MATHEMATICA APPLICANDA
Vol. 46(2) 2018, p. 273–291
doi: 10.14708/ma.v46i2.6352

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Selected applications of differential equations in Vanilla Options valuation.

Abstract The purpose of this paper is to present the applications of differential equations in Vanilla Options pricing. At the beginning we introduce main assumptions of the Black-Scholes Model with necessary comments, which as a norm can not be easily found in literature. In section 2, we show what in common have the diffusion equation and the fair price of European Option. Please note that these considerations were originally presented as proof of Black-Scholes Formula. In sections 3 and 4 we explain why valuation of the American option can be coming down to Free Boundary Problem. Note: It is very interesting that the same mathematical model is well known as Stefan Problem describing temperature distribution in homogeneous medium undergoing a phase change. At the end, we introduce a Finite Difference Method which will be used to solve problem numerically. We will describe the main features of the method showing potential threats, which could happen as a result of using this method without a thorough understanding of its structure. At the end we make a comparison with other, widely used methods.

This paper has as a goal to illustrate the potential importance of deterministic approach in financial engineering.

2010 Mathematics Subject Classification: Primary: 91G20; Secondary: 90B25; 60G40; 93E20.

Key words and phrases: the Black-Scholes Model, numerical methods, the American Option, the European Option, the Finite Difference Method, the Free Boundary Problem.

1. Introduction In financial models one of the basic assumptions about investors is that they want to gain as much as it is possible but they have aversion to take the risk. Each investing strategy can be considered as a compromise between willing of profit and fear of losses - usually possible profit increase with probability of loss. Option can be considered as some kind of insurance - a more prudent speculator might want to reduce the maximal loss by a quantity $K > 0$. He thus will buy an option which would correspond to the strike price K. For him (option holder) it is the way to protect himself against the risk, for option issuer it is possibility to profit by selling this financial product. The fundamental question is what is the value of this security? The answer has essential meaning in financial world and global economy.

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The first record of an option contract can be found in [20]. According to the story Greek philosopher Thales profited by option-type agreement around the 6th century B.C. The problem of fair valuing this kind of financial instrument was not formalized until 1900.

At this year L.Bachelier by his pioneering thesis [2] began the theory of option pricing. In the same work he initiated the study of diffusion processes five years before recognized as the groundbreaking works of A.Einstein[4], M.Smoluchowski [14] and decades before famous works of K.Itô [7], P.Lévy [8] and N.Wiener [18]. Bachelier as the first developed the theory of Brownian motion and found practical application of this concept in financial engineering.

The culminating event in developing theory of option pricing was 1973 when Black, Scholes and Merton found consistent formulas for the fair prices of European options[3],[9]. The discovery was of such great importance that the autors were awarded the Nobel Prize in Economics in 1997. Very interesting is fact that for short time to maturity formulas of Bachelier are very close to results of Black, Merton and Scholes[13].

Until today, there is no knowledge of any analytical formula of American option fair price, which could have any practical application. In order to determine this value, as a rule it is given main importance to Monte Carlo methods. Usually they are easier to implement, but require more time or are related to higher numerical errors than deterministic methods.

2. Vanilla Options Option put/call is a contract, in which holder can buy/sell some property for fixed price K (it is fixed in time $t = 0$). Other characteristics of option is time to maturity T and value of underlying asset at moment t which is denoted by S_t . Underlying asset can be action, stock exchange indexes, foreign currency, futures contracts, obligations. The options most frequently used are the European and American options (both called "Vanilla options"). Payement profile for the European call option is equal $\max(S_T - K, 0)$, whereas for the European put equals $\max(K - S_T, 0)$. The European option can be realized only at time T. The American option, can be realized in any time t between 0 and T, its profile for type call equals $\max(S_t - K, 0)$, for put equals $\max(K - S_t, 0)$. It is necessary to understand that at time of option valuation (we can assume that it is time $t = 0$) we know the value of underlying assets only at time $t = 0$.

2.1. Assumptions of the Black-Scholes Model

To begin, we will recall assumptions of model to be considered [3]. They are as follow:

1. **Market liquidity.**
   By this assumption, we will consider the balance between supply and demand - in other words, at the moment of decision of selling some good,

there is a will to buy it; similarly, at the moment of buying some product there is someone who wants to sell it. Of course, purchase and sell are done at the price of equal value of considered goods (“market price“). In actual practice, this assumption is unacceptable only in extreme cases like possibility of bankruptcy (if a company goes bankrupt, each of the investors will want to sell their stocks, but not many will willingly take the risk to buy said stock, even at a lower price).

