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Piotr Kruczek (Wrocław)
Wojciech Żuławiński (Wrocław)
Patrycja Pagacz (Wrocław)
Agnieszka Wyłomańska (Wrocław)

Fractional lower order covariance based-estimator for Ornstein-Uhlenbeck process with stable distribution

Abstract The Ornstein-Uhlenbeck model is one of the most popular stochastic processes. It has found many interesting applications including physical phenomena. However, for many real data, the classical Ornstein-Uhlenbeck process cannot be applied. It is related to the fact that for many phenomena the vectors of observations exhibit the so-called heavy-tailed behaviour. In such cases, the modifications of the classical models need to be used. In this paper, we analyze the Ornstein-Uhlenbeck process based on stable distribution. This distribution is one of the most classical members of the heavy-tailed class of distributions. In the literature, one can find various applications of stable processes. However, the heavy-tailed property implies that the classical methods of estimation and statistical investigation cannot be applied. In this paper, we propose a new method of estimation of the stable Ornstein-Uhlenbeck process. This technique is based on the alternative measure of dependence, called fractional lower order covariance, which replaces the classical covariance for infinite-variance distribution. The proposed research is a continuation of the authors' previous studies, where the measure called covariation was proposed as the base for the estimation technique. We introduce the stable Ornstein-Uhlenbeck process and remind its main properties. In the main part, we define the new estimator of the parameters for discrete representation of the Ornstein-Uhlenbeck process. Its effectiveness is checked by Monte Carlo simulations.

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1. Introduction. Continuous time models are popular in various applications. They seem to be the most natural for modelling high-frequency data, which appear for instance in finance. One of the most famous examples of continuous time models is the Ornstein-Uhlenbeck (O-U) process. It was introduced by Uhlenbeck and Ornstein [40] and its classical version is related to the ordinary Brownian motion. This process is considered as a stationary solution for the classical Klein-Kramers dynamics [20]. One can find different applications of the Ornstein-Uhlenbeck process among which the most known are the economic studies. In finance, it is known as the Vasiček model, [41], which was one of the stochastic models used to describe the interest rate data. The Ornstein-Uhlenbeck process exhibits the so-called mean reversion feature which is very often visible in the economic phenomena. One can find many other applications of the Ornstein-Uhlenbeck process, like physical and biological phenomena [7, 9, 37].

However, the analysis of real-life data shows that the classical Ornstein-Uhlenbeck process very often cannot be directly applied. This is related to the fact that many real phenomena exhibit non-Gaussian behaviour and the distribution of the data belongs rather to the heavy-tailed family of distributions. The probability distribution is called heavy-tailed if its tail (1-cumulative distribution function) has power-law behaviour. In such a case, there is a higher probability of large observations than in the Gaussian case. One of the most popular distributions with this property is a stable one (called also α-stable or Lévy stable). In the literature one can find many different applications of this distribution e.g. in finance [6, 29, 32], biology [5], physics [25] and signal processing [31]. This distribution is considered as the generalization of the Gaussian one and possesses many useful properties. However, one of its main drawbacks is that for most of the cases (except the Gaussian case) the second moment and variance do not exist. Therefore, for models based on stable distribution, the covariance cannot be considered as the measure of dependence. Moreover, for most of the members of the stable distribution family, the probability density function is not given in the closed form. Therefore, this distribution is described in the language of the characteristic function.

One can consider the stochastic processes with the stable distribution. The Ornstein-Uhlenbeck process based on stable distribution [33] is one of the examples. This process arises as the classical Ornstein-Uhlenbeck model for which the Brownian motion is replaced by the process with stationary independent increments having stable distribution [4]. This process is widely discussed in the literature in different aspects [11, 12, 16, 17, 18, 24, 35, 38]. For such a process, the typical measure of dependence i.e. autocorrelation, does not exist. Therefore, alternative measures have to be used [43]. A few popular choices are: covariation [35], codifference [44] and fractional lower order covariance (FLOC) [23].

In this paper, we consider the Ornstein-Uhlenbeck process based on stable distribution and propose the new estimation method for its parameters. The idea of the proposed technique is based on the discrete representation of the analyzed process which in the statistical literature is called an autoregressive model of order 1 (AR(1)). For the discrete version of the Ornstein-Uhlenbeck process based on the stable distribution, we propose to apply the modified Yule-Walker method. The classical Yule-Walker method, which is used for the estimation of the autoregressive models' parameters, is based on the autocovariance function of the given process [3]. Because for the models based on stable distribution the autococovariance function is not properly defined, we propose to replace it by one of the alternative measures of dependence mentioned above. This paper is a continuation of the authors' previous research presented in [21], where the modified Yule-Walker method based on autocovariation was defined. In this article, a similar approach is described. We are going to focus on fractional lower order covariance. This measure is an extension of covariance for infinite variance stable distributions. In the literature one can find interesting applications of this statistic, see e.g. [22, 34, 46]. The proposed technique is straightforward and it can be easily adapted to a modified Yule-Walker equation. It can be applied to general autoregressive processes with stable distribution, however, we demonstrate its effectiveness for an autoregressive model with order 1, as the discrete version of the Ornstein-Uhlenbeck process.

In the literature one can find different approaches used in the problem of the parameters' estimation for the stable Ornstein-Uhlenbeck process. We only mention here the selected approaches, such as the trajectory fitting method and the weighted least squares approach [14, 15, 45] as well as methods based on the discrete version of the Ornstein-Uhlenbeck process [8, 28, 47]. The proposed approach is a new idea which extends the existing methods.

The rest of the paper is structured as follows: In section 2 the stable distribution is recalled. Moreover, we describe the alternative measures of dependence and their basic properties. Section 3 consists of the definition of the O-U process with Gaussian and stable distribution. Then, the novel estimator of the O-U process parameters is proposed in section 4. The performance of this estimator is tested on simulated data (section 5). The last section summarises the article and contains the conclusions.

