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Generalized relative spectra

In [3] relative spectra for normalable elements of a linear lattice are defined and subsequently are used as a tool in proving many important results concerning linear lattices.

Relative spectra were originally defined for continuous linear lattices in [2]. Amemiya attempted to generalize them for arbitrary linear lattices in [1]. However, his method is different from that in [2]. If we use cluster lattices as in [5], we can generalize relative spectra for arbitrary linear lattices by the same method as in [2], using the cluster $\{x\}^{\perp\perp}$ instead of the projector $[x]$. This is the purpose of this paper.

All proofs similar to ones found in [3] will be omitted.

1. Representation spaces. We begin by letting L denote a linear lattice and C its associated cluster lattice as defined in [5]. Since C is a complete Boolean algebra we can define the representation space $\mathfrak{C}$ as in [3], where $\mathfrak{C}$ consists of all maximal ideals $\mathfrak{p} \subset C$ and where all sets of the form $U_X = \{\mathfrak{p}\colon X \in \mathfrak{p}, X \in C\}$ determine a unique compact, Hausdorff topology on $\mathfrak{C}$. It is easily shown that U_X is an open compact set for every $X \in C$ and, moreover, as a consequence of the general theory for representation spaces we have the following elementary properties:

(1.1) $$U_X = \emptyset \textit{ if and only if } X = \{0\};$$

(1.2) $$U_X \subset U_Y \textit{ if and only if } X \subset Y;$$

(1.3) $$U_X \cup U_Y = U_{X \vee Y} \quad \textit{and} \quad U_X \cap U_Y = U_{X \wedge Y};$$

(1.4) $$U_X - U_Y = U_{X - X \wedge Y};$$

(1.5) $$\mathfrak{C} = \bigcup_{X \in C} U_X;$$

(1.6) $$U_{X^\perp} = (U_X)'.$$

By Theorem 5.17 in [4] we have

(1.7) *If A is a normalable manifold of L, then for any manifold B we have $A^{\perp\perp} \cap B^{\perp\perp} = [A]B^{\perp\perp} = ([A]B)^{\perp\perp}$ and $U_{A^{\perp\perp}} \cap U_{B^{\perp\perp}} = U_{([A]B)^{\perp\perp}}$.*

2. Properties of proper neighborhoods. In this section it is convenient to employ the notational device of denoting the basic set $U_{\{x\}^{\perp\perp}}$ by U_x and U_x for $x \in L$ will be called a *proper neighborhood.* The proofs of the following collection of properties are for the most part straightforward and are therefore omitted with the exception of (2.6):

(2.1) $$U_{|a|} = U_a \quad \text{and} \quad U_{aa} = U_a \quad \text{for } a \neq 0;$$

(2.2) $$U_a \cup U_b = U_{\{a,b\}^{\perp\perp}} = U_{|a| \vee |b|} = U_{|a|+|b|};$$

(2.3) $$U_a \cap U_b = U_{|a| \wedge |b|};$$

(2.4) $$U_a \cap U_b = \emptyset \text{ if and only if } a \perp b;$$

(2.5) $$U_a = U_{a^+} \cup U_{a^-} \quad \text{and} \quad U_{a^+} \cap U_{a^-} = \emptyset;$$

(2.6) *If* $a = \bigvee_{\lambda \in \Lambda} a_\lambda$ *and* $a_\lambda \geqslant 0$ $(\lambda \in \Lambda)$, *then* $U_a = \left(\bigcup_{\lambda \in \Lambda} U_{a_\lambda}\right)^-$.

