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ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
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K. WIERTELAK (Poznań)

A note on the remainder term of the prime-number formula for arithmetical progressions

1. Let us write

$$\psi(x, k, l) = \sum_{\substack{n \equiv l \pmod{k} \\ n \leqslant x}} \Lambda(n),$$

where k, l denote integers, $k \geqslant 1$, $0 < l \leqslant k$, $(l, k) = 1$.

Consider the remainder term

$$\tilde{\Delta}(x, k, l) \stackrel{\text{def}}{=} \psi(x, k, l) - \frac{x}{\varphi(k)}.$$

Using te idea of T. Tatuzawa we can prove (see [4] and [2], p. 354-360):

If $\prod_{\chi \bmod k} L(s, \chi) \neq 0$, $s \neq \beta_1$, *in the region*

$$\sigma \geqslant 1 - \frac{b_1}{\max\{\log k, \log^{\gamma}(|t|+3)\}}, \qquad 0 < \gamma \leqslant 1,$$

then

$$\tilde{\Delta}(x, k, l) = 0\left(\frac{x}{\varphi(k)}\exp\{-b_2 \log^{1/(1+\gamma)} x\}\right)$$

for

$$x \geqslant \exp(k^{2(1+\gamma)/\gamma}),$$

where $b_1, b_2, \ldots$ *are numerical constants.*

The subject of this note is to converse this theorem in the case $l = 1$ (see [7]). We prove the following

THEOREM. *If* $0 < \gamma \leqslant 1$,

$$\tilde{\Delta}(x, k, 1) < c_1 \frac{x}{\varphi(k)} \exp(-c_2 \log^{1/(1+\gamma)} x) \tag{1.1}$$

for

$$x \geqslant \exp(k^{2(1+\gamma)/\gamma}),$$

where c_1, c_2 *are constants depending only on* γ, *then* $\prod_{\chi \bmod k} L(s,\chi) \neq 0$ *in the region*

$$\sigma > 1 - \frac{1}{c_0}\left(\frac{c_2}{2}\right)^{1+\gamma}\frac{1}{\log^{\gamma} t}, \qquad t \geqslant \max\left(c_3, \exp\frac{c_2}{2}k^{2/\gamma}\right), \tag{1.2}$$

where $c_3, c_4, \ldots$ *are constants depending only on* c_1, c_2 *and* γ, *and* c_0 *is numerical.*

The case $k = 1$ was solved by P. Turán (see [5], [6]).

2. The proof of the theorem will be based on the following Turán's Theorem (see [5], p. 52):

Let $z_1, z_2, \ldots, z_h$ *be complex numbers such that*

$$|z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_h|, \; |z_1| \geqslant 1,$$

and let $b_1, b_2, \ldots, b_h$ *be any complex numbers.*

Then, if m *is positive and* $N \geqslant h$, *there exists an integer* ν *such that* $m \leqslant \nu \leqslant m+N$ *and*

$$|b_1 z_1^{\nu} + b_2 z_2^{\nu} + \ldots + b_h z_h^{\nu}| \geqslant \left(\frac{1}{48e^2}\,\frac{N}{2N+m}\right)^N \min_{1\leqslant j\leqslant h} |b_1 + b_2 + \ldots + b_j|.$$

We will use also the following lemmas:

I. *Let* k *be a positive integer*

$$a_n = \begin{cases} 1, & n \equiv 1 (\mathrm{mod}\, k), \\ 0, & n \not\equiv 1 (\mathrm{mod}\, k). \end{cases}$$

Then for $\sigma > 1$, $\xi > 1$

$$\Bigg|(-1)^r \sum_{n\geqslant\xi} \frac{a_n \Lambda(n) \log^{r+1}(n/\xi)}{n^s (r+1)!} + \frac{1}{\varphi(k)}\left(\sum_{\varrho} \frac{\xi^{\varrho-s}}{(\varrho-s)^{r+2}} - \frac{\xi^{1-s}}{(1-s)^{r+2}}\right)\Bigg| < b_4 \frac{\log\{k(|t|+2)\}}{\xi^{3/2}},$$

where r *is a positive integer and* ϱ *runs over all zeros of* $\prod_{\chi \bmod k} L(s,\chi)$ *in the strip* $0 \leqslant \sigma \leqslant 1$ (see [3]).