2. The stable distribution. The stable distribution can be considered as an extension of the Gaussian one. This distribution is characterized by four parameters: the stability index α, the scale parameter σ, the skewness parameter β and the shift parameter μ. The theory of stable distribution was introduced by Paul Lévy and Aleksander Khinchine in the 1920s and 1930s. These random variables can be defined in several ways, one of the most useful definition is based on the characteristic function. Let X be a random variable, it is said to have a stable distribution ($X \sim S(\alpha, \sigma, \beta, \mu)$) if its characteristic function has the following form [35]:

$$\phi_X(t) = \mathbb{E}e^{itX} = \begin{cases} \exp\left\{-\sigma^\alpha |t|^\alpha \left(1 - i\beta \operatorname{sign}(t) \tan\left(\frac{\pi\alpha}{2}\right)\right) + i\mu t\right\} & \alpha \neq 1, \\ \exp\left\{-\sigma|t| \left(1 + i\beta \frac{2}{\pi} \operatorname{sign}(t) \ln\left(|t|\right)\right) + i\mu t\right\} & \alpha = 1, \end{cases} \quad (1)$$

where parameters $\alpha \in (0, 2]$, $\beta \in [-1, 1]$, $\sigma > 0$ and $\mu \in \mathbb{R}$.

In the case of the stability index equal to $\alpha = 2$ and the skewness parameter $\beta = 0$, the stable distribution is reduced to the Gaussian one. Furthermore, the stability index informs about the property of the tail of the distribution.

Indeed, in the case of $\alpha < 2$ the considered distribution has heavy tails, which means that there is a higher probability to obtain observation far from the mean. It is worth mentioning that, usually the probability density functions for the stable distribution are not given in the closed form. There are a few exceptions like Gaussian distribution ($S(2,\sigma,0,\mu)$, Cauchy distribution ($S(1,\sigma,0,0)$) or Lévy distribution ($S(1/2,\sigma,1,\mu)$). One can observe that the sum of random variables from the stable distribution is also stable. Let us assume that X_1 and X_2 are independent and $X_i \sim S(\alpha,\sigma_i,\beta_i,\mu_i)$ and $i=1,2$. Then,

$$X_1 + X_2 \sim S(\alpha,\sigma,\beta,\mu),$$

where $\sigma = (\sigma_1^\alpha + \sigma_2^\alpha)^{1/\alpha}$, $\beta = \frac{\beta_1\sigma_1^\alpha + \beta_2\sigma_2^\alpha}{\sigma_1^\alpha + \sigma_2^\alpha}$ and $\mu = \mu_1 + \mu_2$.

This fact easily follows from the property of independence and the definition of the stable distribution. Finally, we can formulate another important property of the stable distribution. Let $X \sim S(\alpha,\sigma,\beta,\mu)$ and $a,b \in \mathbb{R}$ are some constants, then:

$$\begin{aligned}
&aX + b \sim S(\alpha, |a|\sigma, \operatorname{sign}(a)\beta, a\mu + b) && \alpha \neq 1,\\
&aX + b \sim S(\alpha, |a|\sigma, \operatorname{sign}(a)\beta, a\mu - 2/\pi a\sigma\beta \ln|a| + b) && \alpha = 1.
\end{aligned}$$

This fact is also satisfied due to the definition and properties of the characteristic function. Finally, we would like to define the symmetric stable distribution ($S\alpha S$). Let X be a real-valued random variable, then it follows $S\alpha S$ distribution when its characteristic function is given by:

$$\phi_X(t) = \exp(-\sigma^\alpha |t|^\alpha).$$

In this case, the characteristic function depends only on two parameters (α and σ), remaining parameters are equal to zero. For the stable distribution with $\alpha < 2$ the variance and second moment do not exist. However, the Fractional Lower Order Moments (FLOM) can be introduced. Let X be a random variable, the FLOM of order $0 < p < 2$ is defined as:

$$\mathrm{FLOM}(X,p) = \mathbb{E}|X|^p.$$

It is worth mentioning, that FLOM exists for each order $p < \alpha$. Therefore, this statistic is well-defined for the stable distribution. Furthermore, for $S\alpha S$ random variable FLOM satisfies the following equation [35]:

$$\mathrm{FLOM}(X,p) = C(p,\alpha)\sigma^{\frac{p}{\alpha}} \quad \text{for } 0 < p < \alpha,$$

where

$$C(p,\alpha) = \frac{2^{p+1}\Gamma(\frac{p+1}{2})\Gamma(\frac{-p}{\alpha})}{\alpha\sqrt{\pi}\Gamma(\frac{-p}{2})},$$

where $\Gamma(\cdot)$ is a Gamma function.

The stability index is also related to the rate of the distribution tail decay. For $\alpha < 2$ the distribution exhibits a power-law behaviour:

$$\begin{cases} \lim_{x\to\infty} x^{\alpha} P\{X > x\} = C_{\alpha}\frac{1+\beta}{2}\sigma^{\alpha}, \\ \\ \lim_{x\to\infty} x^{\alpha} P\{X < -x\} = C_{\alpha}\frac{1-\beta}{2}\sigma^{\alpha}, \end{cases} \tag{2}$$

where $C_{\alpha} = \left(\int_0^{\infty} x^{-\alpha}\sin(x)dx\right)^{-1} = \frac{1}{\pi}\Gamma(\alpha)\sin(\frac{\pi\alpha}{2})$.

2.1. Measures of dependence for infinite variance processes. One of the most common measures of dependence are undoubtedly covariance and correlation. They can be used for any distribution with finite variance and second moment (e.g. Gaussian). Therefore, they are not well defined for the stable distribution with $\alpha < 2$. In such case the alternative measures should be used.

2.1.1. Covariation This measure is well defined for $S\alpha S$ random variables with the stability index $\alpha > 1$. Let X_1 and X_2 follow $S\alpha S$ distribution with $\alpha > 1$. Furthermore, Γ_s is a spectral measure of the random vector (X_1, X_2). Then, the covariation $CV(X_1, X_2)$ is defined as [35]:

$$CV(X_1, X_2) = \int_{S_2} s_1 s_2^{<\alpha-1>}\Gamma_s(ds), \tag{3}$$

where S_2 is the unit sphere in $\mathbb{R}$ and $a^{<p>} = |a|^p \operatorname{sign} a$ is a signed power. Covariation can be also defined using the joint moment of random variables X_1, X_2 of order $p \in (1, \alpha)$. The formula (3) can be equivalently written [10]:

$$CV(X_1, X_2) = p\frac{\mathbb{E}(X_1 X_2^{<p-1>})\sigma_{X_2}^{\alpha}}{\mathbb{E}|X_2|^p}, \tag{4}$$

where σ_{X_2} is a scale parameter for random variable X_2. Let us present some properties of this measure. It is worth mentioning, that covariation is not symmetric. Furthermore, it is linear and additive with respect to the first argument. Let us assume that X_1, X_2, Y are $S\alpha S$ random variables and parameters $a, b \in \mathbb{R}$. Then, following equation is satisfied:

$$CV(aX_1 + bX_2, Y) = aCV(X_1, Y) + bCV(X_2, Y).$$

On the other hand, the additivity in the second argument $CV(X, Y_1 + Y_2) = CV(X, Y_1) + CV(X, Y_2)$ is satisfied if and only if random variables Y_1 and Y_2 are independent. Furthermore, the covariation has the scaling property. Indeed, for $S\alpha S$ random variables X and Y and real numbers a, b, the following condition holds [35]:

$$CV(aX, bY) = ab^{<\alpha-1>}CV(X, Y).$$

One of the main properties of the covariation is formulated for independent random variables. Let us assume that X and Y are independent $S\alpha S$, then $CV(X,Y)=0$. However, for $1<\alpha<2$ it is possible to have $CV(X,Y)=0$ with dependent X and Y. Finally, for $\alpha=2$ the covariation is equal to half of the covariance. Moreover, for $\alpha>1$, the covariation induces a norm on the linear space of jointly $S\alpha S$ random variables. If X is a $S\alpha S$ random variable with $\alpha>1$, then:

$$||X||_\alpha = (CV(X,X))^{1/\alpha}.$$

The autocovariation of the stationary process $\{X(t)\}$ for lag k is defined as the covariation of random variables $X(t)$ and $X(t-k)$:

$$CV(X(t),X(t-k)) = \frac{\mathrm{E}(X(t)X(t-k)^p \operatorname{sign}(X(t-k))}{\sigma^\alpha_{X(t-k)}\,\mathrm{E}\,|X(t-k)|}.$$

In the literature, one can find several methods for estimation of the covariation and the autocovariation [10, 21].

2.1.2. Codifference Another alternative measure of dependence that can be considered instead of classical covariance is the codifference [35]. Let X_1 and X_2 be the infinitely divisible random variables, then the codifference is defined as:

$$\begin{aligned} CD(X_1,X_2) = -\quad & \log(\mathbb{E}\exp(i(X_1-X_2)) + \log(\mathbb{E}\exp(iX_1)) \\ & + \log(\mathbb{E}\exp(-iX_2)). \end{aligned}$$

On the other hand the codifference can be also represented alternatively as:

$$CD(X_1,X_2) = \quad \log\left(\frac{\Phi_{X_1-X_2}(1)}{\Phi_{X_1}(1)\Phi_{X_2}(1)}\right),$$

where $\Phi_{X_u}(t)$ is the characteristic function of the random variable X_u for point t. Using this formula one can see that the codifference is always well defined for all infinitely divisible random variables (e.g. $S\alpha S$). Moreover, $CD(X_1,X_2)=CD(X_2,X_1)$, for symmetric random variables. Finally, there is a relation between the codifference and the covariation. Let us assume that $S\alpha S$ with $\alpha>1$, then the codifference can be expressed by the means of the covariation norm, namely:

$$CD(X_1,X_2) = ||X_1||^\alpha_\alpha + ||X_2||^\alpha_\alpha - ||X_1-X_2||^\alpha_\alpha. \tag{5}$$

Furthermore, for independent random variables X and Y the codifference is equal to zero $CD(X,Y)=0$. It is worth mentioning that for Gaussian random variables codifference is reduced to covariance. In particular, for $X,Y\sim S(2,\sigma,\beta,\mu)$ we obtain $CD(X,Y)=Cov(X,Y)$. It is worth mentioning that, this measure is more general and is well defined for any infinitely divisible process.

The autocodifference of the stationary process $\{X(t)\}$ for lag k is defined as the codifference of random variables $X(t)$ and $X(t-k)$:

$$CD(X(t), X(t-k)) = \log\left(\frac{\Phi_{X(t)-X(t-k)}(1)}{\Phi_{X(t)}(1)\Phi_{-X(t-k)}(1)}\right). \tag{6}$$

The estimator of codifference is described for instance in [44].

2.1.3. Fractional lower order covariance (FLOC). FLOC is a natural extension of the covariance. It can be computed for $S\alpha S$ random variables with the stability index $\alpha \leq 2$. Let X_1 and X_2 be $S\alpha S$ random variables, then the FLOC is defined as follows [23]:

$$\mathrm{FLOC}_{X_1,X_2}(A,B) = \mathbb{E}[X_1^{<A>} X_2^{<B>}], \tag{7}$$

where parameters $A+B<\alpha$ and $A,B \geq 0$. Therefore, FLOC can be computed for any $S\alpha S$ random variable, even for stability indexes smaller than 1. However, this statistic depends on parameters A, B and each set of parameters changes the value of FLOC. Furthermore, for Gaussian random variables, FLOC reduces to covariance when $A=B=1$. Three properties of FLOC are presented below, these results were calculated by Authors, thus the proofs are also included. The main property of the FLOC is following.

Fact 2.1 *Let us assume that X, Y are independent $S\alpha S$ random variables with the stability index α. Then, for all $A+B<\alpha$ we obtain $FLOC_{X,Y}(A,B) = 0$.*

Proof

$$\mathrm{FLOC}_{X,Y}(A,B) = \mathbb{E}[X^{<A>}]\mathbb{E}[Y^{<B>}] =$$
$$= \left(\mathbb{E}[|X|^A] * \frac{1}{2} - \mathbb{E}[|X|^A] * \frac{1}{2}\right)\left(\mathbb{E}[|Y|^B] * \frac{1}{2} - \mathbb{E}[|Y|^B] * \frac{1}{2}\right) = 0$$

Furthermore, it can be shown that for some real parameter c FLOC has scaling property:

Fact 2.2 *Let us assume that X, Y are random variables with the stability index α and $c \in \mathbb{R}$. Then, for all $A+B<\alpha$ the following formulas hold:*

$$\begin{cases} FLOC_{cX,Y}(A,B) = c^{<A>} FLOC_{X,Y}(A,B) \\ \\ FLOC_{X,cY}(A,B) = c^{<B>} FLOC_{X,Y}(A,B). \end{cases} \tag{8}$$

Proof We are going to prove only the first equation from system (8). The second one follows exactly the same. We have the following:

$$\begin{aligned} \mathrm{FLOC}_{cX,Y}(A,B) &= \mathbb{E}[(\mathrm{sign}(cX)|cX|^A Y^{<B>}] \qquad (9) \\ &= \begin{cases} c^A \mathbb{E}[(\mathrm{sign}(X)|X|^A Y^{<B>}] & c>0 \\ -c^A \mathbb{E}[(\mathrm{sign}(X)|X|^A Y^{<B>}] & c<0 \end{cases} \\ &= c^{<A>} \mathrm{FLOC}_{X,Y}(A,B). \end{aligned}$$

FACT 2.3 *Let us assume that X, Y_1, Y_2 are random variables with the stability index α. Furthermore, Y_1 and Y_2 are independent. Then, for all $A+B<\alpha$:*

$$FLOC_{X,Y_1+Y_2}(A,B) = FLOC_{X,Y_1}(A,B) + FLOC_{X,Y_2}(A,B).$$

The proof for this fact is analogous to covariation, which is derived in Property 2.7.15 in [35]. It is worth mentioning that the independence of Y_1 and Y_2 is crucial here. In the case of dependent variables the above fact is not satisfied.