2. **Possibility of short selling.**
   ”Short selling“ refers to the selling of assets which does not belong to us - for example, a broker, on behalf of a client, borrows a considered asset from another client and sells at the market price; after predetermined time, we are obligated to return the borrowed goods and bear the costs. In actuality, even if short selling is allowed, it is not for free (in the Black-Scholes Model we usually assume that this opportunity is for free).

3. **Possibility of purchase/sell at each moment any (even irrational) number of underlying assets, and free-risk assets.**
   Of course, in real life we can not buy/sell nor integer number of stock. Without this assumption it is impossible to show the existence of only ”fair price”.

4. **No transaction costs and taxes.**
   This assumption is unrealistic - in the financial world both of them are present. If we will take these costs into account, the issue of valuation would be much more difficult (problem of optimal control). However they can often be omitted, because in the case of “big“ assets, investors may have some discounts on the costs of the transaction (considered costs are negligible as compared to ”big” assets).

5. **Absence of arbitrage (there is no possible gain without taking the risk).**
   In the real world, there is the possibility of arbitrage - eg. having information about any given company, which is not widely available. However, the arbitrage quickly disappears so assumption about its absence is acceptable. Here it is hidden assumption that investors have the same (total) access to the information.

6. **The interest rate r is constant over the time; moreover, it is the same for lending and capital investment.**
   Of course, usually the interest rate for deposits is different than for loans; however, this condition is a consequence of the lack of arbitrage, market liquidity and the absence of transaction costs. As for the stability in time, in the case of “short” time intervals (eg. 1 month) is acceptable,

but if we want to deal with an option for a period longer than year we should count the errors resulting from this assumption. The interest rate r is not constant over time, usually we model r via stochastic differential equations - eg. Vasicek Model, Model CIR, HJM Model.

7. **The price of the underlying instrument is described by the Geometric Brownian Motion (GBM):**

$$\begin{cases} S(t) = S(0)\exp(\mu t + \sigma B_t), \\ S(0) = S_0, \end{cases}$$

where:
$S(t)$ - the price of the underlying instrument, μ - drift (constant), σ - volatility (constant), B_t- Brownian motion.

Note: this assumption is not acceptable in the case of sudden jumps in the stock market. Aggressive behavior of investors and stock market crashes are not predictable in this model. The basic Black-Scholes Model assumes that the stocks do not bring dividends, but there is a simple generalization into the model covering this case. It is worth to note that the Black-Scholes Model is a model based of continuity of trading in stock exchanges - including the time when the exchange is closed.

Although, strictly speaking, none of these assumptions is fulfilled 100%, but the essence of the mathematical model is to take the situation we want to model into a simplified and idealized case. Due to its simplicity and practicity, the Black-Scholes Model is one of the most widely used in option pricing.

3. European option In the case of European option, starting from the diffusion equation, we can analytically determine a fair price. Before we will discover the Black-Scholes Formula we will present a theorem which is crucial in our considerations.

THEOREM 1 (the Feymann-Kac Formula)[15]
Let S_t follows

$$\begin{cases} dS_t = \mu(S_t)\,dt + \sigma(S_t)\,dB_t, \\ S_0 = x. \end{cases}$$

Let us consider following partial differential equation (PDE):

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \dfrac{1}{2}\sigma^2(x)\dfrac{\partial^2 u(x,t)}{\partial x^2} + \mu(x)\dfrac{\partial u(x,t)}{\partial x} + q(x)\,u(x,t), \\ u(x,0) = f(x), \end{cases} \tag{1}$$

where $q : R \longrightarrow R$ is continuous and bounded and $f : R \longrightarrow R$ is continuous and $f \in O(|x|)$ (when $x \longrightarrow \mp\infty$).

Then function

$$u(x,t) \equiv E_x \left(f(S_t) \exp\left(\int_0^t q(S_u)\, du \right) \right)$$

is solution of (1) and this solution is unique in class of functions having at most power-law growth.

Let us return do the Black-Scholes Model; then, we will conduct a reasoning for the valuation for call options. For "put" this is completely analogous.