II. *Denoting by* $N(T)$ *the numbers of zeros of all* $L(s,\chi)$, $\chi \bmod k$ *in the region* $0 \leqslant \sigma < 1$, $|t| \leqslant T$ *we have*

$$N(T+1) - N(T) < b_1 k \log k(T+2),$$

where $k \geqslant 1$, $T \geqslant 0$, *and* b_1 *is a numerical constant* (see [2], Theorem (3.3), p. 220).

III. *If* $\sigma \geqslant \frac{1}{2}$, $t > e$, $k \geqslant 1$, *then*

$$|L(s, \chi)| < b_6 \log kt\{1 + (kt)^{1-\sigma}\},$$

(see [1]).

3. Let t be positive, a_n defined as above and

$$f_{N_1N_2}(k, t) = \sum_{N_1 \leqslant n \leqslant N_2} a_n \Lambda(n) \exp(-it \log n). \tag{3.1}$$

Then

$$a_n \Lambda(n) = \left(\tilde{\Delta}(n, k, 1) - \tilde{\Delta}(n-1, k, 1)\right) + \frac{1}{\varphi(k)}.$$

Hence

$$\begin{aligned} (3.2) \quad & |f_{N_1N_2}(k, t)| \\ & \leqslant \frac{1}{\varphi(k)} \left| \sum_{N_1 \leqslant n \leqslant N_2} \exp(-it \log n) \right| + \\ & + \left| \sum_{N_1 \leqslant n \leqslant N_2} \left(\tilde{\Delta}(n, k, 1) - \tilde{\Delta}(n-1, k, 1)\right) \exp(-it \log n) \right| \\ & = \frac{1}{\varphi(k)} I_1 + I_2. \end{aligned}$$

By partial summation, we get

$$I_2 \leqslant |\tilde{\Delta}(N_2, k, 1)| + |\tilde{\Delta}(N_1 - 1, k, 1)| + \\ + \sum_{N_1 \leqslant n \leqslant N_2 - 1} |\tilde{\Delta}(n, k, 1)| \left|1 - \exp\left(-it \log(1 + 1/n)\right)\right|.$$

Let t be such that for the above c_2 and γ

$$1 + t^2 \leqslant \exp\left\{\left(\frac{2}{c_2} \log t\right)^{1+\gamma}\right\}, \tag{3.3}$$

and N_1, N_2 such that

$$\exp\left\{\max\left(\left(\frac{2}{c_2} \log t\right)^{1+\gamma}, k^{(2+\gamma)/\gamma}\right)\right\} \leqslant N/2 \leqslant N_1 < N_2 \leqslant N. \tag{3.4}$$

Applying (1.1) and noticing that the right-hand side of (1.1) is increasing with $x \geqslant N/2$, we get

$$I_2 < c_4 \frac{Nt}{\varphi(k)} \exp(-c_2 \log^{1/(1+\gamma)} N).$$

From (3.4) we get

$$\exp(-c_2 \log^{1/(1+\gamma)} N) \leqslant 1/t^2.$$

Hence

$$I_2 < \frac{c_5}{\varphi(k)}\,\frac{N}{t}. \tag{3.5}$$

We have also

$$\frac{1}{\varphi(k)}\cdot I_1 < \frac{c_6}{\varphi(k)}\cdot\frac{N}{t} \tag{3.6}$$

(see [5], p. 152-153).