Moreover, the autoFLOC of the stationary process $\{X(t)\}$ for lag k is defined as FLOC of random variables $X(t)$ and $X(t-k)$:

$$\mathrm{FLOC}(k,A,B) = \mathrm{FLOC}_{X(t),X(t-k)}(A,B). \tag{10}$$

Finally, we would like to introduce the estimator of the autoFLOC. It is derived directly from equation (7). Let us assume that

$$\mathbf{X} = \{X(1), X(2), \ldots X(N)\}$$

is a trajectory of length N, then the estimator of FLOC based on $\mathbf{X}$ is defined as [23]:

$$\widehat{\mathrm{FLOC}}(k,A,B) = \frac{\sum_{n=L_1}^{L_2} |X(n)|^A |X(n+k)|^B \,\mathrm{sign}[X(n)X(n+k)]}{L_2 - L_1}, \tag{11}$$

where $L_2 = \min(N, N-k)$ and $L_1 = max(0,-k)$.

3. The Ornstein-Uhlenbeck process The Ornstein-Uhlenbeck process can be defined as a stationary solution of the Langevin equation with respect to Brownian motion. Therefore, it satisfies the following equation:

$$dX(t) = \theta(\mu - X(t))dt + \sigma dB(t), \tag{12}$$

where $\theta, \sigma > 0$, $\mu \in \mathbb{R}$ and $\{B(t)\}$ is a Brownian motion. One of the main properties of this process is a mean reversion. In a significant long term, the process will have values around the long term mean. For instance, when the process is below the equilibrium level, the drift is positive, otherwise, it is negative and pulls the process down to reach the mean. As it was mentioned, the O-U process is especially popular in financial mathematics and it is typically used to model interest rates.
One can obtain the unique solution of equation (12):

$$X(t) = X(0)e^{-\theta t} + \mu(1-e^{-\theta t}) + \sigma \int_0^t e^{-\theta(t-s)} dB(s). \tag{13}$$

Therefore, $\{X(t)\}$ is a Gaussian process. Moreover, for the fixed value of the starting parameter $X(0)$ the distribution tends ($t \to \infty$) to the stationary distribution $N(\mu, \frac{\sigma^2}{2\theta})$.

The approach proposed in this paper is based on the discrete version of the O-U process, which is based on the Euler scheme discretization and takes the form:

$$X(t+\Delta t) - X(t) = \theta(\mu - X(t))\Delta t + \sigma\sqrt{\Delta}B_t, \tag{14}$$

where $t = 0, \Delta t, \ldots,$ Δt is a time step and $B_t = B(t+\Delta t) - B(t)$. Taking Δt equal to 1 and $\mu = 0$ we obtain the time series AR(1).

The problem of discretization of the continuous-time processes is widely discussed in the literature. One can find differences related related to the continuous and the discrete version of the given process and the influence of the discretization, for instance, on the estimation results [1, 13, 27, 30, 36, 39, 45]. The proposed discretization is the simplest one. However, this approach is not new in the literature. As it was mentioned in the Introduction, in the literature one can find few methods of the Ornstein-Uhlenbeck process parameters estimation which are based on its discrete version.

3.1. The Ornstein-Uhlenbeck process with stable distribution. The classical O-U process assumes that data follows the Gaussian distribution. However, in real cases very often data reveals the property of heavy tails, which is not related to Gaussian behaviour. Therefore, in the literature, one can find the extension of the O-U process with the stable distribution [33]. In such cases, some significant jumps and changes would be present in the data. In order to apply this, the distribution in equation (12) has to be changed. The O-U process driven by stable Lévy process is defined by the following stochastic differential equation:

$$dX(t) = \theta(\mu - X(t))dt + \sigma dZ(t), \tag{15}$$

where $\{Z(t)\}$ is a Lévy process, i. e. process with independent, stationary and $S\alpha S$ increments. The Euler discretization scheme can be defined similarly to the classical case. It is given by the formula:

$$X(t+\Delta t) - X(t) = \theta(\mu - X(t))\Delta t + \sigma(\Delta t)^{1/\alpha}Z_t, \tag{16}$$

where $t = 0, \Delta t, \ldots$ and $Z_t = Z(t+\Delta t) - Z(t)$. Taking Δt equal to 1 and $\mu = 0$ we obtain the time series AR(1) with the stable distribution. Indeed, we obtain the following expression for AR(1):

$$X_t - \phi X_{t-1} = \sigma Z_t, \tag{17}$$

where $\phi = 1 - \theta$ and $\{Z_t\}$ constitute sequence of independent, identically distributed (i.i.d.) $S\alpha S$ random variables. For the AR time series with stable innovations several estimation methods are described in the literature e.g. modified Yule-Walker [21], normalized covariation [10] and Whittle method [2]. Under such assumptions it is proved in [35] that the system satisfying equation (17) has a unique, stationary solution of the following form:

$$X_t = \sum_{i=0}^{\infty} c_i Z_{t-i}, \quad t \in \mathbb{Z}, \quad Z_t \sim S\alpha S, \quad a.s \tag{18}$$

where real c_i's satisfy $|c_i| < Q^{-1}$, where $Q > 1$ if and only if polynomial $\phi(z)$ has no roots in the unit disc $z : |z| \leq 1$. The sequence $\{X_t\}$ is then a stationary process and has $S\alpha S$ distribution.

4. Estimation of the stable O-U parameters based on FLOC. The main aim of the paper is to present a new method of estimation for parameters of the O-U process with the stable distribution based on fractional lower order covariance. The new method is a modification of the classical Yule-Walker method. However, instead of the autocovariance function, the FLOC is used. The need for the substitution is lack of the theoretical finite second moment for random variables with $S\alpha S$ distribution. The FLOC is well defined for the class of such variables, therefore it can be successfully used in that case. In this article, we are going to analyse the discretized version of the O-U process, which is an AR(1) model. The estimation procedure of this process consists of several steps.