Corollary 1 *A fair price of the call option $u(x,t)$ is solution of:*

$$\begin{cases} u_t = \frac{1}{2}\sigma^2 x^2 u_{xx} + rxu_x - ru, \\ u(x,0) = f(x) = \max(x-K,0), \end{cases} \tag{2}$$

for $x = S_0$ and $t = T$. □

Proof Let Q-martingale measure, $\hat{B}_t$ - Brownian motion with respect to measure Q; S_t - stock price; then

$$\begin{cases} dS_t = rS_t dt + \sigma S_t d\hat{B}_t, \\ S_0 = x. \end{cases}$$

A fair price of the call option is

$$C_0 = E_x^Q[f(S_T)e^{-rT}] = e^{-rT}E_x^Q[\max(S_T - K,0)] = u(x,T),$$

where f is the function of payment - for call option $f(x) = \max(x-K,0)$. Using notations from the previous theorem we have:

$$q(x) = -r = const,$$

$$\mu(x) = rx,$$

$$\sigma(x) = \sigma x.$$

By the Feymann-Kac Formula the proof is completed. ■

Proposition 1 *(the Black-Scholes Formula) The fair price of European call option is equal*

$$x\Phi\left(\frac{\log\frac{x}{K}+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)-Ke^{-rT}\Phi\left(\frac{\log\frac{x}{K}+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right),$$

where r - interest rate, σ - volatility, T - time horizont, K - strike, x-value of underlying instrument at time 0 (also noted as S_0). □

Proof Using 3 changing of variables we will conduct (2) into diffusion equation which solution is well known. Let us introduce functions g, y, h:

$$g(x,T)=u(e^x,T),$$

$$y(x,T)=e^{rT}g(x,T),$$

$$h(x,T)=y\left(x-\left(r-\frac{1}{2}\sigma^2\right)T,T\right).$$

Now (2) has a form of classical diffusion equation:

$$\begin{cases} h_T=\frac{1}{2}\sigma^2 h_{xx}, \\ h(x,0)=\max(e^x-K,0). \end{cases}$$

The solution of (2) has a form

$$h(x,T)=\int_R \frac{1}{\sqrt{2\pi\sigma^2 T}}e^{\frac{(y-x)^2}{2\sigma^2 T}}\max(e^y-K,0)\,dy=$$

$$=\frac{1}{\sqrt{2\pi\sigma^2 T}}\int_{\log K}^{\infty}e^{-\frac{(y-x)^2}{2\sigma^2 T}}e^y dy-K\frac{1}{\sqrt{2\pi\sigma^2 T}}\int_{\log K}^{\infty}e^{-\frac{(y-x)^2}{2\sigma^2 T}}dy.$$

After change of variable $y-x=z$ we get that:

$$h(x,T)=e^{x+\frac{\sigma^2 T}{2}}\Phi\left(\frac{x-\log K+\sigma^2 T}{\sigma\sqrt{T}}\right)-K\Phi\left(\frac{x-\log K}{\sigma\sqrt{T}}\right).$$

With the function h we return to the function u using inverse transformations of previously applied changing of variables.

$$y(x,T)=h\left(x+\left(r-\frac{1}{2}\sigma^2\right)T,T\right),$$

$$g(x,T)=e^{-rT}y(x,T),$$

$$u(x,T) = g(\log x, T).$$

By this way we discovered the Black-Scholes Formula for European call option, analogically as it was done first time [10]:

$$u(x,T) = x\Phi\left(\frac{\log\frac{x}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\log\frac{x}{K} + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right).$$

It is worth nothing that this is the exact price in term of the considered model. Here it is not taken into account the profit for the issuer of option.

Digression 1 Obviously to determine the fair price of European put option we can use exactly the same way as before. But having the fair price of European call, we can get the fair price for put at once. The relationship between European call and put is defined by the following theorem:

Theorem 2 The put-call parity
In the Black-Scholes Model for fair price of European call and put options (at any time t between 0 and T) true is following formula [11]:

$$C_t - P_t = S_t - Ke^{-r(T-t)}.$$

Above we assumed, that present value of zero-coupon bond that matures to 1 unit of money at time T is equal $e^{-r(T-t)}$. If we assume other value of a bond, noted as $B(t,T)$, then the put-call parity has a form:

$$C_t - P_t = S_t - KB(t,T).$$

It is worth observing that the zero-coupon bond that matures to 1 unit of money at time T, is the present value for K.

4. American option Because the American option can be realized at each moment before T (with T), it is clear that fair price (it means price which does not give opportunity of arbitrage) can not be lower than fair price of its European analogue. Indirectly we can say that the American option gives more opportunities to profit. The question is how to valuate this kind of options, and if exists any formula analogous to the Black-Scholes Formula. Pricing of the American option is much more complicated than in the European case because for each moment that we have to find value of option, and for each value S_t we have to decide if it is worth it to realize such option (we recall that value of S_t is not given). This kind of problem is known as free boundary problem.

Digression 2 Very interesting is the fact that if dividend rate $d = 0$, then the value of American call option is equal to its European analogue. To show this identity, it is enough to argue that it is not worth to realize the American option before T (then it is optimal to realize option only at T; for what authorizes the European option as well).