Using (3.5), (3.6) with (3.2) we infer that

$$|f_{N_1N_2}(k,t)| < \frac{c_8}{\varphi(k)}\cdot\frac{N}{t}. \tag{3.7}$$

Let us apply (3.7) to estimate from above the sum $|\sum_{N_1\leqslant n\leqslant N_2} a_n\Lambda(n)/n^s|$ in the region

$$1<\sigma\leqslant\tfrac{3}{2},\qquad t>2. \tag{3.8}$$

By partial summation from (3.7) we get

$$\Bigl|\sum_{N_1\leqslant n\leqslant N_2} a_n\frac{\Lambda(n)}{n^s}\Bigr| < \frac{c_9}{\varphi(k)}\,\frac{N^{1-\sigma}}{t}. \tag{3.9}$$

4. Choosing

$$\eta\geqslant\exp\Bigl\{\max\Bigl(\Bigl(\frac{2}{c_2}\log t\Bigr)^{1+\gamma},\ k^{2(1+\gamma)/\gamma}\Bigr)\Bigr\} \tag{4.1}$$

and using (3.9) with

$$N_1^j=\eta\cdot 2^j,\qquad N_2^j=\eta\cdot 2^{j+1},\qquad j=0,1,2,\ldots,$$

we get

$$\Bigl|\sum_{n\geqslant\eta} a_n\frac{\Lambda(n)}{n^s}\Bigr| < c_{10}\frac{\eta^{1-\sigma}}{\varphi(k)t(\sigma-1)}. \tag{4.2}$$

If

$$\xi\geqslant\exp\Bigl\{\max\Bigl(\Bigl(\frac{2}{c_2}\log t\Bigr)^{1+\gamma},\ k^{2(1+\gamma)/\gamma}\Bigr)\Bigr\} \tag{4.3}$$

and r denotes a positive integer, then multiplying (4.2) by $\eta^{-1}\cdot\log^r\eta\xi^{-1}$ and integrating over $\langle\xi,+\infty)$ we get

$$\Bigl|\sum_{n\geqslant\xi} a_n\frac{\Lambda(n)}{n^s}\int_\xi^n\frac{1}{\eta}\log^r\frac{\eta}{\xi}\,d\eta\Bigr| \leqslant \frac{c_{10}}{\varphi(k)t(\sigma-1)}\int_\xi^\infty\eta^{-\sigma}\log^r\frac{\eta}{\xi}\,d\eta.$$

Since

$$\int_\xi^\infty\eta^{-\sigma}\log^r\frac{\eta}{\xi}\,d\eta = r!\,\frac{\xi^{1-\sigma}}{(\sigma-1)^{r+1}},\qquad \int_\xi^n\frac{1}{\eta}\log^r\frac{\eta}{\xi}\,d\eta=\log^{r+1}\Bigl(\frac{n}{\xi}\Bigr)\cdot\frac{1}{r+1},$$

we have

$$\left|\sum_{n\geqslant\xi}\frac{a_n\Lambda(n)}{n^s}\log^{r+1}\frac{n}{\xi}\right|<\frac{c_{10}}{\varphi(k)}\cdot\frac{(r+1)!\,\xi^{1-\sigma}}{t(\sigma-1)^{r+1}}. \tag{4.4}$$

Applying I to (4.4) we get

$$\left|\sum_{\varrho}\frac{\xi^{\varrho-s}}{(\varrho-s)^{r+2}}-\frac{\xi^{1-s}}{(1-s)^{r+2}}\right|<\frac{c_{10}}{t}\,\frac{\xi^{1-\sigma}}{(\sigma-1)^{r+2}}+c_{11}\,\frac{k\log\{k(t+2)\}}{\xi^{3/2}}.$$

Since

$$\left|\frac{\xi^{1-s}}{(s-1)^{r+2}}\right|\leqslant\frac{\xi^{1-\sigma}}{t(\sigma-1)^{r+1}},$$

we have

$$\left|\sum_{\varrho}\frac{\xi^{\varrho-s}}{(\varrho-s)^{r+2}}\right|<c_{10}\,\frac{\xi^{1-\sigma}}{t(\sigma-1)^{r+2}}+c_{11}\frac{k\log\{k(t+2)\}}{\xi^{3/2}}. \tag{4.5}$$