THEOREM 1 Let us assume that the vector $\{X_1, ..., X_N\}$ is a realization of a discrete version of the O-U process given by equation (17) with the sequence $\{Z_t\}$ of an i.i.d. $S\alpha S$ random variables. Moreover, we assume $\alpha > 1$ and $\sigma = 1$. Then, the estimator of the parameter ϕ is given by the following formula:

$$\hat{\phi} = \frac{\sum_{n=1}^{N-1} |X_n|^A |X_{n+1}|^B \operatorname{sign}[X_n X_{n+1}]}{\sum_{n=1}^{N-1} |X_n|^{A-1} |X_{n+1}|^{B+1}}, \tag{19}$$

where A and B are parameters satisfying inequalities $0 < A + B < \alpha$.

PROOF Let us assume that $S_t = \operatorname{sign}(X_t)$ and

$$FLOC(k, A, B) = E[X_t^{<A>} X_{t-k}^{<B>}].$$

Let us multiply the equation (17) by $S_t S_{t-1}$

$$X_t S_t S_{t-1} - \phi X_{t-1} S_t S_{t-1} = Z_t S_t S_{t-1}$$

It is clear that $X_t S_t = |X_t|$:

$$|X_t| S_{t-1} - \phi |X_{t-1}| S_t = Z_t S_t S_{t-1}$$

Let us multiply the above equation by: $|X_t|^{a-1} |X_{t-1}|^b$:

$$|X_t|^a |X_{t-1}|^b S_{t-1} - \phi |X_t|^{a-1} |X_{t-1}|^{b+1} S_t = Z_t S_t S_{t-1} |X_t|^{a-1} |X_{t-1}|^b$$

Let us multiply the above equation by S_t

$$|X_t|^a|X_{t-1}|^b S_{t-1}S_t - \phi|X_t|^{a-1}|X_{t-1}|^{b+1}S_t^2 = Z_t S_t^2 S_{t-1}|X_t|^{a-1}|X_{t-1}|^b$$

It is clear that $S_t^2 = 1$:

$$|X_t|^a|X_{t-1}|^b S_{t-1}S_t - \phi|X_t|^{a-1}|X_{t-1}|^{b+1} = Z_t S_{t-1}|X_t|^{a-1}|X_{t-1}|^b$$

By taking the expectation we obtain:

$$\mathbb{E}[|X_t|^a|X_{t-1}|^b S_{t-1}S_t] - \phi\mathbb{E}[|X_t|^{a-1}|X_{t-1}|^{b+1}] = \mathbb{E}[Z_t S_{t-1}|X_t|^{a-1}|X_{t-1}|^b]$$

Let us use the representation of $\{X_t\}$ given by equation (18):

$$\begin{aligned} & \mathbb{E}[|X_t|^a|X_{t-1}|^b S_{t-1}S_t] - \phi\mathbb{E}[|X_t|^{a-1}|X_{t-1}|^{b+1}] = \\ = \; & \mathbb{E}[Z_t \left|\left(\sum_{i=0}^{\infty} c_i Z_{t-i-1}\right)\right|^{a-1} |(\sum_{j=0}^{\infty} c_j Z_{t-j-1})|^b S_{t-1}]. \end{aligned}$$

The left side of the above equation can be rewritten as follows:

$$\begin{aligned} & \mathbb{E}[|X_t|^a|X_{t-1}|^b S_{t-1}S_t] - \phi\mathbb{E}[|X_t|^{a-1}|X_{t-1}|^{b+1}] \\ & \qquad = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \mathbb{E}[Z_t|c_i Z_{t-i}|^{a-1}|c_j Z_{t-j-1}|^b S_{t-1}]. \end{aligned}$$

The random variables Z_t, Z_{t-i} and Z_{t-j-1} are not independent if and only if $t = t-i = t-j-1$. It is clear that the above equality is always false. Therefore, the random variables are independent and any of them has $EZ_t = \mu = 0$, if the assumption that $\alpha > 1$ and $\{Z_t\} \sim S\alpha S$ is valid. After applying that remark and the definition of FLOC we get an explicit form for the parameter ϕ:

$$\phi = \frac{\mathbb{E}[|X_t|^a|X_{t-1}|^b S_{t-1}S_t]}{\mathbb{E}[|X_t|^{a-1}|X_{t-1}|^{b+1}]} = \frac{FLOC(1,a,b)}{\mathbb{E}[|X_t|^{a-1}|X_{t-1}|^{b+1}]}. \tag{20}$$

Then, after substitution of the theoretical FLOC with its estimator and theoretical fractional moments with empirical ones we get the estimator of parameter ϕ.

Additionally, it is worth mentioning that the estimator of parameter θ can be easily calculated as $\hat{\theta} = 1 - \hat{\phi}$.

5. Monte Carlo simulations.

5.1. Optimal A, B parameters for the estimator of the stable O-U process parameter with fixed θ and α In order to test the effectiveness of O-U process parameters' estimators, the analysis of the simulated data was performed. The FLOC-based estimation using Theorem 1 was done. We analyse the O-U process which satisfies the following equation:

$$dX(t) = -0.9X(t)dt + Z(t), \tag{21}$$

where $\{Z(t)\}$ is the stable Lévy process. The Euler discretization of equation (21) takes the form:

$$X_t - 0.1X_{t-1} = Z_t, \tag{22}$$

where $\{Z_t\}$ constitutes a sample of i.i.d. random variables from $S\alpha S$ distribution with the stability parameter $\alpha = 1.9$ and the scale parameter $\sigma = 1$. Using the Monte Carlo method we generate the trajectory of length 1000 of the model given by equation (21) and estimate the discrete time O-U process parameter using FLOC-based estimator. The sample trajectory of such a model is presented in Fig. 1.