Proof. Trivially $U_a \supset \left(\bigcup_{\lambda \in \Lambda} U_{a_\lambda}\right)^-$. If there exists $\mathfrak{p} \in U_a \cap \left(\bigcup_{\lambda \in \Lambda} U_{a_\lambda}\right)^{-\prime}$, then there exists a positive element $b \in L$ such that $\mathfrak{p} \in U_b \subset U_a \cap \left(\bigcup_{\lambda \in \Lambda} U_{a_\lambda}\right)^{-\prime}$, and $b \wedge a_\lambda = 0$ for all $\lambda \in \Lambda$. Thus we must have $b \wedge a = 0$. This cannot be possible because $U_a \cap U_b = U_b \neq \emptyset$. Therefore $U_a = \left(\bigcup_{\lambda \in \Lambda} U_{a_\lambda}\right)^-$.

3. Proper values. As in [3] we are motivated to define the *proper value* $(a, \mathfrak{p})$ of an element a in L at the point $\mathfrak{p}$ in $\mathfrak{C}$ by

$$(a, \mathfrak{p}) = \begin{cases} +\infty & \text{for } \mathfrak{p} \in U_{a^+}, \\ 0 & \text{for } \mathfrak{p} \in U_a', \\ -\infty & \text{for } \mathfrak{p} \in U_{a^-}. \end{cases}$$

For proper values we have trivially by definition

(3.1) $$(0, \mathfrak{p}) = 0 \quad \text{for any } \mathfrak{p} \in \mathfrak{C},$$

(3.2) $$(a, \mathfrak{p}) \geqslant 0 \text{ if and only if } \mathfrak{p} \notin U_{a^-},$$

(3.3) $$(a, \mathfrak{p}) \leqslant 0 \text{ if and only if } \mathfrak{p} \notin U_{a^+},$$

(3.4) $$(a, \mathfrak{p}) \neq 0 \text{ if and only if } \mathfrak{p} \in U_a.$$

By definition and (1.7), it easily follows that

(3.5) $$(a, \mathfrak{p}) = ([A]a, \mathfrak{p}), \quad \text{if } A \text{ is normalable and } \mathfrak{p} \in U_{A^{\perp\perp}}.$$

For arbitrary elements a and b in L such that $a \geqslant b$, it is always true that $a^+ \geqslant b^+ \geqslant 0$ and $0 \leqslant a^- \leqslant b^-$. From these inequalities it follows that $U_{b^+} \subset U_{a^+}$ and $U_{a^-} \subset U_{b^-}$ so that we obtain

(3.6) $$a \geqslant b \text{ implies } (a, \mathfrak{p}) \geqslant (b, \mathfrak{p}).$$

We can easily prove

(3.7) $(\alpha a, \mathfrak{p}) = \alpha(a, \mathfrak{p})$ *for all real* α *and* $a \in L$ *if we use the convention* $0(\pm\infty) = 0$.

In the following proof sums and differences of the forms $\infty - \infty$ and $-\infty - (-\infty)$ do not make sense. By (2.2) and the identity

$$(a+b)^+ + a^- + b^- = (a+b)^- + a^+ + b^+$$

we have

$$(A) \quad U_{(a+b)^-} \cup U_{a^-} \cup U_{b^-} = U_{(a+b)^-} \cup U_{a^+} \cup U_{b^+}.$$