Let us assume that the American call option holder has intention to realize it at time $t < T$. He will do that only if it would be profitable, it means if $S_t > K$ (price for buying underlying asset is higher than price for which one underlying asset can be bought by holding option). Then, he gains $S_t - K$. Option holder can also proceed following strategy: at time t short sell an underlying asset for S_t, next buy underlying asset for $\min(S_T, K)$ at time T. Then his profit is $S_t - \min(S_T, K)$ and of course is higher than $S_t - K$. So never is worth realize the American call before time T.

Remark 1 We assumed that dividend rate $d = 0$; only under this condition the conclusion above is true. In all this paper we assume that $d = 0$. By analogical way we can show that if $r = 0$, then it is not worth to realize the American put before T, so in this case the value of American put is equal to its European equivalent.

5. Valuation of the American option using PDE

5.1. Valuation of the American option as Free Boundary Problem

Similarly as in the case of European option we want to find a fair price using the Black-Scholes Equation. As we mentioned before, we do not know the analitical solution; but using numerical methods we can find aproximate solution.
We consider the Black-Scholes Equation (with boundary condition determining put option):

$$\begin{cases} \dfrac{\partial u(S,t)}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 u(S,t)}{\partial S^2} + rS\dfrac{\partial u(S,t)}{\partial S} - ru(S,t) = 0, \\ u(S,T) = f(S,T). \end{cases}$$

Where $f(S,t) = \max(K - S, 0)$. By this notation we recall that the payoff function is dependent on time (note that in this chapter $S = S_t$; $t \in [0,T]$).

At each moment it is necessary to make the decision if it is worth to use option, or keep it on. Mathematically we can describe it as:

$$u(S,t) = f(S,t),$$

if we realize option or

$$\frac{\partial u}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 u}{\partial S^2}+rS\frac{\partial u}{\partial S}-ru=0,$$

if we keep it on. This can be written in the following manner:

$$\left(u\left(S,t\right)-f\left(S,t\right)\right)\cdot\left(\frac{\partial u}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 u}{\partial S^2}+rS\frac{\partial u}{\partial S}-ru\right)=0.$$

Notice that at each moment t we can gain at least $\max(K-S,0)$ (and maybe even more). It follows us to inequality:

$$u\left(S,t\right)\geq f\left(S,t\right).$$

Because after optimal exercise moment $u\left(S,t\right)$ can not describe value of underlying instrument, true is following inequality:

$$\frac{\partial u}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 u}{\partial S^2}+rS\frac{\partial u}{\partial S}-ru\leq 0.$$

Remark 2 Assumption of lack of arbitrage says that expected profit of option (calculated at time 0) is equal to value which we have to pay for option. It is worth to recall this observation in the case of American option, because here the situation is more complicated than in the European case.

The first boundary condition is as follows: for price of underlying instrument sufficiently high (in general form - converging to infinity) we will not use the option. So:

$$\lim_{S\to\infty}u\left(S,t\right)=0.$$

By analogical way if price of underlying instrument will be "very small" (more general - converging to 0) we will use option selling underlying instrument for K:

$$\lim_{S\to 0}u\left(S,t\right)=K.$$

Now, a problem of American (put) option valuation is transformed into Free Boundary Problem:

$$\begin{cases}\dfrac{\partial u}{\partial t}+\dfrac{1}{2}\sigma^2S^2\dfrac{\partial^2 u}{\partial S^2}+rS\dfrac{\partial u}{\partial S}-ru\leq 0,\\ u\left(S,t\right)\geq f\left(S,t\right),\\ \left(u\left(S,t\right)-f\left(S,t\right)\right)\cdot\left(\dfrac{\partial u}{\partial t}+\dfrac{1}{2}\sigma^2S^2\dfrac{\partial^2 u}{\partial S^2}+rS\dfrac{\partial u}{\partial S}-ru\right)=0,\\ \lim\limits_{S\to\infty}u\left(S,t\right)=0,\\ \lim\limits_{S\to 0}u\left(S,t\right)=K,\\ u\left(S,T\right)=f\left(S,T\right).\end{cases}$$

An equivalent way to introduce the formulation of American option pricing can be found, for example in [19].

