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# IDENTITY, EQUALITY, NAMEABILITY AND COMPLETENESS*

**Abstract**

This article is an extended promenade strolling along the winding roads of identity, equality, nameability and completeness, looking for places where they converge.

We have distinguished between identity and equality; the first is a binary relation between objects while the second is a symbolic relation between terms. Owing to the central role the notion of identity plays in logic, you can be interested either in how to define it using other logical concepts or in the opposite scheme. In the first case, one investigates what kind of logic is required. In the second case, one is interested in the definition of the other logical concepts (connectives and quantifiers) in terms of the identity relation, using also abstraction.

The present paper investigates whether identity can be introduced by definition arriving to the conclusion that only in full higher-order logic a reliable definition of identity is possible. However, the definition needs the standard semantics and we know that with this semantics completeness is lost.

We have also studied the relationship of equality with comprehension and extensionality and pointed out the relevant role played by these two axioms in Henkin's completeness method. We finish our paper with a section devoted to general semantics, where the role played by the nameable hierarchy of types is the key in Henkin's completeness method.

*Keywords:* first-order logic, type theory, identity, equality, indiscernibility, comprehension, completeness, translations, nameability

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# 1. Introduction

*Identity as a logical primitive* is the title of an expository paper published in 1975 by Henkin in *Philosophia* [9], in a volume entirely devoted to identity. Identity is analyzed from a variety of perspectives, including some where modalities, agents, descriptions, intentionality, etc. are taken into consideration. At the start Henkin declares: "By the relation of identity we mean that binary relation which holds between any object and itself, and which fails to hold between any two distinct objects."[1] Identity is a global relation on the mathematical universe; usually we are not interested in the whole identity relation but in a subset of it, namely, the diagonal of the Cartesian product of a particular domain, say $\mathbf{A}$. We can refer to it with $Id_{\mathbf{A}} = \{\langle \mathbf{x}, \mathbf{x} \rangle \mid \mathbf{x} \in \mathbf{A}\}$.

By equality we mean a symbolic relation between terms which is reflexive, transitive and symmetric.

Due to the central role the notion of identity plays in logic, we are interested either in how to define it using other logical concepts; in particular, what kind of logic is required.

# 2. First-order Logic

Our first-order languages contain a set (possibly empty) of individual constants, *CONST*, a set (possibly empty) of functional symbols, *FUNC*, and a set (possibly empty) of relational symbols, *REL*. The language also includes logical symbols. The structures used to interpret them have this form:

$$\mathcal{A} = \left\langle \mathbf{A}, \langle c^{\mathcal{A}} \rangle_{c \in CONST}, \langle f^{\mathcal{A}} \rangle_{f \in FUNC}, \langle R^{\mathcal{A}} \rangle_{R \in REL} \right\rangle$$

and $=^{\mathcal{A}}$ is not listed here.

It is rather obvious that among the possible relations on $\mathbf{A}$ there are three categories: some relations are listed in the structure—those that have *proper names* in the language—, others are *definable* using the language, and finally, when the universe is infinite, there are relations that are undefinable in the structure using the formal language. As an easy example of the last category, let us take the standard structure of natural numbers,

$$\mathcal{N} = \langle \mathbb{N}, 0, \sigma, +, \bullet \rangle$$

[1] In [9], p. 31.

and enumerate the countable set of definable subsets of $\mathbb{N}$, say $\mathbf{X}_0, \mathbf{X}_1, \ldots,$ $\mathbf{X}_n, \ldots$ If we take the set

$$\mathbf{Y} = \{n \mid n \notin \mathbf{X}_n\}$$

we shortly realize that it cannot be definable. The reason being that in case it were, $\mathbf{Y} = \mathbf{X}_m$ and we then arrive to the contradiction $m \in \mathbf{X}_m$ iff $m \notin \mathbf{X}_m$.

In fact, the same argument applies for any logic with a countable vocabulary because the set of formulas is countable while the set of elements in $\wp\mathbb{N}$ is uncountable.

## 2.1. Identity and Equality in FOL

Now we wonder about the identity relation. Is it definable? In first-order logic we usually take the equality symbol $=$ as primitive and stipulate that the expression $\tau = t$ is true under the interpretation $\mathfrak{I} = \langle \mathcal{A}, g \rangle$ when the two terms denote the same object, $\mathfrak{I}(\tau) = \mathfrak{I}(t)$,[2] that is, when $\langle \mathfrak{I}(\tau), \mathfrak{I}(t) \rangle \in Id_{\mathbf{A}}$. Therefore, the identity relation has a proper name in the language, even though $=^{\mathcal{A}}$ is not listed in the structure.

We do not include the binary relation of identity, as we do with the denotation of other predicate symbols, the reason being that the denotation of the equality symbol in any structure is always identity on the structure's domain and it could be understood as a logical constant.

This semantic stipulation must be complemented with axioms and rules so that in the calculus one can derive the usual laws of reflexivity, symmetry and transitivity as logical theorems. Very often we take reflexivity and equals substitution as primitive inference rules:[3]

$$\textbf{REF} \qquad \frac{}{\hookrightarrow \tau = \tau}$$

$$\textbf{ES} \qquad \frac{\Omega \hookrightarrow \varphi\frac{\tau}{x}}{\Omega,\ \tau = t \hookrightarrow \varphi\frac{t}{x}}$$

[2]As you see, we are being loose as we are using $=$ for both, the equality symbol between terms, as in $\tau = t$, and the identity relation between objetcs, as in $\mathfrak{I}(\tau) = \mathfrak{I}(t)$. One way out of this confusion is to use different notations, one for the equality (the symbol in the object language) and another for identity (metalanguage). We prefer to use just "=" for both.

[3]This is the option taken in [13] where we defined a sequent calculus.

and derive from them the rest of properties. Among these derivable properties are these two:

$$\begin{aligned}
\forall x_1 ... x_n y_1 ... y_n ( \bigwedge_{i \in \{1,...,n\}} x_i &= y_i \to (\varphi(x_1 ... x_n) \to \varphi(y_1 ... y_n))) \\
\forall x_1 ... x_n y_1 ... y_n ( \bigwedge_{i \in \{1,...,n\}} x_i &= y_i \to (\tau(x_1 ... x_n) = \tau(y_1 ... y_n)))
\end{aligned}$$

However, all these axioms alone are not enough to determine the precise nature of the interpretation of the equality symbol. These two schemes guarantee that equality will help to define a congruence relation with respect to all the definable relations. Reflexivity, symmetry and transitivity state that equality obeys the rules of an equivalence relation. We know that an equivalence relation does not need to be genuine identity and that is why we explicitly adopt the convention that the equality sign is to be interpreted as genuine identity.