If we suppose that $(a, \mathfrak{p}) = \pm\infty$ and $(b, \mathfrak{p}) = \mp\infty$, then $(a, \mathfrak{p}) + (b, \mathfrak{p})$ does not make sense. Next let us suppose that $(a, \mathfrak{p}) = +\infty$ and $(b, \mathfrak{p}) \geqslant 0$. In this case clearly we have $(a, \mathfrak{p}) + (b, \mathfrak{p}) = +\infty$. On the other hand, $\mathfrak{p} \in U_{a^+}$ and $(b, \mathfrak{p}) \geqslant 0$ imply $\mathfrak{p} \notin U_{a^-}$ and $\mathfrak{p} \notin U_{b^-}$. Since $\mathfrak{p} \in U_{a^+}$, by (A) we conclude $\mathfrak{p} \in U_{(a+b)^+}$ which means $(a+b, \mathfrak{p}) = +\infty$. If $(a, \mathfrak{p}) = -\infty$ and $(b, \mathfrak{p}) \leqslant 0$, then replace a by $-a$ and b by $-b$ in the previous argument and apply (3.7) to obtain $(a+b, \mathfrak{p}) = (a, \mathfrak{p}) + (b, \mathfrak{p})$. Finally, if $(a, \mathfrak{p}) = 0$ and $(b, \mathfrak{p}) = 0$, then by definition of proper value we have $\mathfrak{p} \notin U_a$ and $\mathfrak{p} \notin U_b$. By (2.5) this means $\mathfrak{p} \in U_{a^\pm}$ and $\mathfrak{p} \notin U_{b^\pm}$. If we suppose that $\mathfrak{p} \notin U_{(a+b)^+}$, then by $(A)\, p \in U_{(a+b)^-}$ because $\mathfrak{p} \notin U_{a^+}$ and $\mathfrak{p} \notin U_{b^+}$. This is impossible by (2.5). By similar reasoning we cannot have $\mathfrak{p} \in U_{(a+b)^-}$. Thus $\mathfrak{p} \notin U_{(a+b)}$ and $(a+b, \mathfrak{p}) = 0$ by definition. In all cases we have verified that

(3.8) $(a+b, \mathfrak{p}) = (a, \mathfrak{p}) + (b, \mathfrak{p})$ *under the assumption that the right-hand side of the equality makes sense.*

Since $(a \vee b)^+ = a^+ \vee b^+$ and $(a \vee b)^- = a^- \wedge b^-$, applying (2.2) and (2.3) we obtain $U_{(a \vee b)^+} = U_{a^+} \cup U_{b^+}$ and $U_{(a \vee b)^-} = U_{a^-} \cap U_{b^-}$. By the definition of proper values, $(a \vee b, \mathfrak{p}) = +\infty$ is equivalent to $(a, \mathfrak{p}) = +\infty$ or $(b, \mathfrak{p}) = +\infty$. Hence $(a \vee b, \mathfrak{p}) = \mathrm{Max}\{(a, \mathfrak{p}), (b, \mathfrak{p})\} = +\infty$ in this case. Similarly $(a \vee b, \mathfrak{p}) = -\infty$ is equivalent to $\mathfrak{p} \in U_{a^-} \cap U_{b^-}$. Therefore $\mathrm{Max}\{(a, \mathfrak{p}), (b, \mathfrak{p})\} = -\infty$ in this case. Finally, if $\mathfrak{p} \notin U_{a \vee b}$, then $\mathfrak{p} \notin U_{a^-} \cap U_{b^-}$. This means $(a, \mathfrak{p})$ and $(b, \mathfrak{p})$ are not both $-\infty$. Also by (3.6) and the fact that both a and b are $\leqslant a \vee b$, we have that $(a, \mathfrak{p})$ and $(b, \mathfrak{p})$ are both $\leqslant 0$. Thus either $(a, \mathfrak{p}) = 0$ or $(b, \mathfrak{p}) = 0$ and we have verified that in all cases

(3.9) $$(a \vee b, \mathfrak{p}) = \mathrm{Max}\{(a, \mathfrak{p}), (b, \mathfrak{p})\}.$$

On replacing a by $-a$ and b by $-b$ it follows that

(3.10) $$(a \wedge b, \mathfrak{p}) = \mathrm{Min}\{(a, \mathfrak{p}), (b, \mathfrak{p})\}.$$

4. Relative spectra. Paralleling [3], we can prove that for any two elements a and b of L and $\mathfrak{p} \in U_a$ there exists a unique extended real number λ_0 for which $(\lambda a - b, \mathfrak{p}) = (a, \mathfrak{p})$ when $\lambda > \lambda_0$ and $(\lambda a - b, \mathfrak{p})$

