether the algebra $CDF = \langle CDF, \wedge, \vee, {}^d, \oplus, \otimes \rangle$ can be endowed with the residuation operation with respect to $\otimes$, that is, we ask whether there exists a binary operation $\Rightarrow$ on CDF, such that for any cumulative distribution functions F, G and X it is the case that

(res) $F \otimes X \leq G$ if and only if $X \leq F \Rightarrow G$.

We now consider the algebra

$$CDF_0 = \langle CDF, \to, \wedge, \vee, \oplus, \otimes, {}^d \rangle$$

of cumulative distribution functions augmented with the operation $\rightarrow$ defined as in Section 6. (The convolution operation is discarded.) CDF_0 is similar to the Łukasiewicz algebra A_c.

It is appropriate to look at the structure CDF0 from a more general algebraic perspective. We shall apply the terminology and notation adopted in [7, p. 398].

An abstract algebra $\langle A, \wedge, \vee, \{o_i\}_{i \in I}\rangle$ is called a distributoid ([7, p. 398]) if $\langle A, \wedge, \vee\rangle$ is a distributive lattice, and each $f \in \{o_i\}_{i \in I}$ is a (finitary) operation on A that "distributes" in each of its places over at least one of $\wedge$ and $\vee$, leaving the lattice operation unchanged or switching it with its dual.

Theorem 3.9. *The algebra* $CDF_0 = \langle CDF, \rightarrow, \wedge, \vee, \oplus, \otimes, {}^d\rangle$ *is a distributoid. More exactly, if* $F, G, H \in CDF$*, then:*

(i) $(F \vee G) \oplus H = (F \oplus H) \vee (G \oplus H)$,
(ii) $(F \wedge G) \otimes H = (F \otimes H) \wedge (G \otimes H)$,
(iii) $(F \vee G) \rightarrow H = (F \rightarrow H) \wedge (G \rightarrow H)$,
(iv) $H \rightarrow (F \wedge G) = (H \rightarrow F) \vee (H \rightarrow G)$.
(v) $(F \vee G)^d = F^d \wedge G^d$ *and* $(F \wedge G)^d = F^d \vee G^d$.

Proof. See [6]. $\Box$

4. The infinite-valued Łukasiewicz logic and cumulative distribution functions

We define:

$$\Delta := \{F \in CDF : F(0) = 1\}.$$

Thus, if $F \in CDF$, then $F(x) = 1$, for all $x \geq 0$. The set Δ is not interesting from the viewpoint of probabilty theory, because it contains cumulative distributions of probabilistic measures concentrated on the the set of negative real numbers. But Δ is interesting due to its relationship with Łukasiewicz logics.

Theorem 4.1. Δ *satisfies the following conditions:*

(a) $F \rightarrow F \in \Delta$,
(b) *if* $F \in \Delta$ *and* $F \leq G$*, then* $G \in \Delta$,

(c) *if* $F \in \Delta$ *and* $G \in \Delta$, *then* $F \otimes G \in \Delta$,

(d) *if* $F \in \Delta$ *and* $G \in \Delta$, *then* $F \wedge G \in \Delta$,

(e) *if* $F \in \Delta$ *and* $F \to G \in \Delta$, *then* $G \in \Delta$,

for all $F, G \in CDF$.

Conditions (b) and (c) state that Δ is a filter in the strong sense. According to (b) and (d), Δ is also a "standard" lattice-theoretic filter. In turn, (a) and (e) state that Δ validates the identity axiom and the detachment rule corresponding to $\to$.

Proof. See [6]. □

The following theorem is an immediate consequence of the definitions of the above two algebras:

Theorem 4.2. *The mapping* $h : CDF \longrightarrow I$ *given by* $h(F) := F(0)$ *is a homomorphism from the algebra* CDF_0 *onto the Łukasiewicz algebra* Ac.

Moreover the filter Δ *is the pre-image of* $\{1\}$ *with respect to* h, *that is,* $\Delta = \{F \in CDF : h(F) = 1\}$.

The pair $\langle CDF_0, \Delta\rangle$ is a logical matrix for the language L of Łukasiewicz logics. (The negation connective $\neg$ is interpreted in the distributoid CDF_0 as the operation ^{d}. The other connectives are interpreted in CDF_0 in the obvious way.