From conditions (3.8), (4.3) it follows

$$\frac{1}{\xi}\cdot\frac{k\log k(t+2)}{\xi^{1/2}}<\frac{\xi^{1-\sigma}}{t}\cdot\frac{2\log\xi\cdot\exp\log^{1/2}\xi}{\xi^{1/2}}.$$

Hence from (4.5) we get the estimate

$$\left|\sum_{\varrho}\frac{\xi^{\varrho-s}}{(\varrho-s)^{r+2}}\right|<c_{12}\,\frac{\xi^{1-\sigma}}{t(\sigma-1)^{r+2}}. \tag{4.6}$$

5. If Theorem (1.1), (1.2) is false, then there exist such zeros $\varrho^*=\sigma^*+$ $+it^*$ for which

$$\sigma^*>1-\frac{1}{40}\left(\frac{c_2}{2}\right)^{1+\gamma}\cdot\frac{1}{\log^{\gamma}t^*},\qquad t^*\geqslant\max\left(c_3,\exp\left(\frac{c_2}{2}k^{2/\gamma}\right)\right), \tag{5.1}$$

where c_3 is an arbitrary large constant. Let t^* be such that

$$t^*\geqslant\max\left\{e^e,\exp\left(\frac{2}{c_2}\right)^{1+\gamma},\exp\left(40^{1/\gamma}\left(\frac{c_2}{2}\right)^{(1+\gamma)/\gamma}\right),\exp\frac{c_2}{2}k^{2/\gamma}\right\}\overset{\text{def}}{=}T_0. \tag{5.2}$$

We apply estimate (4.6) with

$$s=s_1=1+\left(\frac{c_2}{2}\right)^{1+\gamma}\cdot\frac{1}{\log^{\gamma}t^*}+it^*\equiv\sigma_1+it^*, \tag{5.3}$$

$$\xi=e^{(r+2)\omega}, \tag{5.4}$$

where

$$\log t^*\leqslant r+2\leqslant\tfrac{5}{4}\log t^* \tag{5.5}$$

and

$$\omega = \left(\frac{2}{c_2}\right)^{1+\gamma} \log^{\gamma} t^* . \tag{5.6}$$

It is easily seen that conditions (3.3), (4.3) are satisfied and that (3.8), owing to (5.2), is of the form

$$1 < \sigma_1 = 1 + \left(\frac{c_2}{2}\right)^{1+\gamma} \cdot \frac{1}{\log^{\gamma} t^*} < 1 + \left(\frac{c_2}{2}\right)^{1+\gamma} \cdot \frac{1}{40\cdot(c_2/2)^{1+\gamma}} = 1 + \frac{1}{40}. \tag{5.7}$$

Multiplying (4.6) by

$$|\xi^{s_1-\varrho^*}(s_1-\varrho^*)^{r+2}| = \xi^{\sigma_1-\sigma^*}(\sigma_1-\sigma^*)^{r+2},$$

we get

$$\left| \sum_{\varrho} \xi^{\varrho-\varrho^*} \left(\frac{s_1-\varrho^*}{s_1-\varrho}\right)^{r+2} \right| < c_{13} \frac{\xi^{1-\sigma^*}}{t^*} \left(\frac{\sigma_1-\sigma^*}{\sigma_1-1}\right)^{r+2}. \tag{5.8}$$

From (5.1), (5.3), (5.5) it follows

$$\left(\frac{\sigma_1-\sigma^*}{\sigma_1-1}\right)^{r+2} < \left(1+\frac{1}{40}\right)^{r+2} < t^{*\,1/32}.$$

Therefore from (5.8) we get the estimate

$$\left| \sum_{\varrho} \xi^{\varrho-\varrho^*} \left(\frac{s_1-\varrho^*}{s_1-\varrho)}\right)^{r+2} \right| < c_{14} \frac{\xi^{1-\sigma}}{t^{*\,31/32}}. \tag{5.9}$$