$= -(a, \mathfrak{p})$ when $\lambda < \lambda_0$. Henceforth the unique value λ_0 is called the *relative spectrum* of b by a at $\mathfrak{p}$ and is denoted by $(b/a, \mathfrak{p})$. All of the theorems in section 18 of [3] generalize directly with the exception of 18.4 which should be changed to read $(b/a, \mathfrak{p}) = ([A]b/a, \mathfrak{p}) = (b/[A]a, \mathfrak{p}) = ([A]b/[A]a, \mathfrak{p})$ for a normalable manifold A and $\mathfrak{p} \in U_a \cap U_{A^{\perp\perp}}$. For convenience we list the analogues to the theorems found in section 18 of [3]:

(4.1) *If* $(\lambda a - b, \mathfrak{p}) = (a, \mathfrak{p}) \neq 0$, *then* $\lambda \geqslant (b/a, \mathfrak{p})$ *and if* $(\lambda a - b, \mathfrak{p}) = -(a, \mathfrak{p}) \neq 0$, *then* $\lambda \leqslant (b/a, \mathfrak{p})$;

(4.2) *For every real* α, $(\alpha a/a, \mathfrak{p}) = \alpha$, *where* $\mathfrak{p} \in U_a$;

(4.3) $(b, \mathfrak{p}) = 0$ *implies* $(b/a, \mathfrak{p}) = 0$ *for* $\mathfrak{p} \in U_a$;

(4.4) $(b/a, \mathfrak{p}) = ([A]b/a, \mathfrak{p}) = (b/[A]a, \mathfrak{p}) = ([A]b/[A]a, \mathfrak{p})$ *for a normalable manifold* A *and* $\mathfrak{p} \in U_a \cap U_{A^{\perp\perp}}$;

(4.5) *For every real* α, $(\alpha b/a, \mathfrak{p}) = \alpha(b/a, \mathfrak{p})$, *where* $\mathfrak{p} \in U_a$ *and* $0(\pm\infty) = 0$ *by convention*;

(4.6) *For* $\mathfrak{p} \in U_a$, $\big((b+c)/a, \mathfrak{p}\big) = (b/a, \mathfrak{p}) + (c/a, \mathfrak{p})$ *if the right-hand side makes sense*;

(4.7) *For* $\alpha \neq 0$, $(\alpha b/\alpha a, \mathfrak{p}) = (b/a, \mathfrak{p})$, *where* $\mathfrak{p} \in U_a$;

(4.8) *For every* $\mathfrak{p} \in U_{a+}$ *if* $b \geqslant c$ *we have* $(b/a, \mathfrak{p}) \geqslant (c/a, \mathfrak{p})$;

(4.9) *For every* $\mathfrak{p} \in U_{a+}$, $(b \vee c/a, \mathfrak{p}) = \mathrm{Max}\{(b/a, \mathfrak{p}), (c/a, \mathfrak{p})\}$ *and* $(b \wedge c/a, \mathfrak{p}) = \mathrm{Min}\{(b/a, \mathfrak{p}), (c/a, \mathfrak{p})\}$;

(4.10) *For* $\mathfrak{p} \in U_a$, $|(b/a, \mathfrak{p})| = |(|b|/a, \mathfrak{p})| = |(b/|a|, \mathfrak{p})| = (|b|/|a|, \mathfrak{p})$;

(4.11) $|b| \leqslant \alpha|a|$ *implies* $|(b/a, \mathfrak{p})| \leqslant \alpha$ *for* $\mathfrak{p} \in U_a$;

(4.12) *For* $\mathfrak{p} \in U_a \cap U_b$, $(c/a, \mathfrak{p}) = (c/b, \mathfrak{p})(b/a, \mathfrak{p})$ *if the right-hand side makes sense.*

5. Relative spectra as functions of $\mathfrak{p}$. With a few minor alterations in hypotheses the next group of theorems are analogous to some of the results found in section 19 of [3]. Excepting for (5.3) their proofs are sufficiently different to warrant their inclusion:

(5.1) *For* $0 \leqslant a \in L$, *if* $(b/a, \mathfrak{p}) > 0$ *for all* $\mathfrak{p} \in U_a$ *and* $b \in \{a\}^{\perp\perp}$, *then* $b \geqslant 0$.