It follows from the above theorem that the mapping h is a strict surjective homomorphism (see [15]) from the matrix $\langle CDF_0, \Delta\rangle$ onto the Łukasiewicz matrix $\langle Ac, \{1\}\rangle$. This fact implies:

Corollary 4.3. *The consequence operation determined by the matrix* $\langle CDF_0, \Delta\rangle$ *in the language* L *coincides with the consequence operation determined by the Łukasiewicz matrix* $\langle A_c, \{1\}\rangle$. □

But there are more links between cumulative distribution functions and Łukasiewicz logics. To elucidate them, we introduce further consequence operations determined by the above algebras of cumulative distribution functions.

The symbol $Hom(A,B)$ marks the set of all homomorphisms $h : A \longrightarrow B$ from an algebra A to a similar algebra B.

The relation of probabilistic entailment $\models^{\leq}$ on L is defined as follows. For any $n \geq 1$ and any formulas $\alpha_1, \ldots, \alpha_n, \beta$ of L: we put:

$$\alpha_1, \ldots, \alpha_n \models^{\leq} \beta \overset{df}{\Longleftrightarrow} (\forall h \in Hom(L, CDF_0))h(\alpha_1) \wedge \ldots \wedge h(\alpha_n) \leq h(\beta).$$

If X is an infinite set of formulas, we assume that

$X \models^{\leq} \beta \Leftrightarrow_{df} \alpha_1, \ldots, \alpha_n \models^{\leq} \beta$ for some $n \geq 1$ and some formulas $\alpha_1, \ldots, \alpha_n \in X$.

Moreover, it is declared that $\emptyset \models^{\leq} \beta$ for *no* formula β. Thus the above "probabilistic" logic does not possess tautologies.

Let $CDF_{[-1,1]}$ be the set of all cumulative distribution functions F such that F is continuous throughout an open interval O that includes the closed interval $[-1, 1]$ and moreover F is constant on $[-1, 1]$. (Each such set O individually depends on F.)

It is not difficult to see that $CDF_{[-1,1]}$ forms a subalgebra of CDF_0 (denoted by $CDF_{[-1,1]}$). Moreover, for any real number $r \in I$, the mapping $\varphi_r : CDF_{[-1,1]} \longrightarrow A_c$ given by $\varphi_r(F) := F(r)$, $F \in CDF_{[-1,1]}$, is a homomorphism between the two algebras. E.g.,

$\varphi_r(F^d) = F^d(r) = 1 - F((-r)-)$
$= 1 - F(-r)$ (because F is continuous on some open set that contains $[-1, 1]$)
$= 1 - F(r)$ (because F is constant on $[-1, 1]$)
$= \neg F(r)$
$= \neg(\varphi_r(F))$ in A_c.

Note. We may also define the subalgebra of $CDF_{[-1,1]}$ consisting of all continuous cumulative distribution function being constant on $[-1, 1]$. The above reasoning carries over to this subalgebra. $\square$

$\models_{[-1,1]}$ marks the entailment relation determined on L by the algebra $CDF_{[-1,1]}$. Thus, for any $n \geq 1$ and any formulas $\alpha_1, \ldots, \alpha_n, \beta$ of L:

$$\alpha_1, \ldots, \alpha_n \models^{\leq}_{[-1,1]} \overset{df}{\Longleftrightarrow} (\forall h \in Hom(L, CDF_{[-1,1]}))h(\alpha_1) \wedge \ldots \wedge h(\alpha_n) \leq h(\beta).$$

The remaining cases are defined in a similar way as for $\models^{\leq}$.

Since $CDF_{[-1,1]}$ is a subalgebra of CDF_0, we obtain:

THEOREM 4.4. $\models^{\leq}_{[-1,1]}$ *is stronger than* $\models^{\leq}$, *i.e.*, $\models^{\leq}\ \leq\ \models^{\leq}_{[-1,1]}$. $\square$

(This is an instantiation of a more general phenomenon holding for semilattice based logics preserving degrees of truth.)