6. Let us estimate the part of the sum (5.9) for which $t_\varrho > t^*+8$. Owing to II and (5.5) we have

$$S_1 = \left| \sum_{\substack{\varrho \\ t_\varrho > t^*+8}} \xi^{\varrho-\varrho^*} \left(\frac{s_1-\varrho^*}{s_1-\varrho}\right)^{r+2} \right| < \sum_{n=8}^{\infty} \xi^{1-\sigma^*} \left(\frac{1}{32n}\right)^{r+2} \sum_{\substack{\varrho \\ t^*+n < t_\varrho \leqslant t^*+n+1}} 1$$

$$\leqslant c_{15} \xi^{1-\sigma^*} \sum_{n=8}^{\infty} \frac{kn \log kt^*}{n^{r+2}(32)^{r+2}} < c_{16} \xi^{1-\sigma^*} \frac{k \log kt^*}{(t^*)^{\log 32}}.$$

Similarly, we can prove that the sums for

$$t^*+6(\sigma_1-\sigma^*) \leqslant t_\varrho \leqslant t^*+8; \qquad 0 \leqslant t_\varrho \leqslant t^*-6(\sigma_1-\sigma^*)$$

and

$$|t_\varrho - t^*| \leqslant 6(\sigma_1-\sigma^*), \qquad \sigma_\varrho < 1-3(\sigma_1-\sigma^*)$$

are absolutely less than

$$c_{17} \xi^{1-\sigma^*} \frac{\log^{3/2} t^*}{t^{*\,\log 3.97}}.$$

Therefore, owing to (5.9) and (5.4), we have

$$V \equiv \Bigg| \sum_{\substack{\varrho \\ |t_\varrho - t^*| \leqslant 6(\sigma_1 - \sigma^*) \\ \sigma_\varrho \geqslant 1 - 3(\sigma_1 - \sigma^*)}} \left(e^{\omega(\varrho - \varrho^*)} \frac{s_1 - \varrho^*}{s_1 - \varrho} \right)^{r+2} \Bigg| < c_{18} \frac{\xi^{1-\sigma^*}}{t^{*\,31/32}}. \tag{6.1}$$

7. Let us estimate V from below. We begin with the estimate of the number of terms in V.

Let

$$\left(\frac{c_2}{2}\right)^{1+\gamma} \frac{1}{\log^\gamma t^*} = \mu,$$

N_1 stands for the number of zeros of $\prod\limits_{\chi \bmod k} L(s, \chi)$ in the circle $|s - s_1| \leqslant 8\mu$.

Since $16\mu < \frac{1}{2}$, we have owing to III and applying (5.2), in the circle $|s - s_1| \leqslant 16\mu$ the estimate

$$\log \left| \frac{L(s, \chi)}{L(s_1, \chi)} \right| \leqslant 15\mu \log kt^* + c_{19} \log\log kt^*.$$

Hence, owing to Jensen inequality,

$$N_1 < 25k\mu \log kt^* + c_{19} k \log\log kt^*$$
$$< 25 \left(\frac{c_2}{2}\right)^{1+\gamma/2} \frac{\log t^*}{\log^{\gamma/2} t^*} + c_{20} \log^{\gamma/2} t \cdot \log\log t^*.$$

Since the maximal distance of a point in

$$|t_\varrho - t^*| \leqslant 6(\sigma_1 - \sigma^*), \qquad \sigma_\varrho \geqslant 1 - 3(\sigma_1 - \sigma^*) \tag{7.1}$$

from s_1 is

$$d = \big| s_1 - (1 - 3(\sigma_1 - \sigma^*)) + (t^* - 6(\sigma_1 - \sigma^*)i) \big|,$$

in view of $1 - \sigma^* < \frac{1}{40}(\sigma_1 - 1)$, $\sigma_1 - \sigma^* < \frac{41}{40}(\sigma_1 - 1)$ we get $d < 7.4(\sigma_1 - 1) < 8\mu$. So the region (7.1) is contained in the circle $|s - s_1| \leqslant 8\mu$. Therefore the number of terms in V is less than

$$25 \left(\frac{c_2}{2}\right)^{1+\gamma/2} \frac{\log t^*}{\log^{\gamma/2} t^*} + c_{20} \log^{\gamma/2} t^* \cdot \log\log t^*. \tag{7.2}$$