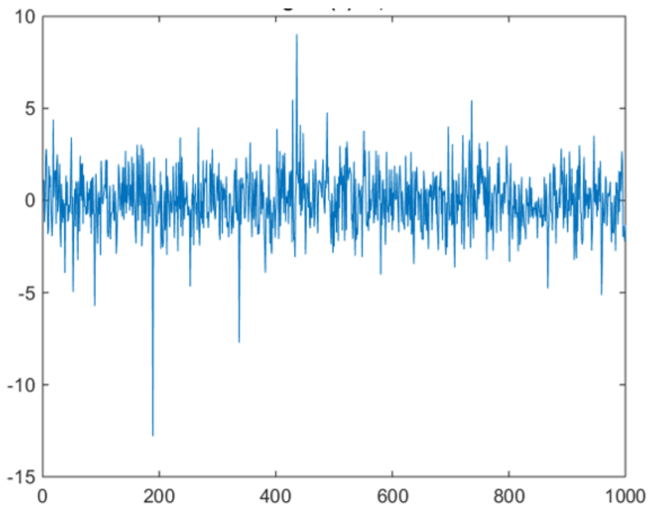

Figure 1: A sample trajectory of the discrete time O-U process model given by equation (21)

In order to find the parameters A and B which will be the most optimal the estimations for different pairs were performed. We fix the values of $\theta = 0.9$ and $\alpha = 1.9$. Then, for each pair of $A, B = 0, 0.1, \dots, \alpha,\ \ A + B < \alpha$ the parameter θ is estimated. For each pair of parameters A, B 10000 Monte Carlo simulations were performed and then the means of estimated values were calculated. The results are presented in Tab. 1.

Table 1: The table of means of 10000 estimated values $\hat{\theta}$ for the stable time O-U process with different values of parameters A and B, $\theta = 0.9$, $\sigma = 1$ and $\alpha = 1.9$.

A/B	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7
0	0.988	0.988	0.988	0.988	0.989	0.989	0.989	0.989	0.989	0.989	0.989	0.99	0.99	0.99	0.99	0.991	0.991	0.99
0.1	0.983	0.983	0.983	0.983	0.983	0.984	0.984	0.984	0.984	0.985	0.985	0.985	0.986	0.986	0.987	0.987	0.987	0.99
0.2	0.976	0.976	0.976	0.977	0.977	0.977	0.978	0.978	0.979	0.979	0.98	0.98	0.981	0.981	0.982	0.983	0.983	
0.3	0.967	0.968	0.968	0.968	0.969	0.969	0.97	0.971	0.971	0.972	0.973	0.974	0.975	0.976	0.977	0.977		
0.4	0.957	0.958	0.958	0.959	0.96	0.96	0.961	0.962	0.963	0.964	0.965	0.967	0.968	0.969	0.97			
0.5	0.947	0.947	0.948	0.949	0.949	0.95	0.952	0.953	0.954	0.956	0.957	0.959	0.96	0.962				
0.6	0.935	0.935	0.936	0.937	0.938	0.939	0.941	0.942	0.944	0.946	0.948	0.95	0.952					
0.7	0.922	0.923	0.924	0.925	0.926	0.928	0.929	0.931	0.933	0.936	0.938	0.94						
0.8	0.908	0.909	0.91	0.911	0.913	0.915	0.917	0.919	0.921	0.924	0.927							
0.9	0.893	0.894	0.895	0.896	0.898	0.9	0.902	0.905	0.908	0.911								
1	0.876	0.877	0.878	0.879	0.881	0.883	0.886	0.889	0.893									
1.1	0.857	0.857	0.858	0.859	0.861	0.863	0.867	0.87										
1.2	0.834	0.834	0.834	0.835	0.837	0.84	0.843											
1.3	0.806	0.805	0.805	0.805	0.807	0.81												
1.4	0.773	0.77	0.769	0.769	0.77													
1.5	0.731	0.728	0.725	0.724														
1.6	0.682	0.676	0.671															
1.7	0.621	0.614																
1.8	0.549																	

The best pairs A, B (for which the means of estimated values are closest to $\theta = 0.9$) were highlighted in the Tab. 1. In particular, for $A = 0.9$ and $B = 0.5$ we obtain the most appropriate results.

5.2. Optimal A, B for the stable O-U process with random θ fixed α A similar study was performed for different values of the θ parameter in order to check how sensitive are A, B for different values of θ. Therefore, the following model was considered:

$$dX(t) = -\theta X(t)dt + Z(t). \tag{23}$$

where $\{Z(t)\}$ is a stable Lévy process. In Tab. 2 the obtained results for $0 < \theta < 1$ and in Tab. 3 for $1 < \theta < 2$ are presented. Moreover, there are also illustrated mean errors (ME) calculated for any θ from the following formula:

$$\mathrm{ME} = \left| \frac{1}{n} \sum_{i=1}^{n} \hat{\theta}_i - \theta \right|,$$

where n is equal to 1000 and it is the number of Monte Carlo simulations, θ is the theoretical value of the parameter and $\hat{\theta}_i$ is its i-th estimator. Similarly, the mean percentage error (MPE) is calculated using the following formula:

$$\mathrm{MPE} = \left| \frac{\frac{1}{n} \sum_{i=1}^{n} \hat{\theta}_i - \theta}{\theta} \right|.$$

We are looking for the pair of parameters $A, B = 0, 0.05, \ldots, \alpha,\ A + B < \alpha$, which minimise the ME and MPE. In Tab. 2 and Tab. 3 for each value of θ the most suitable A, B, the estimator $\hat{\theta}$, ME and MPE are presented. One can observe that both ME and MPE are small and the estimators are close to the real values of parameter θ. In particular, the optimal values of A, B are varying. However, the optimal A is close to 1 for all values of θ.

Table 2: The table of estimated values $\hat{\theta}$ for the stable O-U process for different values of parameters A and B, $0 < \theta < 1$, $\sigma = 1$ and $\alpha = 1.7$

θ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
A	0.95	0.9	0.95	0.9	0.9	0.85	0.8	0.75	0.7
B	0.55	0.1	0.7	0.4	0.5	0.3	0.1	0	0
$\hat{\theta}$	0.10001	0.20033	0.29984	0.39859	0.50073	0.59996	0.69908	0.79995	0.89978
ME	0.00001	0.00033	0.00016	0.00141	0.00073	0.00004	0.00092	0.00005	0.00022
MPE	0.00009	0.00164	0.00054	0.00353	0.00146	0.00007	0.00131	0.00007	0.00024

5.3. Optimal A, B for stable O-U process with random θ and α In a real case, the values of θ are unknown, thus we would like to test the most suitable A, B for any θ and α. Parameters A, B have to be fixed in the estimation procedure, thus the following test is proposed. Indeed, in

Table 3: The table of estimated values $\hat{\theta}$ for the stable O-U process for different values of parameters A and B, $1 < \theta < 2$, $\sigma = 1$ and $\alpha = 1.7$