### 2.1.1. The language of identity

We can even have a first order language whose only predicate symbol is equality, *the language of identity*. This language is used to talk about the *Identity structure*, a structure with little "structure" since it has the form $\mathcal{A} = \langle \mathbf{A} \rangle$, where only the universe of discourse needs to be specified. What can be expressed in such an exiguous language? For finite structures we can express the cardinality of the domain using

$$\begin{array}{llll}
\lambda_n := & \exists x_1 ... x_n \bigwedge_{i \neq j} x_i \neq & x_j & i, j \in \{1, ..., n\} \\
\mu_n := & \forall x_1 ... x_{n+1} \bigvee_{i \neq j} x_i = & x_j & i, j \in \{1, ..., n+1\}
\end{array}$$

the first one is saying that there are at least $n$ elements in the domain and the second, that there are at most $n$ elements. A shorter equivalent formula for $\lambda_n$ is

$$\lambda_n^* := \forall x_1 ... x_{n-1} \exists x_n \bigwedge_i x_n \neq x_i \qquad i \in \{1, ..., n-1\}$$

Can we express that the universe has a finite but undetermined number of elements? The answer is no. Even though infinity can be axiomatized by the infinite set of $\lambda_n$ sentences, we know that neither finiteness nor infinity are finitely axiomatizable in *FOL* due to the fact that the logic is compact. The easy argument to show that infinity is not axiomatizable is that in case it were we had $\Lambda \models \varphi$ (where $\Lambda = \{\lambda_n \mid n \geq 1\}$ and $\varphi$ stands for the formalization of infinity). By compactness there should be a finite subset of $\Lambda$, say $\Gamma$, such that $\Gamma \models \varphi$ and this is not possible as finite subsets of $\Lambda$ have always finite models.

Another interesting logical theorem we can prove is that the universe is non empty, $\exists x(x = x)$. This theorem is no longer so when we shift to free-logic where empty universes are allowed and terms might denote outside the domain of quantification.

### 2.1.2. Why identity is needed

Why do we want a language with equality? It is well known that "in any consistent first-order theory (without identity) possessing infinitely many non-logical constants, where such a definition is manifestly impossible, the addition of an identity symbol as a new primitive, with the usual axioms for identity, can never introduce inconsistency."[4]

For pure logical investigations it makes sense to work with *Equality free first-order* languages, but when we want to include function symbols, equality (interpreted as identity) is necessary. Using equality we can formulate that "there exist a unique $x$ such that $\varphi(x)$" by the formula

$$\exists x(\varphi(x) \wedge \forall y(\varphi(y) \rightarrow x = y))$$

or

$$\exists x(\forall y(\varphi(y) \leftrightarrow x = y))$$

that can be abbreviated as $\exists^1 x\varphi(x)$.

The unique existence is compulsory when willing to expand a language with new function symbols by *explicit definitions*; i.e., when we have two languages $L$ and $L^+$ such that $L \subseteq L^+$, and $f$ is an $n-$ary function symbol in the language $L^+$ introduced by

$$\forall \overline{x} y(f(\overline{x}) = y \leftrightarrow \psi(\overline{x} y))$$

In this case we need to prove the *admissibility condition* of the explicit definition, namely the formula

[4] In [9], p. 31

$$\forall \overline{x} \exists^1 y \psi(\overline{x} y)$$

A way of referring to a particular individual in *FOL* is to use individual constants, another way is by means of descriptions. These are closely related to the concept of identity since descriptions can be understood as a way to isolate and identify objects by means of certain expressions. How can we deal with descriptions? With the help of equality the descriptor operator $\iota x \varphi$ could be introduced using a *contextual definition*

$$\psi(\iota x \varphi) \leftrightarrow_{Df} \exists x(\forall y(\varphi(y) \leftrightarrow x = y) \wedge \psi(x)) \tag{2.1}$$

but this equality symbol needs to be actual identity. Moreover, this *iota* operator cannot be given an explicit definition in *FOL*. This solution is in accordance with Russell's theory of descriptions and can be seen as a translation into classical logic.

The solution which is more in accordance with Frege's view is to include $\iota x \varphi$ as a new term, for any formula $\varphi$ and variable $x$. It is clear that we want that under the interpretation $\mathfrak{I} = \langle \mathcal{A}, g \rangle$ this term denotes the unique element satisfying $\varphi$; that is, $\mathfrak{I}(\iota x \varphi)$ is the unique $\mathbf{x} \in \mathbf{A}$ such that $\mathfrak{I}_x^{\mathbf{x}}(\varphi) = T$. The problem is that $\mathfrak{I}(\iota x \varphi)$ is not defined if there isn't exactly one individual satisfying $\varphi$. If there is not such individual, or if there are more than one, the above stipulation does not say how $\iota x \varphi$ should be interpreted. The 'solution' is to include a special *nil entity* in the domain as denotation of the iota term when the uniqueness condition fails, that is

> $\mathfrak{I}(\iota x \varphi)$ is the unique $\mathbf{x} \in \mathbf{A}$ s.t. $\mathfrak{I}_x^{\mathbf{x}}(\varphi) = T$ if there is such thing; otherwise it is the *nil individual*.

In case we also stipulate that $\mathfrak{I}(\psi(\iota x \varphi)) = F$ for all iota terms interpreted as the *nil entity*, the formula 2.1 is a theorem.

We wonder if the last stipulation is in strict accordance with Frege's view as he argues that the reference of a sentence is a truth-value, but admits the possibility that there be sentences that lack a referent:

> sentences which contain proper names without referents will be of this kind. The sentence "Odysseus was set ashore at Ithaca while sound asleep" obviously has a sense. But since it is doubtful whether the name "Odysseus," occurring therein, has a referent, it is also doubtful whether the whole sentence has one. ([4], p. 32).

### 2.2. Equality-free First-order Logic

Why do we take identity as a logical primitive concept in first order logic? Given an Equality free first-order language $\mathcal{L}$ and a $\mathcal{L}$-structure $\mathcal{A}$, is there a formula $\varphi(x, y)$ defining identity in the structure's universe?, i.e., such that

$$Id_{\mathbf{A}} = \{\langle \mathbf{x}, \mathbf{y}\rangle \in \mathbf{A} \mid \models_{\mathcal{A}} \varphi \, [xy/\mathbf{xy}]\}$$

This is equivalent to the demand that the $\mathcal{L}$-formula $\varphi(x, y)$ is such that

$$\models_{\langle \mathcal{A},=\rangle} \forall xy(x = y \leftrightarrow \varphi(x, y))$$

where the new structure is the extension of $\mathcal{A}$ with the identity as interpretation of the new equality symbol.

Does the formula work for any $\mathcal{L}-$structure $\mathcal{B}$?

The answer is negative, even in the best scenario where we only have a finite set of non-logical predicate constants. Of course, for particular cases there could be such a defining formula, as we will see in the following example using a language with only two predicate symbols. In the language $\mathcal{L}$ that only contains the unary predicate constant $R$ and the binary predicate constant $T$ formula

$$(Rx \leftrightarrow Ry) \wedge \forall z(Txz \leftrightarrow Tyz) \wedge \forall z(Tzx \leftrightarrow Tzy)$$

is a good candidate as the formula expresses that $x$ and $y$ cannot be distinguished in our formal language. But we can give two $\mathcal{L}$-structures with the same domain $\mathbf{A}$, one where the formula defines $Id_{\mathbf{A}}$ and another where it defines something else. Take the structure $\mathcal{A} = \left\langle \{1, 2, 3\}, T^{\mathcal{A}}, R^{\mathcal{A}} \right\rangle$, where $R^{\mathcal{A}} = \{1\}$ and $T^{\mathcal{A}} = \{\langle 2, 3\rangle\}$:

$$Id_{\mathbf{A}} = \{\langle \mathbf{x}, \mathbf{y}\rangle \in A \mid \models_{\mathcal{A}} (Rx \leftrightarrow Ry) \wedge \forall z(Txz \leftrightarrow Tyz) \wedge \forall z(Tzx \leftrightarrow Tzy) \; [xy/\mathbf{xy}]\}$$

The other example is the structure $\mathcal{A}^* = \left\langle \{1, 2, 3\}, T^{\mathcal{A}^*}, R^{\mathcal{A}^*} \right\rangle$, with $R^{\mathcal{A}^*} = \{1, 2, 3\}$ and $T^{\mathcal{A}^*} = \varnothing$ where the formula defines $\mathbf{A} \times \mathbf{A}$, not $Id_{\mathbf{A}}$.