We mark by $\models^{\leq}_{c}$ the Łukasiewicz's infinite valued logic preserving degrees of truth determined by the algebra A_c. Thus, for any $n \geq 1$ and any formulas $\alpha_1, \ldots, \alpha_n, \beta$ of L_0

$$\alpha_1, \ldots, \alpha_n \models^{\leq}_{c} \stackrel{df}{\Longleftrightarrow} (\forall h \in Hom(L_0, A_c) h(\alpha_1) \wedge \ldots \wedge h(\alpha_n) \leq h(\beta).$$

The definition of $\models^{\leq}_{c}$ is extended onto infinite sets of formulas and on the empty set in a similar way as in the case of $\models^{\leq}$. (Note that $\models^{\leq}_{c}$ in [10], is defined on $\emptyset$ in a different way; there the set of theses of the resulting logic is non-empty.)

Although the three logical systems $\models^{\leq}, \models^{\leq}_{[-1,1]}$ and $\models^{\leq}_{c}$ are semantically defined by disparate algebras of different proveniences, they, quite suprisingly, can be compared.

THEOREM 4.5. $\models^{\leq}_{[-1,1]}$ *is stronger than the Łukasiewicz system* $\models^{\leq}_{c}$, *i.e.*, $\models^{\leq}_{c}\ \leq\ \models^{\leq}_{[-1,1]}$.

PROOF. Assume $\alpha_1, \ldots, \alpha_n \models^{\leq}_{c}$. Let $h : L \longrightarrow CDF_{[-1,1]}$ be a homomorphism. We claim that $h(\alpha_1) \wedge \ldots \wedge h(\alpha_n) \leq h(\beta)$.

In view of the above remarks, for every $r \in I$, the mapping φ_r : $CDF_{[-1,1]} \longrightarrow A_c$ given by $\varphi_r(F) := F(r)$, is a homomorphism. Hence the assumption $\alpha_1, \ldots, \alpha_n \models^{\leq}_{c} \beta$ implies that that every $r \in I$, $\varphi_r(h(\alpha_1)) \wedge \ldots \wedge \varphi_r(h(\alpha_n)) \leq \varphi_r(h(\beta))$ in the algebra A_c. This means that $\min(h(\alpha_1)(r), \ldots, h(\alpha_n)(r)) \leq h(\beta)(r)$ for all $r \in I$. Thus $h(\alpha_1) \wedge \ldots \wedge h(\alpha_n) \leq h(\beta)$. $\square$

We define the following congruence Φ on the algebra $CDF_{[-1,1]}$. For $F, G \in CDF_{[-1,1]}$:

$F \equiv G(mod\Phi)$ $\stackrel{df}{\Longleftrightarrow}$ there exists an open interval O on R such that $[-1,1] \subset O$, both F and G are continuous on O and agree on $[-1,1]$.

In the light of the above remarks, it is easy to see that the quotient algebra $CDF_{[-1,1]}/\Phi$ is isomorphic with the Łukasiewicz algebra A_c.

Theorem 4.6. *The Łukasiewicz algebra A_c is embeddable into the algebra CDF_0. Consequently, the Łukasiewicz logic $\models_c^{\leq}$ is stronger than $\models^{\leq}$, i.e., $\models^{\leq} \leq \models_c^{\leq}$.*

Proof. For each real number $r \in I$ we define the cumulative distribution function F_r as follows:

$$F_r(x) := \begin{cases} 0 & x < -1 \\ r & -1 \leq x < 1 \\ 1 & x \geq 1. \end{cases}$$

The set $\{F_r : r \in I\}$ is closed with respect to the operations $\to$, $\wedge$, $\vee$, $\oplus$, $\otimes$, ^{d}. In fact, for any $a, b \in I$,

$$\begin{aligned}
&F_a \wedge F_b = F_{\min(a,b)} = F_{a \wedge b} \\
&F_a \vee F_b = F_{\max(a,b)} = F_{a \vee b} \\
&F_a^d = F_{1-a} = F_{\neg a} \\
&F_a \oplus F_b = F_{\min(1,a+b)} = F_{a \oplus b} \\
&F_a \otimes F_b = F_{\max(0,a+b-1)} = F_{a \otimes b} \\
&F_a \to F_b = F_a^d \oplus F_b = F_{1-a} \oplus F_b = F_{\min(1,1-a+b)} = F_{a \to b}.
\end{aligned}$$

As the mapping assigning to each number $r \in I$ the cumulative distribution F_r is injective, it follows from the above equations that the algebra A_c is isomorphic with the subalgebra of CDF_0 whose underlying set is $\{F_r : r \in I\}$.