θ	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
A	0.75	0.75	0.8	0.85	0.85	0.9	0.95	0.9	0.95
B	0.25	0	0.1	0.3	0.15	0.45	0.7	0.1	0.55
$\hat{\theta}$	1.10060	1.20048	1.29992	1.40038	1.49882	1.59836	1.70062	1.80002	1.90009
ME	0.00060	0.00048	0.00008	0.00038	0.00118	0.00164	0.00062	0.00002	0.00009
MPE	0.00055	0.00040	0.00006	0.00027	0.00079	0.00103	0.00037	0.00001	0.00005

the simulation procedure, the parameter θ and the stability index α were drawn from the uniform distribution. It was assumed that $0 < \theta < 2$, $\theta \neq 1$ and $1.5 < \alpha < 1.9$. Therefore, the most appropriate parameters $A, B = 0, 0.05, \ldots, \alpha$, $A + B < \alpha$ are going to be chosen. In order to choose the optimal A, B, the mean absolute error (MAE) is calculated using the following formula:

$$\mathrm{MAE} = \frac{\sum_{i=1}^{n} \left|\hat{\theta}_i - \theta\right|}{n}, \tag{24}$$

where n is the number of Monte Carlo simulation. The results of the test with drawn θ and α parameters are presented in Tab. 4. In the procedure 1000 Monte Carlo simulations were performed and three best sets of A, B are illustrated. One can observe that parameter A should be close to 1 and parameter B is around 0.5. It is worth mentioning that the MAE is small and the proposed estimator can be used even though we do not know the theoretical values of α.

Table 4: The table of estimated values $\hat{\theta}$ for the stable O-U process for drawn values of $0 < \theta < 2$, $\theta \neq 1$, $1.5 < \alpha < 1.9$ and fixed $\sigma = 1$. The best three set of parameters A, B are presented.

	#1	#2	#3
A	0.9	0.9	0.9
B	0.55	0.5	0.45
MAE	0.023777	0.024321	0.025658

5.4. The optimal A,B selection for given α. In the case of real data analysis, it is crucial to properly select the A,B parameters. Indeed, A, B influence the estimation results. It has already been tested, which values of A,B should be used for data with unknown α (Tab. 4). However, it is possible to optimize the estimation procedure. Particularly, it is proposed to choose appropriate A, B parameters for given α. Let us assume that $\{X(t)\}$ is the O-U process with the stable distribution, given by equation (15). Then, this process $\{X(t)\}$ also has a stable distribution. Thus, in the first step, the parameter α can be estimated for $\{X(t)\}$. In the literature one can find many

Table 5: Parameters A, B, for which the estimator $\hat{\theta}$ has the smallest MSE, with α from different intervals.

$\alpha \in$	A	B	MSE	MAE
(1.1, 1.3)	0.75	0	0.0055	0.0549
(1.3, 1.5)	0.85	0.4	0.0018	0.0326
(1.5, 1.7)	0.9	0.55	0.0009	0.0229
(1.7, 1.9)	0.9	0.5	0.0009	0.0236

different estimators of stable distribution parameters i.e. the approximated maximum likelihood method [42], McCulloch's method [26] or the regression method [19]. It is sufficient to apply one of them and then the appropriate A, B parameters for an estimated α can be used. The simulation study for different α was performed and for each interval, the most appropriate A, B were selected. In Tab. 5 the results of the simulation study are presented. It was tested for $\alpha \in (1.1, 1.9)$, for each interval 1000 trajectories of the O-U process were simulated. In each case, the trajectory length was $N = 1000$. In each repetition, the α parameter was drawn from the appropriate interval. In this case, the additional error, namely mean squared error (MSE) is calculated using the following formula:

$$\text{MSE} = \frac{1}{n}\sum_{i=1}^{n}(\hat{\theta}_i - \theta_i)^2. \tag{25}$$

In Tab. 5 the parameters A, B with the smallest MSE and MAE are presented. It can be observed that the bigger α is, the higher values of A, B parameters should be used.

5.5. The comparison with another estimators of the O-U process parameters As it was mentioned, in the literature one can find several different approaches used for the estimation of the parameters of the O-U process with the stable distribution. In this section, the estimator based on FLOC measure is compared with two other estimators, which are proposed for the discrete version of the O-U process. Particularly, we compare the introduced approach with the method based on normalized covariation (NCV) [21] and the Whittle method [28]. Firstly, the MSEs for the three estimation methods are compared. In Tab. 6 the MSE for each method is presented. It was calculated based on simulated 1000 trajectories for each interval of α. In each case, the trajectory length was 1000. Moreover, in the method based on FLOC, we apply the A, B parameters, which are optimal for the corresponding α taken from Tab. 5. One can observe that the smallest value of MSE is obtained for the Whittle method. However, FLOC method outperforms the approach based on NCV. Furthermore, for $\alpha > 1.3$ the results for FLOC and Whittle methods are almost the same. Only for $\alpha \in (1.1,\ 1.3)$ the Whittle

Table 6: The MSE for a different estimation method, random $\alpha \in (1.1, 1.9)$ and $\phi \in (-1, 1)\backslash\{0\}$.

method \ $\alpha \in$	(1.1, 1.3)	(1.3, 1.5)	(1.5, 1.7)	(1.7, 1.9)
FLOC	0.0057	0.0019	0.0008	0.0009
Whittle	0.0004	0.0006	0.0005	0.0006
NCV	0.0240	0.0102	0.0037	0.0024

Table 7: The computational time (in seconds) for $M = 100$ trajectories of the O-U process with a different trajectory length N.

	FLOC(0.8, 0.2)	Whittle	NCV
$N = 100$	0.0071	0.2570	0.0028
$N = 200$	0.0122	0.2646	0.0029
$N = 300$	0.0177	0.2825	0.0032
$N = 400$	0.0232	0.2900	0.0035
$N = 500$	0.0285	0.2993	0.0038
$N = 600$	0.0338	0.3092	0.0040
$N = 700$	0.0398	0.3176	0.0044
$N = 800$	0.0453	0.3286	0.0048
$N = 900$	0.0502	0.3372	0.0049
$N = 1000$	0.0559	0.3499	0.0053
$N = 10000$	0.3066	1.1326	0.0324

method outperforms the other techniques.

Another important property of the estimator is the computational time. In order to test this, the simulation study was performed. We consider here different trajectory lengths of the O-U process based on stable distribution. For each case, we simulate 100 trajectories. In Tab. 7 the computational time is presented for 100 trajectories. The least complex method is the one based on NCV. However, the FLOC outperforms the Whittle method. In Fig. 2 the time in log scale is presented. The results are presented for $\alpha = 1.9$ and $\theta = 0.5$, however, we observe that the α and θ parameters have no influence on the computational time. The only parameter which influences the results is the trajectory length N. Although the Whittle method is more precise, it is more complex and its computational time is longer. On the other hand, the estimators based on the NCV and FLOC methods have simpler formulas and their computation is faster.