In case the formula were used to give an explicit definition of equality

$$\forall xy(x = y \leftrightarrow_{Df} (Rx \leftrightarrow Ry) \wedge \forall z(Txz \leftrightarrow Tyz) \wedge \forall z(Tzx \leftrightarrow Tzy))$$

we see that the binary predicate introduced by the formula obeys the usual rules for equality and expresses the *indiscernibility principle* in equality free first-order logic. The principle is saying that two objects are identical when there is no property able to distinguish them. The formula is the nearest we can come up with in first-order logic to formalize *Leibniz's principle of indiscernibles* in the language $\mathcal{L}$.

In 2012, at the 8th Scandinavian Logic Symposium we presented some of these ideas in a talk entitled: *All identicals are equal, but some equals are more equal than others.*

### 2.3. Quinean structures and indiscernibility

The interplay between identity and equality in first-order logic is studied by Ketland [11] in more detail. In the first place, for a particular structure $\mathcal{A}$, Ketland investigates whether $Id_{\mathbf{A}}$ is first-order definable in a language $\mathcal{L}$ without equality. Structures equipped with such a defining formula are called *Quinean structures.* On the other hand, as far as equality is concerned, a generalization of the first-order definition of equality introduced in our previous example is what he called *first-order indiscernibility formula.* For every $n-$ary predicate constant $P$ in $\mathcal{L}$, he introduces the formula $x \approx_P y$ by the following $n-$ary conjunction

$$\forall \overline{z}_1(Px\overline{z}_1 \leftrightarrow Py\overline{z}_1) \wedge \forall \overline{z}_2(Pz_1xz_3...z_n \leftrightarrow Pz_1yz_3...z_n) \wedge ... \wedge \forall \overline{z}_n(P\overline{z}_nx \leftrightarrow P\overline{z}_ny)$$

where $\overline{z}_i$ is the sequence $z_1 \ldots z_{i-1}z_{i+1} \ldots z_n$ for each $i \in \{1, \ldots, n\}$.

In case language $\mathcal{L}$ contains just a finite set of predicates, say $P_1, ..., P_m$, the conjunction of all these $x \approx_{P_i} y$ is the so named *indiscernibility formula,* $x \approx y$; namely, the formula

$$x \approx y = \bigwedge \{x \approx_{P_i} y \mid P_i \text{ is a primitive symbol of } \mathcal{L}\}$$

Finally, he introduces what he called *Leibniz formulas* as a way to encode the basic properties of equality we added to the calculus of *FOL* (reflexivity of equality **REF** and equals substitution **ES**). These formulas have no particular shape, but are characterized by their behavior in particular structures. A formula $\varphi(xy)$ is a *Leibniz formula* for a structure $\mathcal{A}$ in case

(1) $\models_{\mathcal{A}} \forall x\varphi(xx)$ and
(2) $\models_{\mathcal{A}} \forall xy(\varphi(xy) \rightarrow \forall \overline{z}(\psi(x\overline{z}) \rightarrow \psi(y\overline{z}))$

A Leibniz formula defines an equivalent relation and any formula able to define identity in a structure is a Leibniz formula.[5] For languages with

[5]These lemmas are proved in page 175 of [11].

only a finite number of predicates, Leibniz formulas and indiscernibility formulas are obviously related since both are semantically equivalent. In page 176 of [11], Ketland proves that for any Leibniz formula $\varphi(xy)$ and any structure $\mathcal{A}$

$$\models_{\mathcal{A}} \forall x(\varphi(xy) \leftrightarrow x \approx y)$$

Let $\approx_{\mathcal{A}}$ be the binary relation defined by $x \approx y$ in the structure $\mathcal{A}$, the indiscernibility relation. This relation is an equivalence relation and one can build the quotient structure $\mathcal{A}/\approx_{\mathcal{A}}$ and prove that this one is a Quinean structure which is elementary equivalent to $\mathcal{A}$.

## 3. Higher-order logic with standard semantics

Higher-order logic (*HOL*) in general, and second-order logic in particular, have been victims of strong criticism, often putting their status as "logic" into question. We are not going to quote Quine, as many logicians do. We just try to concentrate on several issues related to identity, equality, nameability and completeness. Of course, the weakness of its computability power in contrast with the strength of its expressiveness are going to appear all along this section.

### 3.1. Equality, Comprehension and Extensionality in SOL

What characterizes second-order logic (*SOL*) is that quantification is no longer restricted to individuals, since we can quantify also over sets and relations. Let us recall that the *standard semantics* is being determined by structures

$$\mathcal{A} = \langle \mathbf{A}, \langle \mathbf{A}_n \rangle_{n \geq 1}, \langle C^{\mathcal{A}} \rangle_{C \in OPER.CONS} \rangle$$

where $\mathbf{A} \neq \varnothing$, $\mathbf{A}_n = \wp(\mathbf{A}^n)$ and the $C^{\mathcal{A}}$ are either elements of $\mathbf{A}$ or functions, sets and relations on $\mathbf{A}$.

In second-order logic with standard semantics equality is no longer introduced as a primitive logical concept as we can define it using Leibniz's principle by

$$\forall xy(x = y \leftrightarrow_{Df} \forall X(Xx \rightarrow Xy))$$

We do not need reflexivity of equality and equals substitution as primitive inference rules, since they are already derived rules. The formula $\forall X(Xx \rightarrow Xy)$ can be used as well to define identity for individuals as the relation defined by it is "genuine" identity in any standard second-order structure. The reason being that in the domain of unary relations (used to interpret unary predicate variables in standard models) all the possible relations are included and so are all the singletons. It could also be defined by the formula

$$\forall Y^2(\forall z\, Y^2zz \rightarrow Y^2xy)$$

because the least reflexive relation is the identity, which we have in the domain of binary relations in any standard structure.

The two formulas are syntactically equivalent as well, since one can prove this theorem in the second-order calculus

$$\vdash_{SOL} \forall xy(\forall X(Xx \leftrightarrow Xy) \leftrightarrow \forall Y(\forall z Yzz \rightarrow Yxy))$$

We can say that in standard second-order logic indiscernibility and identity collapses and equality is a defined concept whose interpretation is identity.