The second statement of the theorem follows from the first one. □

Let $CCDF_{[-1,1]}$ be the subalgebra of $CDF_{[-1,1]}$ consisting of all cumulative distribution functions that are continuous throughout the real line and constant in the interval $[-1, 1]$. Let $\models_{cont,[-1,1]}^{\leq}$ be the entailment relation on L determined by the algebra $CCDF_{[-1,1]}$. It is clear that $\models_{[-1,1]}^{\leq} \leq \models_{cont,[-1,1]}^{\leq}$.

Corollary 4.7. $\models^{\leq} \leq \models_c^{\leq} \leq \models_{[-1,1]}^{\leq} \leq \models_{cont,[-1,1]}^{\leq}$.

Proof. By Theorems 4.4-4.6 and the above remark. □

Thus, according to the above corollary, the Łukasiewicz logic $\models_c^{\leq}$ is bounded from the bottom and from the above by the logical systems $\models^{\leq}$ and $\models_{[-1,1]}^{\leq}$, respectively. The latter systems bear clear probabilistic connotations.

References

[1] P. Billingsley, **Probability and Measure**, John Wiley & Sons, Inc., New York, NY, 1995.

[2] R. L. O. Cignoli, I. M. L. D'Ottaviano, D. Mundici, **Algebraic Foundations of Many-valued Reasoning**, Kluwer, Dordrecht, 2000.

[3] P. Cintula, P. Hájek and Ch. Noguera (eds.), **Handbook of Mathematical Fuzzy Logic** (Studies in Logic, Volumes 37-38), College Publications, London, 2011.

[4] J. Czelakowski, *O probabilistycznej interpretacji predykatw* (Polish), [in:] [5].

[5] A. Wójtowicz and J. Golińska-Pilarek (eds.), *Identyczność znaku czy znak identyczności? (Identity of Sign or the Sign of Identity?)*, Warsaw University Press, Warsaw, 2012.

[6] J. Czelakowski, *Probabilistic Interpretations of Predicates*, [in:] Katalin Bimbó (ed.), J. Michael Dunn on Information Based Logics (Outstanding Contributions to Logic, Volume 8), Springer, Berlin, 2016, pp. 247–278.

[7] J. M. Dunn and G. M. Hardegree, **Algebraic Methods in Philosophical Logic** (Oxford Logic Guides, Oxford Science Publications, Volume 41), Oxford University Press, New York, 2001.

[8] J. M. Font, *Taking degrees of truth seriously*, **Studia Logica** 91 (2009), pp. 383–406.

[9] J. M. Font, J. Gil, A. Torrens V. and Verdú, *On the infinite-valued ukasiewicz logic that preserves the degrees of truth*, **Archiv for Mathematical Logic** 45/7 (2006), pp. 839–868.

[10] J. M. Font and R. Jansana, *Leibniz filters and the strong version of a protoalgebraic logic*, **Archiv for Mathematical Logic** 40 (2001), pp. 437–465.

[11] B. Ganter, G. Stumme, R. Wille, (eds.), **Formal Concept Analysis: Foundations and Applications** (Lecture Notes in Artificial Intelligence, No. 3626), Springer-Verlag, Berlin 2005.

[12] P. Hájek, **Metamathematics of Fuzzy Logics**, Kluwer, Dordrecht, 1998.

[13] G. Malinowski, **Many-Valued Logics**, Clarendon Press, Oxford, 1993.

[14] Y. Shramko and H. Wansing, **Truth and Falsehood. An Inquiry into Generalized Logical Values** (Trends in Logic, Volume 36), Springer, Berlin, 2011.

[15] R. Wójcicki, **Theory of Logical Calculi. Basic Theory of Consequence Operations**, Kluwer, Dordrecht, 1988.

[16] R. Wójcicki and G. Malinowski, (eds.), **Selected Papers on Łukasiewicz Sentential Calculi**, Ossolineum, Wrocław, 1977.

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