6. Finite difference method in the aspect of American put option valuation

Let us introduce an uniform grid of the $[0, T]$ interval:

$$0 = t_0 < t_1 < t_2 < \ldots < t_j < \ldots < t_n = T,$$

where

$$t_{j+1} - t_j = \Delta t,$$

for each $j = 0, \ldots, n-1$.
By analogical way we introduce a grid of the $[0, S_{max}]$ interval:

$$0 = S_0 < S_1 < S_2 < \ldots < S_i < \ldots < S_m = S_{max},$$

where S_{max} is some upper bound for S_t, $t \in [0, T]$ and

$$S_{i+1} - S_i = \Delta S,$$

for each $i = 0, \ldots, m-1$.
Let us introduce following notation

$$u_i^j = u(S_i, t_j).$$

In the Black-Scholes Equation we replace derivatives by their approximations:

$$\frac{u_i^j - u_i^{j-1}}{\Delta t} + \frac{1}{2}\sigma^2 S_i^2 \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{(\Delta S)^2} + rS_i \frac{u_{i+1}^j - u_{i-1}^j}{2\Delta S} - ru_i^j = 0.$$

Then we transform this expression into:

$$u_i^j - u_i^{j-1} + a_i \left(u_{i+1}^j - 2u_i^j + u_{i-1}^j\right) + b_i \left(u_{i+1}^j - u_{i-1}^j\right) - \Delta t r u_i^j = 0,$$

where

$$a_i = \frac{\Delta t \sigma^2 S_i^2}{2(\Delta S)^2},$$

$$b_i = \frac{\Delta t r S_i}{2\Delta S}.$$

Let us note

$$d_i = a_i - b_i,$$

$$m_i = (-2a_i - r\Delta t),$$

$$e_i = a_i + b_i.$$

Now the system of equations can be written as follows:

$$u_i^j - u_i^{j-1} + d_i u_{i-1}^j + m_i u_i^j + e_i u_{i+1}^j = 0,$$

for $i = 1, \ldots, N-1$.

In matrix form:

$$u_{1:N-1}^j - u_{1:N-1}^{j-1} + P u_{0:N}^j = 0, \tag{3}$$

where

$$P = \begin{pmatrix} d_1 & m_1 & e_1 & 0 & \ldots & 0 \\ 0 & d_2 & m_2 & e_2 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & \ldots & d_{N-2} & m_{N-2} & e_{N-2} & 0 \\ 0 & \ldots & 0 & d_{N-1} & m_{N-1} & e_{N-1} \end{pmatrix}.$$

The first and last column correspond to boundary conditions. Now we will divide matrix P (which is represented by set $\Omega \cup \partial\Omega$ of nodes of time and space) into interior Ω and border $\partial\Omega$:

$$P u_{\Omega \cup \partial\Omega}^j = A u_\Omega^j + B u_{\partial\Omega}^j,$$

where

$$u_{\Omega \cup \partial\Omega}^j = u_i^j,$$

for $i = 0, 1, \ldots, N$.

$$u_\Omega^j = u_i^j,$$

for $i = 1, \ldots, N-1$.

$$u_{\partial\Omega}^j = u_i^j,$$

for $i = 0, N$.

Equation (3) we can write as:

$$u_\Omega^j - u_\Omega^{j-1} + A u_\Omega^j + B u_{\partial\Omega}^j = 0. \tag{4}$$

u_Ω^0 we find from:

$$u_\Omega^{j-1} = (I + A)\, u_\Omega^j + B u_{\partial\Omega}^j.$$

Notice that (4) corresponds to an explicit scheme, but

$$u_\Omega^j - u_\Omega^{j-1} + A u_\Omega^{j-1} + B u_{\partial\Omega}^{j-1} = 0 \tag{5}$$

corresponds to implicit scheme. Taking linear combination of (4) and (5) we get the mixed scheme:

$$\theta\left(u_{\Omega}^{j}-u_{\Omega}^{j-1}+Au_{\Omega}^{j-1}+Bu_{\partial\Omega}^{j-1}\right)+(1-\theta)\left(u_{\Omega}^{j}-u_{\Omega}^{j}+Au_{\Omega}^{j}+Bu_{\partial\Omega}^{j}\right)=0.$$

Substituting $A_m = I - \theta A$ and $A_p = I + (1-\theta)\,A$ we get:

$$A_m u_{\Omega}^{j-1} = A_p u_{\Omega}^{j} + \theta B u_{\partial\Omega}^{j-1} + (1-\theta)\,B u_{\partial\Omega}^{j}, \tag{6}$$

for $\theta \in [0, 1]$.

In order to find value u_{Ω}^{0} we have to solve (6) for $j = M, M-1, \dots, 1$ Finally, we are ready to write the algorithm of American (put) option valuation (see Appendix).

The most important example of mixed scheme is the Crank-Nicolson Scheme. The following theorem explains why in our problem this exceptional case is so important:

THEOREM 3 For the considered problem, the Crank-Nicolson Scheme is unconditionally stable and its convergence in respect to ΔS and Δt has 2nd order. [12] [17]

REMARK 3 In the case of mixed schemes we have guaranteed unconditional stability only for $\theta \geq \frac{1}{2}$. For $\theta \in [0, \frac{1}{2})$ we have only conditional stability. Note that if discretization parameters do not fullfill the condition of (conditional) stability, it does not mean that the method is not stable (there is the implication that fullfilled condition implies stability). What we call "condition" is inequality resulting from some estimation, so there is a possibility of formulating it in many different (meaning not equivalent) forms. Good estimations with stability analysis are, for example, in [16], [10].