As in *FOL*, among the possible relations on $\mathbf{A}$ (i.e., the set-theoretical relations) there are at least three categories: some relations are listed in the structure, others are *definable* using the language, and finally, when the universe is infinite, there are relations that are *undefinable* in the structure using the formal language. In [12], pp. 40–47, we also distinguish between first and second-order relations and the notion of *parametrically definable relation* is introduced. In comparison with *FOL*, the status of definable sets and relations in *SOL*-structures improves. We have a formula saying that they all exist, the comprehension schema

$$\exists X^n \forall x_1 ... x_n(X^n x_1 ... x_n \leftrightarrow \varphi)$$

and this formula is a relevant theorem of *SOL*. For instance, we can postulate the existence of the identity relation between individuals

$$\exists X^2 \forall xy(X^2xy \leftrightarrow \forall X(Xx \leftrightarrow Xy))$$

As you will see in section 4, one of the nicest characteristic of higher-order logic is the role played by the definable sets and relations whose existence is a must.

What about identity between sets and relations? Is there any formula $\varphi$ to define this binary relation $Id_{\wp \mathbf{A}^n}$? This identity relation is not a first-order relation but a proper second-order one, and in many second-order languages second-order equality is neither introduced as a primitive logical symbol nor defined using the rest of the symbols in *SOL*.

We wonder if these identities for relations are parametrically definable, like membership and the power set of a given set are

$$\in_1 = \{\langle \mathbf{x}, \mathbf{Y}\rangle \in \mathbf{A} \times \wp\mathbf{A} \mid \models_{\mathcal{A}} Yx\ [xY/\mathbf{xY}]\}$$
$$\wp\mathbf{Y} = \{\mathbf{X} \in \wp\mathbf{A} \mid \models_{\mathcal{A}} \forall z(Xz \ \rightarrow Yz)[XY/\mathbf{XY}]\}$$

Can we use Leibniz's principle to introduce the identity for relations? The answer is negative, since to follow Leibniz's pattern we would need to quantify over third-order variables.

$$\forall X^n Y^n (X^n = Y^n \leftrightarrow_{Df} \forall Z^{(n)}(Z^{(n)}X^n \leftrightarrow Z^{(n)}Y^n))$$

Arguably, the extensionality principle could be used to introduce equality by definition

$$\forall X^n Y^n (X^n = Y^n \leftrightarrow_{Df} \forall x_1 ... x_n (X^n x_1 ... x_n \leftrightarrow Y^n x_1 ... x_n))$$

### 3.2. Type theory

The natural place where these concepts are easily introduced is *Type Theory.* To start with, we define the *type symbols*, which are going to be used as subscripts: (1) 0 and $\iota$ are the basic ones and (2) If $\alpha, \beta$ are type symbols so is $(\alpha\beta)$.

The types are structured in a hierarchy that has the following as basic types: (1) $\mathcal{D}_\iota$ is a non-empty set, the domain of of individuals, (2) $\mathcal{D}_0$ is the domain of truth values (these values are reduced to $T$ and $F$). The other domains are constructed from the basic types as follows: if $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$ have already been constructed, we define $\mathcal{D}_{(\alpha\beta)}$ as the domain formed by all the functions from $\mathcal{D}_\beta$ to $\mathcal{D}_\alpha$.[6]

To talk about this hierarchy, Church's formal language of [3] is introduced in Henkin's 1950 paper [6]. In this language one has variables of all possible types and only four logical constants: $N_{(00)}$ for negation, $A_{((00)0)}$ for disjunction, $\Pi_{(0(0\alpha))}$ is the universal predicate used for quantification and $\iota_{(\alpha(0\alpha))}$ is the descriptor operator. In this language we also have the lambda abstractor operator $\lambda$ as improper symbol.

[6]Nowadays we use the reverse notation, $\mathcal{D}_{(\alpha\beta)}$ contains all the functions from $\mathcal{D}_\alpha$ to $\mathcal{D}_\beta$.

The binary equality predicate is introduced as abbreviation:

$$Q_{((0\alpha)\alpha)} \quad \text{for} \quad (\lambda x_\alpha(\lambda y_\alpha(f_{0\alpha})\,((f_{0\alpha}x_\alpha) \supset (f_{0\alpha}y_\alpha)))) \tag{3.1}$$

where $\supset$ is the conditional connective, defined as usual with negation and disjunction, and $(f_{0\alpha})$ is the universal quantification. The universal quantification is introduced here as

$$(a_\alpha)\,B_0 \quad \text{for} \quad \Pi_{(0(0\alpha))}(\lambda a_\alpha B_0)$$

The following abbreviation is also introduced to facilitate the reading:

$$(A_\alpha = B_\alpha) \quad \text{for} \quad ((Q_{((0\alpha)\alpha)}A_\alpha)(B_\alpha))$$

The definition 3.1 of equality, $Q_{((0\alpha)\alpha)}$, follows Leibniz's pattern and it covers all types. Moreover, as in the second-order case, with standard semantics the formula defines the actual identity relation, in the present case for each type $\alpha$.

Henkin's calculus is also close to Church's. Some of the axioms where equality is involved are *Extensionality*

$$x_0 \equiv y_0 \supset x_0 = y_0$$
$$(x_\beta)(f_{\alpha\beta}x_\beta \supset g_{\alpha\beta}x_\beta) \supset f_{\alpha\beta} = g_{\alpha\beta}$$

and *Choice*

$$f_{0\alpha}x_\alpha \supset f_{0\alpha}(\iota_{(\alpha(0\alpha))}f_{0\alpha})$$

Among the rules of inference, several replacement/substitution rules are the most relevant. *Comprehension* is not an axiom, but a theorem that can be derived using substitution.

Several years later, in [7], Henkin introduces a calculus where some of the substitution rules are simplified and the comprehension axiom is added. As explained in [14] this paper is partially responsible of the translation methodology proposed in [12] and is briefly explained in section 4.3 of the present paper.

### 3.3. Incompleteness and expressiveness

In the theory of types equality for all types can be introduced by the above formula (3.1), and the interpretation in standard structures is actual identity. In *SOL* the definability of equality as identity is restricted to individual terms.

Second-order logic is as well a powerful language where properties like being countable (and uncountable) are finitely axiomatizable; the continuum hypothesis and the generalized continuum hypothesis are other examples.[7] Among the negative results associated to the expressiveness are the failure of: (1) *Compactness*, (2) *Löwenheim-Skolem* theorem and (3) *Completeness* (both weak and strong). It is well known that expressiveness and computability power are not independent variables; in a complete logic, computability and expressiveness are in equilibrium.

Gödel solved the completeness issue positively for first-order logic and negatively for any logical system able to contain arithmetic. *SOL*, with the usual standard semantics, is able to express arithmetic and hence could only be incomplete. A good example of what we are saying is the sentence

$$\varphi := 1P \wedge 2P \wedge 3P \rightarrow \gamma$$

where $1P \wedge 2P \wedge 3P$ is the conjunction of Peano axioms and $\gamma$ is Gödel's self-referent sentence stating its own unprovability. All these sentences are true in the structure of natural numbers. A well known result is the categoricity of these Peano axioms. Therefore, the sentence $\varphi$ is valid but unprovable in any *SOL* calculus. Moreover, the incompleteness phenomenon could not be solved by adding $\varphi$ as an axiom, since we can always find another valid and unprovable sentence.

In [12], pp. 96–114, you can find another proof of incompleteness of *SOL*. The proof uses the fact that the generalized continuum hypothesis is expressible in *SOL* by a formula $\varphi_{GCH}$. The results of Gödel and Cohen that give the validity of the formula in one set-theoretical environment and also the contrary result in another, are taken into consideration. Being different environments, the validity of $\varphi_{GCH}$ in one environment and the validity of $\varphi_{GCH}$ in the other is not problematic. But, assuming completeness, we can show that $\varphi_{GCH}$ must be valid and invalid in the same environment. The substance of the proof rest on the unbalanced relation between $\vdash \varphi_{GCH}$ and $\models \varphi_{GCH}$.