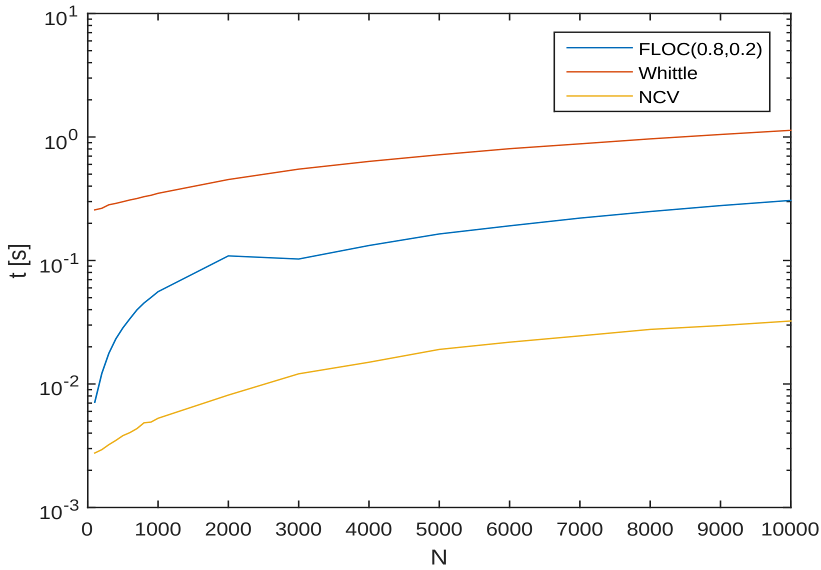

Figure 2: The computational time for a different sample length in log scale.

6. Conclusions. In this paper, we study the Ornstein-Uhlenbeck process based on the stable distribution. It arises after replacement of the ordinary Brownian motion in the classical Ornstein-Uhlenbeck process by the process of stationary independent increments with the stable distribution. The stable processes are widely discussed in the literature and have found various applications. However, their use implies that the classical estimation methods cannot be applied. In this paper, we propose a novel technique of the stable Ornstein-Uhlenbeck process parameters estimation. The new method is based on the fractional lower order covariance, one of the alternative measures of dependence adequate for infinite-variance processes. The presented study is the extension of the authors' previous research where the covariation was proposed as the base for the estimation of the stable autoregressive models. However, in contrast to the covariation-based technique, the proposed in this paper method is simple and the obtained estimator is given in the explicit form. The effectiveness of the proposed method is checked by the Monte Carlo simulations.

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Estymator bazujący na ułamkowych momentach dla procesu Ornsteina-Uhlenbecka z rozkładem stabilnym

Piotr Kruczek,Wojciech Żuławiński, Patrycja Pagacz, Agnieszka Wyłomańska

Streszczenie Proces Ornsteina-Uhlenbecka jest jednym z najbardziej popularnych procesów stochastycznych. Znalazł on wiele ciekawych praktycznych zastosowań. Należy jednak zwrócić uwagę, że klasyczny proces Ornsteina-Uhlenbecka nie może być zastosowany dla wielu danych rzeczywistych, ponieważ często pochodzą one z rozkładów ciężko- ogonwych, dla których nie istnieje drugi moment. W takich przypadkach niezbędna jest modyfikacja klasycznego modelu z wykorzystaniem rozkładu stabilnego. Z powodu zastosowania rozkładu stabilnego niezbędne jest użycie innej metody estymacji niż bazującej na autokowariancji. Zaproponowana została nowa metoda bazująca na ułamkowych momentach. Praca jest kontynuacją wcześniej otrzymanych rezultatów dla innej alternatywnej miary zależności, kowariacji. W pracy przypomniana została definicja stabilnego procesu Ornsteina-Uhlenbecka wraz z propozycją nowych estymatorów dla parametrów tego procesu. W celu sprawdzenia ich właściwości wykonane zostały symulacje Monte Carlo.

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Piotr Kruczek received MSc degree in Mathematics from Wrocław University of Science and Technology in 2015. Currently he is the PhD student. His main research interests are a stochastics process, time series and applied mathematics. References to his research papers can be found in MathSciNet under ID: 1192067.

Wojciech Żuławiński is Master's student of Applied Mathematics at Wrocław University of Science and Technology. His main research interests are a stochastic processes and time series analysis.

Patrycja Pagacz received BSc degree in Applied Mathematics from Wrocław University of Science and Technology in 2018 and BSc degree in Computer Science from WUST in 2019. In 2019 she started her master's degree in Computer Science. Her main research interests are time series analysis and machine learning.

Agnieszka Wyłomańska is a Professor of WUST (Wroclaw University of Science and Technology) at the Faculty of Pure and Applied Mathematics and a member of the Hugo Steinhaus Center for Stochastic Processes. Her area of interest relates to time series analysis, stochastic modeling and statistical analysis of real data (especially technical data related to mining industry, indoor air quality and financial time series). She is an author of more than 100 research papers from the area of the applied and industrial mathematics. She cooperates with industrial companies, especially from the mining industry. References to her research papers can be found in the European Mathematical Society, FIZ Karlsruhe, and the Heidelberg Academy of Sciences bibliography database known as zbMath under ai:wylomanska.agnieszka and in MathSciNet under ID: 738846.

Piotr Kruczek
Wrocław University of Science and Technology
Faculty of Pure and Applied Mathematics
Wybrzeże Wyspiańskiego 27, PL-50-370 Wrocław
E-mail: piotr.kruczek@pwr.edu.pl

Wojciech Żuławiński
Wrocław University of Science and Technology
Faculty of Pure and Applied Mathematics
Wybrzeże Wyspiańskiego 27, PL-50-370 Wrocław
E-mail: wojtek.zulawinski@gmail.com

Patrycja Pagacz
Wrocław University of Science and Technology
Faculty of Pure and Applied Mathematics
Wybrzeże Wyspiańskiego 27, PL-50-370 Wrocław
E-mail: pagacz.patrycja@gmail.com

Agnieszka Wyłomańska
Wrocław University of Science and Technology
Faculty of Pure and Applied Mathematics
Wybrzeże Wyspiańskiego 27, PL-50-370 Wrocław
E-mail: Agnieszka.Wylomanska@pwr.edu.pl

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