REMARK 4 In contrast to Crank-Nicolson Scheme, explicit and implicit schemes (cases $\theta = 0$ and $\theta = 1$) have convergence of 2nd order in respect to ΔS and 1st order in respect to Δt [5].

As conclusion of numerical error analysis in dependence of θ we can state that the best precision should be reached for $\theta = 0.5$, and that for lower θ we can expect large errors (in case of lack of stability). All plots and tables from this paper are the results of our simulations; data determining American Option (S, K, r, T, estimated σ and real price of options) we took from [1].

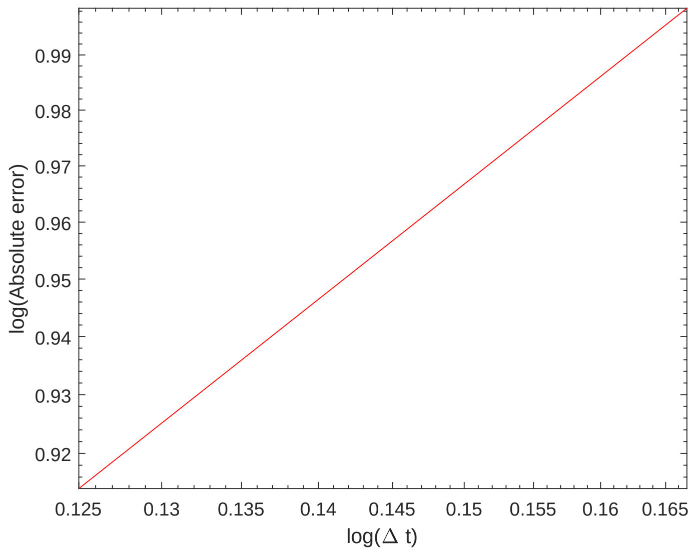


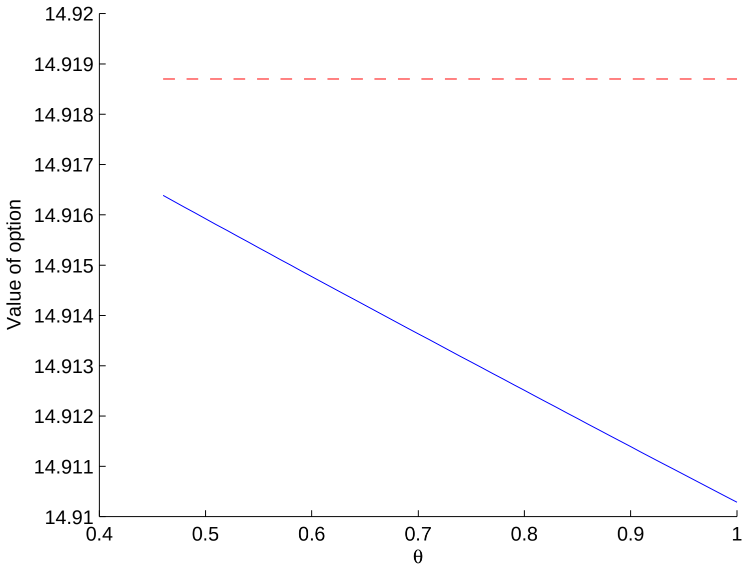
Figure 1: Absolute error in dependence of Δt in Crank-Nicholson scheme (in log-log scale; $S = 80$, $K = 100$, $\sigma = 0.4$, $r = 0.06$, $T = 0.5$). Order of convergence is the linear coefficient of the curve - empirically estimated value is 2.03, exact value is 2.

Strike	80	85	90	95	100	105	110	115
Exact price	21.6059	18.0374	14.9187	12.2314	9.9458	8.0281	6.4352	5.1265
Sim1($\theta = 1/2$)	21.6038	18.0349	14.9159	12.2285	9.9437	8.0252	6.4326	5.1242
Sim2($\theta = 3/4$)	21.6018	18.0324	14.9131	12.2255	9.9407	8.0223	6.4301	5.122
Sim3($\theta = 1$)	21.5998	18.0299	14.9103	12.2225	9.9377	8.0195	6.4275	5.1199
Sim4($\theta = 0$)	20	15	10	5	0	0	0	0