No need to say that the incompleteness phenomenon also embraces Type Theory ($TT$), since the expressive power of $TT$ surpasses that of *SOL*.

[7] These and other examples can be found in the section *More about the expressive power of standard SOL* in [12], pp. 47–60.

## 4. Higher-order logic with general semantics

*HOL* with standard semantics has an extraordinary expressive power but poor logical properties, and when you want to retain logical properties, such as completeness of the calculus, you need to introduce non-standard semantics.

Henkin [6] showed that if the formulas were interpreted in a less rigid way, accepting structures that did not necessarily have to contain all the relations, but at least did contain the definable ones, one can prove strong completeness; all the consequences of a set of hypotheses are provable in the calculus using these hypothesis. The valid formulas with this new semantics, called *general semantics*, form a recursively enumerable set. This set coincides with the set of sentences generated by the calculus rules as logical theorems. Henkin justifies this new semantics as a way of sorting the provable from the unprovable: "we can pick out from among them those which are provable by introducing certain non-classical notions of validity."[8]

The paper *Changing a Semantics: Opportunism or Courage?*, [1], by Andreka, van Benthem, Bezhanishvili and Németi gives a systematic view of general models in mathematical and philosophical terms. Is this move merely an *ad hoc* device? We do not believe so, as often we want to quantify over the relevant properties and they do not coincide with all the set-theoretical possible ones. As we have already pointed out, it is a fact that in case we start with a countable infinite set as the universe of individuals $\mathcal{D}_\iota$, in the universe of subsets of the universe of individuals there will be objects with a name and without one, because with a countable infinite universe of individuals the set of subsets is uncountable, but the sets with a name are only countable. As we will see in section 4.3, Henkin's interest in the nameable types constituted the origin of his completeness proof.

To quantify over the properties we can control with our language seems reasonable, it could be understood as "a critical look at unwarranted 'set-theoretic imperialism' and unquestioning acceptance of set-theoretic structures without a cost-benefit analysis."[9]

This change of semantics shall be associated with a *nominalist* position that came after focussing on the elements of the full hierarchy of types that

[8]In [9], p. 40.

[9]In [1], p. 333.

are definable with lambda abstraction. Henkin himself affirms in *Some Notes on Nominalism*, [8]: "In fact, such an interpretation is implicit in a recent paper[10] discussing the problem of the completeness of the higher-order functional calculi." (p. 22). It is not so clear for us whether this philosophy was only an inspired afterthought or the driving force of his completeness theorem.

### 4.1. Frames and general structures for SOL

The first step to the definition of general semantics is to consider structures, which might be called frames, where the $n$-ary predicate variables run over subsets of $\wp(\mathbf{A}^n)$, we do not force them to take values in the whole power set. Thus, frames are similar to standard structures, with the only exception of the relational universes

$$\mathcal{A} = \langle \mathbf{A}, \langle \mathbf{A}_n \rangle_{n \geq 1}, \langle C^{\mathcal{A}} \rangle_{C \in OPER.CONS} \rangle$$

where $\mathbf{A} \neq \varnothing$ and $\mathbf{A}_n \subseteq \wp(\mathbf{A}^n)$, for each $n$. Clearly, as widening the class of models, the set of valid formulas will decrease and some of the former standard validities are no longer so in frames; in particular, comprehension axiom

$$\exists X^n \forall x_1 ... x_n (X^n x_1 ... x_n \leftrightarrow \varphi)$$

—which is one of the characteristic axioms (or theorems) of *SOL*— might fail. That is why in general models we want the universes to be closed under definability.

#### 4.1.1. Synthesis

Let us synthesize the whole idea involved in this change of semantics. We have $\mathcal{S.S}$, the class of standard structures and the set of validities in this class $\models_{\mathcal{S.S}}$ and we know that there is no complete calculus for $\models_{\mathcal{S.S}}$. On the other hand, we have certain rules (and/or axioms) which are sound, which correspond to a calculus which is an extension of the first-order one obtained by adding new rules to deal with the new quantifiers, call

[10]The paper he refers to is [6], *Completeness in the theory of types.*

it $SOL^-$. We extend this $SOL^-$ calculus to a suitable one for $SOL$, just adding comprehension

$$\textbf{COMP} \quad \frac{}{\hookrightarrow \exists X^n \forall x_1 ... x_n (X^n x_1 ... x_n \leftrightarrow \varphi)}$$

(for any formula where $X^n$ is not free). So we get the set of logical theorems in both calculi: $\vdash_{SOL}$ and $\vdash_{SOL^-}$. Since $\vdash_{SOL}$ is a proper subset of $\models_{\mathcal{S}.\mathcal{S}}$ we widen the class of structures to reduce the set of validities so defining structures in a wider sense (which we call frames) and general structures. They produce two new classes of validities, $\models_{\mathcal{F}}$ and $\models_{\mathcal{G}.\mathcal{S}}$. Since

$$\mathcal{S}.\mathcal{S} \subseteq \mathcal{G}.\mathcal{S} \subseteq \mathcal{F}$$

we have

$$\models_{\mathcal{F}} \subseteq \models_{\mathcal{G}.\mathcal{S}} \subseteq \models_{\mathcal{S}.\mathcal{S}}$$

and it happens (not by chance) that they are exactly the sets $\vdash_{SOL^-}$ (of logical theorems of the extension of first-order logic with new quantifiers) and $\vdash_{SOL}$(of logical theorems of second-order logic). That is, $\models_{\mathcal{F}} = \vdash_{SOL^-}$ and $\models_{\mathcal{G}.\mathcal{S}} = \vdash_{SOL}$. Therefore, both logics are complete with respect to the appropriate semantics.

#### 4.1.2. The little mermaid

At the present time it is obvious that you have to choose between expressive power or complete calculus; in the latter case, the old shadow of Skolem's paradox is back and we get non-standard models of arithmetic, as Henkin explains at the end of his 1950 paper: "The Peano axioms are generally thought to characterize the number-sequence fully in the sense that they form a categorical axiom set any two models for which are isomorphic. As Skolem points out, however, this condition obtains only if 'set' is interpreted with its standard meaning."[11]

In *The Little Mermaid*, [15], we ended the paper, devoted to second-order logic, saying:

> It is clear that you cannot have both: expressive power plus good logical properties. You cannot be a mermaid and have an immortal soul. [...] And the little mermaid got two beautiful

[11] In [6], p. 89.

legs (with a lot of pain, as you might know). But even in stories everything has a price; you know, the poor little mermaid lost her voice. (p. 107).

### 4.1.3. Indiscernibility is no longer identity

How this semantic change affects definability of identity? Clearly, while widening the class of models, the set of valid formulas decreases and some of the former standard validities are no longer so in non-standard models. This is good because the unprovable sentence $\varphi := 1P \wedge 2P \wedge 3P \rightarrow \gamma$ is in the set of validities in standard structures, $\models_{\mathcal{S}.\mathcal{S}}$, but not in the set of validities in general structures, $\models_{\mathcal{G}.\mathcal{S}}$ or frames, $\models_{\mathcal{F}}$. Unfortunately, this is also the case of the proposed definitions for equality for individuals. The first formula

$$\forall X(Xx \leftrightarrow Xy)$$

is equivalent to

$$\forall Y(\forall zYzz \rightarrow Yxy)$$

in a standard model, and even in a general model but not in frames.