Let us estimate V from below applying Turán's Theorem. Owing to (7.2) we can choose

$$N \overset{\text{def}}{=} 25 \left(\frac{c_2}{2}\right)^{1+\gamma/2} \cdot \frac{\log t^*}{\log^{\gamma/2} t^*} + c_{20} \log^{\gamma/2} t^* \cdot \log\log t^*,$$

$$z_j \overset{\text{def}}{=} e^{\omega(\varrho - \varrho^*)} \frac{s_1 - \varrho^*}{s_1 - \varrho}, \qquad m \overset{\text{def}}{=} \log t^*.$$

Since, owing to (5.2), we have for $t^* > c_{20}(\gamma)$ the estimate

$$N < \tfrac{1}{4}\log t^* = \tfrac{1}{4}m,$$

the interval $(m, m+N)$ is contained in (5.5) for

$$t^* \geqslant \max\big(T_0, c_{20}(\gamma)\big) \overset{\text{def}}{=} T_1.$$

Defining the exponent $(r+2)$ in the sense of Turán's Theorem, for $t^* \geqslant T_1$, we get

$$V \geqslant \left(\frac{1}{48e^2}\cdot\frac{N}{2N+m}\right)^N > \left(\frac{25}{72e^2}\left(\frac{c_2}{2}\right)^{1+\gamma/2}\frac{1}{\log^{\gamma/2}t^*}\right)^{25\left(\frac{c_2}{2}\right)^{1+\gamma/2}\cdot\frac{\log t^*}{\log^{\gamma/2}t^*}}\times$$

$$\times\left(\frac{c_{20}}{72e^2}\cdot\frac{\log^{\gamma/2}t^*\log\log t^*}{\log t^*}\right)^{c_{20}\log^{\gamma/2}t^*\cdot\log\log t^*}$$

$$> \exp\left(-12.5\gamma\left(\frac{c_2}{2}\right)^{1+\gamma/2}\frac{\log t^*\cdot\log\log t^*}{\log^{\gamma/2}t^*}\right)\cdot\exp\left(-c_{21}\frac{\log t^*(\log\log t^*)^2}{\log^{\gamma/2}t^*}\right)$$

$$= \exp\left\{-12.5\gamma\left(\frac{c_2}{2}\right)^{1+\gamma/2}\log^{1-\gamma/2}t^*\cdot\log\log t^* - c_{21}\log^{1-\gamma/2}t^*\cdot(\log\log t^*)^2\right\}.$$

Hence, owing to (6.1), it follows

$$\xi^{1-\sigma^*} > \frac{t^{*\,31/32}}{c_{18}}\exp\left\{-12.5\gamma\left(\frac{c_2}{2}\right)^{1+\gamma/2}\log^{1-\gamma/2}t^*\log\log t^* -\right.$$

$$\left.-c_{21}\log^{1-\gamma/2}t^*\cdot(\log\log t^*)^2\right\} > t^{*\,1/2}$$

for $t^* > \max\big(T_1, c_{23}(\gamma)\big)$. Taking into account (5.4), (5.5), (5.6) we get

$$\tfrac{1}{2}\log t^* < (1-\sigma^*)\left(\frac{2}{c_2}\right)^{1+\gamma}\tfrac{5}{4}\log^{1+\gamma}t^*.$$

Hence

$$1-\sigma^* > \tfrac{2}{5}\left(\frac{c_2}{2}\right)^{1+\gamma}\frac{1}{\log^{\gamma}t^*}.$$

and this leads to a contradiction, which proves (1.1), (1.2).

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