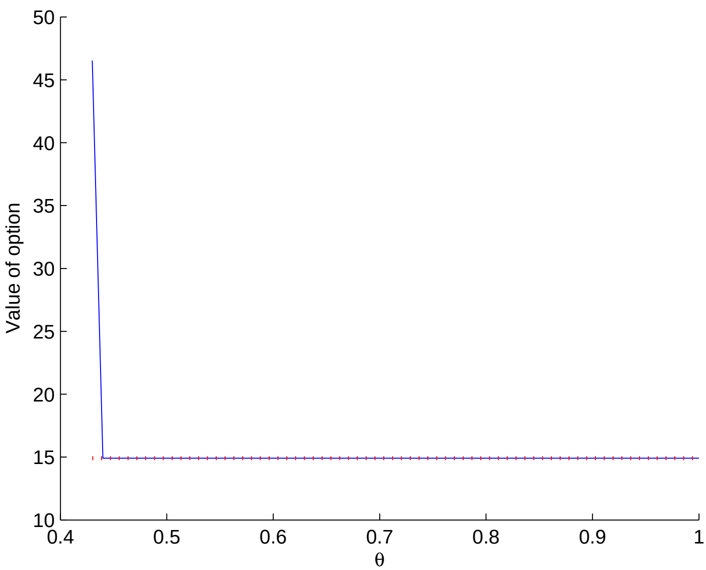
Table 1: Simulations of price for American Option using different values of θ. The best precicion is reached for $\theta = 0.5$; for $\theta = 0$ we see the consequence of lack of stability when $\theta \in [0, \frac{1}{2})$.

time[s]	Strike	80	85	90	95	100	105	110	115
-	Exact price	21.606	18.037	14.919	12.231	9.946	8.028	6.435	5.127
0.92	FD	21.604	18.035	14.916	12.229	9.944	8.025	6.434	5.124
4.41	CRR	21.607	18.037	14.918	12.23	9.944	8.029	6.436	5.126
2.74	LSM	24.128	18.158	16.941	13.067	8.907	7.61	7.153	5.54

Table 2: Comparison of different methods of American Option valuation. FD-presented in this paper Finite Difference Method($\theta = 0.5$), CRR-binomial tree (Cox-Ross-Rubinstein Model; number of nodes=1000), LSM- Longstaff Schwartz Method.



(a)



(b)

Figure 2: Approximation of American option fair price for θ in different intervals ($S = 90$). Colour blue - approximation; red - real price. In a) can be observed increasing precision along with getting closer to the value $\theta = \frac{1}{2}$; in b) is visible explosive behavior for $\theta < \frac{1}{2}$.

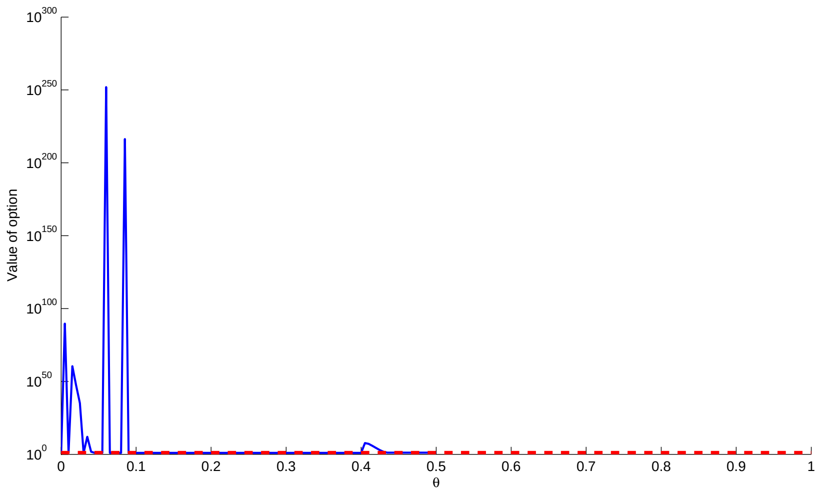


Figure 3: Approximation of American option fair price for $\theta \in [0, 1]$ $(S = 90)$. Colour blue - approximation; red - real price. If conditional stability is not provided we can expect big errors for $\theta \in [0, \frac{1}{2})$.

As we can see in the Table 2 the Finite Difference Method returns results with almost the same (very good) precicion as CRR, but in almost 4 times shorter time. In this experiment Longstaff Schwartz Method gives worse results than previous methods; in almost 3 times longer time than FD and less than 2 times shorter than CRR. As "time" we mean average time calculated for all strikes. Precicion of LSM could be improved at the expense of the time. Similar remark has place for the CRR - if we want to reduce time of computation it can be done by loss of precicion. Simulations for CRR we made using function "AmericanOptCRR" [6], for LSM we used function "AmericanOptLSM" [6] $(M = N = 40)$.