What would happen if we jacked around with semantics in *Equality-free SOL* switching to frames or general models, but retaining the definition of equality (for individuals) as indiscernibility?

Well, we are back to the situation we encountered in *Equality-free FOL* which lacks equality defined as a primitive logical concept denoting identity. We can fashion a frame $\mathcal{A}$ and an interpretation

$$\mathfrak{I} = \langle \mathcal{A}, g \rangle \text{ such that } \mathfrak{I}(x = y) = T \text{ but } \mathfrak{I}(x) \neq \mathfrak{I}(y)$$

For example, the structure $\mathcal{A}$ where $\mathbf{A} = \{1, 2, 3\}$ and $\mathbf{A}_1 = \{\varnothing, A\}$.

Within non-standard structures, the Leibniz's principle defines an equivalence relation of indiscernibility but it could be different from identity. Therefore, if you want the prototypical identity as interpretation of the equality symbol, you should either have it as primitive, or define the concept of *non-standard normal structure* as well. All we have said should serve to warn you that the possibility of defining identity is lost as long as there is no guarantee of having all possible sets as denotation, specifically, all the singletons. As we will see later, comprehension and extensionality should work together in a context where the interpretation of equality is identity.

As we already said, the extensionality principle could be used to introduce equality by definition

$$\forall X^n Y^n (X^n = Y^n \leftrightarrow_{Df} \forall x_1 ... x_n (X^n x_1 ... x_n \leftrightarrow Y^n x_1 ... x_n))$$

If we do, the formula $\Pi^n = \Psi^n$ should be understood as

$$\forall x_1 ... x_n (\Pi^n x_1 ... x_n \leftrightarrow \Psi^n x_1 ... x_n)$$

but this statement alone does not guarantee that the denotation of $\Pi^n$ and $\Psi^n$ are extensionally the same. Being just a definition of equality, the formula stops working as an extensionality principle, it does not work as a mechanism to avoid non-extensional models.

Formula

$$\forall x_1 ... x_n (X^n x_1 ... x_n \leftrightarrow Y^n x_1 ... x_n)$$

defines an equivalent relation among $n$-ary relations of a given model and the binary relation defined by

$$\{\langle \mathbf{X}, \mathbf{Y} \rangle \in \wp \mathbf{A}^n \times \wp \mathbf{A}^n \mid \models_{\mathcal{A}} \forall x_1 ... x_n (X^n x_1 ... x_n \leftrightarrow Y^n x_1 ... x_n) \ [X^n Y^n / \mathbf{XY}]\}$$

has the same problem, it could be different from identity.

In section 4.2 we take equality as a primitive concept and the extensionality principle

$$\forall X^n Y^n (X^n = Y^n \leftrightarrow \forall x_1 ... x_n (X^n x_1 ... x_n \leftrightarrow Y^n x_1 ... x_n))$$

acts as an axiom avoiding non-extensional models.

### 4.2. SOL with equality as primitive

In [12], María Manzano proposes to take identity as a logical primitive for both, individuals and sets and relations, and to use it to interpret equality of terms and predicates. Axioms of Comprehension and Extensionality are added

$$\textbf{EXT} \quad \frac{}{\hookrightarrow \forall X^n Y^n (X^n = Y^n \leftrightarrow \forall x_1 ... x_n (X^n x_1 ... x_n \leftrightarrow Y^n x_1 ... x_n))}$$

The combination of the two axioms allows the derivation of the uniqueness existence

$$\vdash_{SOL} \exists^1 X^n \forall x_1 ... x_n (X^n x_1 ... x_n \leftrightarrow \varphi)$$

What is the relevance of this theorem? Well, having Extensionality and Comprehension in a logic where the identity is primitive, allows a "conservative" extension of the language with a lambda abstraction operator with predicates of this form $\lambda x_1 ... x_n \varphi$ interpreted as

$$\Im(\lambda x_1 ... x_n \varphi) = \{\langle \mathbf{x}_1 ... \mathbf{x}_n \rangle \mid \models_{\mathcal{A}} \varphi \; [x_1 ... x_n / \mathbf{x}_1 ... \mathbf{x}_n]\}$$

Written in lambda notation, comprehension is now

$$\forall x_1 ... x_n (\langle \lambda z_1 ... z_n \varphi \rangle \, x_1 ... x_n \leftrightarrow \varphi)$$

In [12], another calculus for *SOL* is introduced, the so called $\lambda$-*SOL*. In this calculus identity and lambda are primitives. The new $\lambda$ rules are *Introducing Abstraction in the Consequent*

$$\textbf{IAC} \;\; \frac{\Omega \hookrightarrow \varphi \frac{\tau_1 ... \tau_n}{x_1 ... x_n}}{\Omega \hookrightarrow \lambda x_1 ... x_n \varphi \tau_1 ... \tau_n}$$

and *Introducing Abstraction in the Antecedent*

$$\textbf{IAA} \;\; \frac{\Omega \;\; \varphi \frac{\tau_1 ... \tau_n}{x_1 ... x_n} \hookrightarrow \psi}{\Omega \;\; \lambda x_1 ... x_n \varphi \tau_1 ... \tau_n \hookrightarrow \psi}$$

In this calculus we can prove as a theorem the comprehension schema

$$\vdash_{\lambda - C_2} \exists X^n \forall x_1 ... x_n (X^n x_1 ... x_n \leftrightarrow \varphi)$$

Another interesting theorem is the following equivalence

$$\vdash_{\lambda - C_2} \forall x \varphi \leftrightarrow \lambda x \varphi = \lambda x \; x = x$$

saying that individual quantification can be expressed with lambda and equality using the formula saying that the predicates $\lambda x \varphi$ and $\lambda x \; x = x$ are equal; namely, that the predicate $\lambda x \varphi$ is universal as all individuals satisfy $\lambda x \varphi$.

Nevertheless, in non-classical logic we might want non extensional general models where predicates are not identified with their set-theoretical extensions. This is rather convenient in intensional logic when we want to distinguish predicates from their extensions in set theory. In that case the axiom of extensionality should be dropped.

### 4.3. General models for TT

As in the previous case for *SOL*, frames and general structures are hierarchies of types where $\mathcal{D}_{(\alpha\beta)}$ is some class of functions defined over $\mathcal{D}_\beta$ with values in $\mathcal{D}_\alpha$. The requirement for a frame to be a general model is that "for each assignment $\varphi$ and wff $A_\alpha$ of type $\alpha$, the value $V_\varphi(A_\alpha)$ given by the rules (i), (ii) and (iii) is an element of $D_\alpha$. Since this definition is impredicative, it is not immediate clear that any non-standard models exist."[12]

With this remark Henkin opens the door to better definitions. Andrews' proposal in [2] is to force the interpretation of equality to be identity. In [12], page 197, an algebraic definition is given for relational general structures, and also there, equality is identity.

#### 4.3.1. Andrews improved definition

Andrews showed in *General Models and Extensionality*, [2], pp. 395–397, that there is a non-standard general model, according to the previous definition, in which the axiom of extensionality fails and so the soundness of the calculus fails too. The completeness proof remains unaltered. The reason is that "the sets in this model are so sparse that the denotation of the defined equality formula $Q_{0\alpha\alpha}$ is not the actual equality relation."[13] The solution he offers to solve the problem is to guarantee that we have all the singletons in $\mathcal{D}_{0\alpha}$. In this way, the Leibniz definition of equality gives identity. In order to obtain that, he forces $\mathcal{D}_{0\alpha\alpha}$ to contain identity. Let us quote Andrews:

> We suggest that the definition of general models in [6] should be modified by adding the following requirement:
>
> ($\mathrm{a}_0$) For each $\alpha$, $\mathcal{D}_{0\alpha\alpha}$ contains the identity relation $\mathfrak{q}_{0\alpha\alpha}$ on $\mathcal{D}_\alpha$ (and hence $\mathcal{D}_{0\alpha}$ contains the unit set $\mathfrak{q}_{0\alpha\alpha}\mathfrak{x}_\alpha$ for each $\mathfrak{x}_\alpha \in \mathcal{D}_\alpha$.
>
> [...]
>
> Moreover, the model constructed in the proof of Theorem 1 of [6] actually satisfies ($\mathrm{a}_0$), since it can be seen that $\Phi([Q_{0\alpha\alpha}]) = \mathfrak{q}_{0\alpha\alpha}$ (in the notation of that proof). Thus Theorem 2 of [6]

[12]See [6], p. 84. The items (i), (ii) and (iii) constitute the definition by induction of the interpretation $V_\varphi(A_\alpha)$ of any expression $A_\alpha$.

[13]In [1], p. 70.

> becomes correct under the new definition of general model. ([2], p. 397).

### 4.3.2. Nameability and completeness

Henkin wrote in 1996 a very interesting paper, [10], telling us the process that led him into his discovery of completeness for $TT$ with the new semantics. We like to recall that such a discovery is strongly rooted on the already mentioned categories of nameable sets and relations that appear in the standard hierarchy, and that both identity and descriptions, play important roles. Henkin said: "I was specially attracted by the neatness and shortness of the formula expressing the axiom of choice. [...] I decided to try to see just which objects of the hierarchy of types did have names in $\mathcal{T}$."[14]

The nameable types

$$\mathcal{D}_a^n = \{f \in \mathcal{D}_a \mid \text{ there is a } F_a \in \mathit{cwff} \text{ such that } V(F_a) = f\}$$

(where $V$ stands for the interpretation of the formal language in the hierarchy of types) form a proper subset of the standard hierarchy and Henkin wanted to know if this restricted class itself formed a hierarchy.

> We observe that any element of $\mathcal{D}_{ab}^n$ is a function that maps $\mathcal{D}_b^n$ to $\mathcal{D}_a^n$, so the set of all domains $\mathcal{D}_a^n$ itself forms a hierarchy of types. Looking at it, I thought I should make it a little neater by "trimming the fat" from each function in any domain $\mathcal{D}_{ab}^n$. By this I meant that each element of $\mathcal{D}_{ab}^n$ has $\mathcal{D}_b$, rather than $\mathcal{D}_b^n$, as its domain, so I thought I should replace each element $f$ of $\mathcal{D}_{ab}^n$ by $f^*$ the restriction of $f$ to $\mathcal{D}_b^n$, and then work with the resulting sets, say $\mathcal{D}_{ab}^{n*}$, to get a neater representation of the hierarchy of nameable functions.[15]
>
> There was, however, a problem with this idea: What if the hierarchy contracted under the proposed reduction of the domains of functions? In other words, could there be distinct

[14] See [10], p. 146.

[15] Of course $\mathcal{D}_0^n = \mathcal{D}_0$ and $\mathcal{D}_1^n = \mathcal{D}_1$, so we may as well set $\mathcal{D}_0^{n*} = \mathcal{D}_0$ and $\mathcal{D}_1^{n*} = \mathcal{D}_1$ too.

> functions $f$ and $g$ in some $D^n_{(\alpha\beta)}$ such that $f^* = g^*$? ([10], p. 149).

The answer was *NO*, the proof involves the axiom of choice.[16] It is also remarkable that to identify objects named by both $M_\alpha$ and $N_\alpha$ he made use of a criterion based on the calculus, namely, the fact that $\vdash (M_\alpha = N_\alpha)$. The construction seemed to work smoothly, with the only exception of the universe of truth values, $\mathcal{D}_0$. He then realized that to reduce the universe of objects named by propositions (the truth values) to only two, the set of axioms had to be expanded until it constituted a maximal consistent set, say $\Gamma$. Two cwffs $M_\alpha$ and $N_\alpha$ of type $\alpha$ will be called *equivalent* iff $\Gamma \vdash M_\alpha = N_\alpha$.

This maximal consistent set gives Henkin the clue for the whole process. On page 86 of [6], Henkin says:

> We now define by induction on $\alpha$ a frame of domains $\{D_\alpha\}$, and simultaneously a one-one mapping $\Phi$ of equivalent classes onto the domains $D_\alpha$ such that $\Phi([A_\alpha])$ is in $D_\alpha$.
> $D_0$ is the set of two truth values and $\Phi([A_0])$ is $T$ or $F$ according as $A_0$ or $\sim A_0$ is in $\Gamma$
> [...]
> $D_\iota$ is simply the set of equivalence classes $[A_\iota]$ of all cffs of type $\iota$. And $\Phi([A_\iota])$ is $[A_\iota]$
> [...]
> Now suppose that $D_\alpha$ and $D_\beta$ have been defined, as well as the value of $\Phi$ for all equivalence classes of formulas of type $\alpha$ and $\beta$ and that every element of $D_\alpha$, or $D_\beta$, is the value of $\Phi$ for some $[A_\alpha]$, or $[B_\beta]$ respectively. Define $\Phi([A_{\alpha\beta}])$ to be the function whose value, for the element $\Phi([B_\beta])$ of $D_\beta$ is $\Phi([A_{\alpha\beta}B_\beta])$.

The equivalent classes mentioned in the above quote came as a result of a *'subjective'* but extremely *'consistent'* equality relation, that of being '*equals at the eyes of our oracle* $\Gamma$', this maximal consistent set whose model is built by using the very detailed description it offers.

In order to prove that $\Phi$ is a function on equivalent classes and does not depend on the particular representative chosen, Henkin uses choice and extensionality, and both axioms are also related to identity. In particular, he uses this theorem

[16] An explanation is given in [14], p. 155.

$$\begin{array}{ll} \vdash & A_{\alpha\beta}(\iota_{(\beta(0\beta))}(\lambda x_\beta(\sim (A_{\alpha\beta}x_\beta = A'_{\alpha\beta}x_\beta)))) \\ = & A'_{\alpha\beta}(\iota_{(\beta(0\beta))}(\lambda x_\beta(\sim (A_{\alpha\beta}x_\beta = A'_{\alpha\beta}x_\beta)))) \supset .A_{\alpha\beta} = A'_{\alpha\beta} \end{array}$$

We might wonder what the elements of $D_\iota$ are, as far as we do not have individual constants in this language. The selector operators $\iota_{\beta(0\beta)}$ acting on expressions of the appropriate type, say $X_{0\beta}$, produce elements of any type $\beta$; identity is again present, of course!