7. Conclusion A deterministic approach could be succesfully undertaken in option pricing and more general in financial engineering. Using partial differential equations, we can obtain the Black-Scholes Formula and find very good approximation for the price of American option. For numerical method from this paper, is always best to choose the Crank-Nicholson Scheme in order to provide the best possible order of convergence and unconditional stability. As we have seen, using this method should not take place without the knowlegde of whether unconditional stability is provided. Presented modelling is very important alternative for Monte Carlo methods widely associated with Vanilla Options pricing.

8. Appendix Algorithm of American put option pricing:

```
Input data: financial data (S0,sigma,r,T,K);
parameters for discretization: Smin=0, Smax, Delta t, Delta S,
N - number of nodes, M - number of time steps;
theta (parameter determining scheme);

The procedure:
Let Svec be a vector of length N+1 discretizating space
  Compute the time and space increments
 dt = T/M;
 dS = (Smax-Smin)/N;
 Compute elements of sub-matrix Omega of matrix P:
(m=elements of the main diagonal)
m =-2*((dt*sigma^2*Svec(2:N).^2)./(2*dS^2))-dt*r;
(d=elements under the main diagonal)
d=(dt*sigma^2*Svec(2:N).^2)./(2*dS^2)-(dt*r*Svec(2:N))/(2*dS);
 (e=elements above the main diagonal)
 e=(dt*sigma^2*Svec(2:N).^2)./(2*dS^2)+(dt*r*Svec(2:N))/(2*dS);
%from these elements we create a matrix N-1 on N-1.
 Create matrix P from matrix Omega and vectors w1, w2
P=[w1 Omega w2]
where
w1=[d(1),0,..,0]^{T}
w2=[0,0,..,0, e(N-1)]^{T}
d(1)=(dt*sigma^2*Svec(1)^2)/(2*dS(1)^2)-(dt*r*Svec(1))/(2*dS);
e(N-1)=(dt*sigma^2*Svec(N-1)^2)/(2*dS^2) + (dt*r*Svec(N-1))/(2*dS);

Compute matrix A and B, based on them calcule matrices Am and Ap:
Am=I-theta A  Ap=I+(1-theta)A

Compute vector of the payoff function v1=max(K-Svec,0):
 for j=1:1:M
v2=v1;
v1(1)=K;
b = Ap*v2(2:N) + theta *B*v1([1 N-1]) + (1-theta)*B*v2([1 N-1]);
Solve in respect to w equation:
Am*w=b; %%For example using LU algorithm.
Into v1(2:N) put the largest element of corresponding
elements of the payoff function components and vector w.
end
 Using value v1 in points Svec
 interpolate value of the payoff function in S0.
Return this value.
```

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Wybrane zastosowania równań różniczkowych w wycenie opcji waniliowych.

Grzegorz Krzyżanowski

Streszczenie Celem niniejszego artykułu jest przedstawienie zastosowań równań różniczkowych w wycenie opcji waniliowych. Na początku przedstawiamy główne założenia modelu Blacka-Scholesa z niezbędnymi komentarzami, których jako normę nie można łatwo znaleźć w literaturze.
Następnie pokazujemy, co wspólnego ma równanie dyfuzji ze sprawiedliwą ceną opcji europejskiej. Należy pamiętać, że te rozważania zostały pierwotnie przedstawione jako dowód formuły Blacka-Scholesa. Kolejno wyjaśniamy, dlaczego wycena opcji amerykańskiej może zostać sprowadzona do problemu ze swobodnym brzegiem, znanym jako problem Stefana opisującym propagację temperatury w niejednorodnym ośrodku ulegającym przemianom fazowym.
W końcowej części pracy wprowadzamy metodę różnic skończonych, która posłuży do numerycznego rozwiązania problemu. Opisujemy główne cechy metody pokazując potencjalne zagrożenia, które mogą pojawić się w wyniku zastosowania tej metody bez dokładnego zrozumienia jej struktury. Dokonujemy również porównania z innymi, szeroko stosowanymi metodami. Główny cel tego opracowania to pokazanie potencjalnego znaczenia podejścia deterministycznego w inżynierii finansowej.

2010 Klasyfikacja tematyczna AMS (2010): Primary: 91G20; Secondary: 90B25; 60G40; 93E20.

Słowa kluczowe: model Blacka-Scholesa, metody numeryczne, opcja amerykańska, opcja europejska, metoda różnic skończonych, problemy ze swobodnym brzegiem.

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Communicated by: Krzysztof Szajowski

(Received: 22nd of February 2018; revised: 10th of March 2018)