Being $\Gamma$ a maximal consistent set, it is not difficult to prove that the interpretation of any expression is the value of its equivalent class under function $\Phi$:

LEMMA 4.1. *For every $\phi$ and $B_\beta$ we have $V_\phi(B_\beta) = \Phi([B^\phi_\beta])$*

Using this construction Henkin was able to achieve his completeness result.

THEOREM 4.2. *If $\Lambda$ is any consistent set of cwffs (sentences), there is a general model (in which each domain $\mathcal{D}_\alpha$ of $\mathcal{M}$ is denumerable) with respect to which $\Lambda$ is satisfiable.*

To prove this theorem, the set $\Lambda$ is extended to a maximal consistent set $\Gamma$ which serves both as an oracle and as building blocks for the model.

A similar procedure can be applied for *SOL* and *FOL* to obtain completeness. In any case, the completeness result rests upon what can be described and built from a detailed description made in the formal language. Such a description is provided by the sentences in the maximal consistent set. In particular, the universes of *SOL* should include all of sets and relations that the comprehension axiom affirms to exist.

### 4.3.3. The completeness of FOL

In his 1996 paper, [10], Henkin states that he obtained the proof of completeness of first-order logic readapting the argument found for the theory of types. It was clear that to do this he had to get rid of the axiom of choice; as we have already explained, this axiom plays a relevant role in the construction of the hierarchy:

> But when I wrote down details of the proof [...], I saw that the axiom of choice is needed there in a more general way [...] to show that whenever we have a wff M such that $\vdash (\exists x_b)M_0$, then we also have $\vdash (\lambda x_b M_0)(\iota_{b(0b)}(\lambda x_b M_0))$. The fact that this

> condition holds is a direct consequence of having Axiom Schema $11^b$ [...], that schema is trivially equivalent to $(\exists x_b f_{0b} x_b) \supset f_{0b}(\iota_{b(0b)} f_{0b})$
>
> It did not take me very long to notice that in fact, the form of the wff following $(\lambda x_b M_0)$ played no role in the completeness proof; it is only necessary to have some cwff $N_b$ such that $\vdash (\lambda x_b M_0) N_b$ holds if $\vdash (\exists x_b) M_0$ holds. ([10], p. 152).

That is why he extends the consistent set $\Lambda$ not just to a maximal consistent set, but to one containing witnesses.

> ii) If a formula of the form $(\exists x)A$ is in $\Gamma_\omega$ then $\Gamma_\omega$ also contains a formula $A'$ obtained from the wff $A$ by substituting some constant $u_{ij}$ for each free occurrence of the variable $x$. ([5], p. 163).

The model is built using the set $\Gamma_\omega$ as oracle. The natural restriction to definable predicates proved to be useful in this case as well. The universe of the model is the set of constants modulo the equivalence relation of equality according to $\Gamma_\omega$ standards. The relation symbols are interpreted as $n-$ary relations on this universe, according to what our oracle declares.

### 4.3.4. SOL and MSL: completeness via translation

Somehow, frames are just a particular kind of many-sorted models and the *SOL* language a kind of special many-sorted logic (*MSL*).[17] This idea materializes via translation of *SOL* into *MSL*, afterwards the translation is exploited to get metalogical results; for instance, completeness of *SOL* (both weak and strong) with general semantics is transferred from completeness of *MSL*. Therefore, we do not need to follow the whole procedure to prove completeness for *SOL*, we just prove completeness for *MSL*.

What is the difference between a many-sorted structure and a second-order one? The only one is that in *SOL* the domains are no longer independent sets, since they are sets and relations over the individuals domain. That is why we add a membership relation to the many sorted structures when we want them to serve as models for *SOL*. The syntactical translation from *SOL* to *MSL* leaves every formula the same except the atomic formulas $X^n x_1 ... x_n$ that are translated as $\varepsilon x_1 ... x_n X^n$.

[17]In [12], pp. 277–290 you can find a detailed development of this idea.

The final result, as proved in [12], is strong completeness via translation

$$\begin{array}{cccr}
\Pi \models_{\mathcal{G}.\mathcal{S}} \varphi & \Longleftrightarrow & Trans(\Pi) \cup \Delta \models Trans(\varphi) & (*) \\
 & & \Updownarrow & (**) \\
\Pi \vdash_{SOL} \varphi & \Longleftrightarrow & Trans(\Pi) \cup \Delta \vdash_{MSL} Trans(\varphi) & (***)
\end{array}$$

the set $\Delta$ axiomatizes in *MSL* what characterizes *SOL*. Arrow (**) is just completeness and soundness of *MSL*. Arrow (*) is the *Main theorem.* Arrow (***) is easy because we added in $\Delta$ the required axioms. And it should not be surprising that in $\Delta$ we have comprehension, extensionality and disjoint universes, all of them axioms where identity and nameability are involved.

#### 4.3.5. Comprehension versus substitution

Henkin published *Banishing the Rule of Substitution for Functional Variables*, [7], in 1953, whose main result is the introduction of substitution free calculi for classical logics to replace Church's.

The first one is only a calculus for *FOL*. Henkin mentions that the natural extension of this calculus to the second order case, which he calls $F^*$ —namely, extending the quantifier rules to deal with the new predicate variables— is not a sound and complete one.

That is why Henkin proposed the calculus $F^{**}$ whose advantage over the Church's calculus is that it takes comprehension schema as an axiom, so avoiding the troublesome rule of substitution in the problematic cases:

> $F^{**}$ is obtained from $F^*$ by adding the axiom schema
> (iv) $(\exists \mathfrak{c})(\mathfrak{a}_1)...(\mathfrak{a}_n)(\mathfrak{c}(\mathfrak{a}_1,...,\mathfrak{a}_n) \equiv \mathfrak{B})$,
> where $\mathfrak{B}$ is any wff, $\mathfrak{a}_1,...,\mathfrak{a}_n$ are any distinct individual variables, and $\mathfrak{c}$ is any n-adic functional variable not occurring freely in $\mathfrak{B}$. ([7], p. 203).

There is another idea, appearing explicitly in [7], which is also useful:

> This observation suggests in a natural way consideration of certain subsystems of $F^{**}$ containing $F^*$, which can be defined by weakening axiom schema (iv). (pp. 206–207).

In [12], some of the ideas of these two Henkin's papers [6] and [7], produce a treatment of completeness via translation into *MSL*. The central idea is as simple as that: if we weaken comprehension (for instance, for first-order formulas, or for translations of dynamic or modal formulas, or any other recursive set), then we obtain a calculus between $F^*$ and $F^{**}$. And one should find a semantics for the logic thus defined. Of course, this class of structures is placed between $\mathfrak{F}$ and $\mathfrak{G}\mathfrak{S}$. The new logic, call it $XL$, will also be complete.

As María Manzano said in [14]:

> However, I do not wish to be misleading. In his 1950 paper we do not find translations of formulas or the overt appearance of a many-sorted calculus. Regarding higher-order logic, something close to a many-sorted calculus was introduced in the paper of 1953, but many-sorted logic was still not mentioned and neither were translations between logical systems. (p. 270).

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