Proof. It follows immediately from the hypotheses and (1.2) that $U_a = U_{a+}$ and $U_b \subset U_{a+}$. Trivially $U_{b-} \subset U_b$. Suppose there exists $\mathfrak{p} \in U_{b-}$. For this $\mathfrak{p}$ we have $(b, \mathfrak{p}) = -\infty$ and $(a, \mathfrak{p}) = +\infty$. By (3.8) we obtain $(\lambda a - b, \mathfrak{p}) = +\infty$ for all $\lambda > 0$ and $\mathfrak{p} \in U_{b-}$. Clearly we can find a real $\lambda_0 > 0$ such that $0 < \lambda_0 < (b/a, \mathfrak{p})$. By definition of relative spectrum for λ_0 we have $(\lambda_0 a - b, \mathfrak{p}) = -(a, \mathfrak{p}) = -\infty$, contradicting that $(\lambda a - b, \mathfrak{p}) = +\infty$ for $\lambda > 0$. Thus $U_{b^-} = \emptyset$ and by (1.2) $b^- = 0$ establishing (5.1).

An element $a \in L$ is said to be *archimedean* if $\bigwedge_{\xi > 0} \xi |a| = 0$. With this definition we have

(5.2) *If a is a positive, archimedean element of L, $(b/a, \mathfrak{p}) \geqslant 0$ for all $\mathfrak{p} \in U_a$ and $b \in \{a\}^{\perp\perp}$, then $b \geqslant 0$.*

Proof. By (4.2) and (4.6), $\big((b+\varepsilon a)/a, \mathfrak{p}\big) = (b/a, \mathfrak{p}) + \varepsilon > 0$ for arbitrary $\varepsilon > 0$. By (5.1), $b + \varepsilon a \geqslant 0$. The conclusion follows immediately because a is archimedean.

The proof of Theorem 19.2 in [3] is available for

(5.3) *$(b/a, \mathfrak{p})$ is continuous as a function on U_a;*

(5.4) *For an archimedean $b \in L$ $(b/a, \mathfrak{p})$ is almost finite on U_a.*

Proof. It is required to show that there exists an open dense subset of U_a, where $(b/a, \mathfrak{p})$ is finite. Let $A = \{\mathfrak{p} \in U_a \colon |(b/a, \mathfrak{p})| < +\infty\}$. Because $(b/a, \mathfrak{p})$ is continuous, A is an open set. Suppose $U_a \cap A^{-\prime} \neq \emptyset$. There exists $X \in C$ such that $U_X \subset U_a \cap A^{-\prime}$ and $X \neq \{0\}$. Thus there exists $y \in X$ such that $y \neq 0$. It follows from (1.2) that $U_y \subset U_X \subset U_a \cap A^{-\prime}$ and $U_{|a| \wedge |y|} = U_y$ by (2.3). By (4.10) we have $|(b/a, \mathfrak{p})| = (|b|/|a|, \mathfrak{p}) = +\infty$ for all $\mathfrak{p} \in U_y$. Therefore by definition of relative spectra $(\lambda |a| - |b|, \mathfrak{p}) = -(|a|, \mathfrak{p}) = -\infty$ for any λ and $\mathfrak{p} \in U_y$. From this relationship we conclude $U_y \subset U_{(\lambda|a| - |b|)^-}$ for arbitrary real λ. Thus $U_y \cap U_{(\lambda|a|-|b|)^+} = \emptyset$ and this implies by (2.3) $|y| \wedge (\lambda|a| - |b|)^+ = 0$ for all λ. Since b is archimedean by assumption, we have $\left(|a| - \frac{1}{\nu}|b|\right)^+ \Big\uparrow_{\nu=1}^{\infty} |a|$ and we obtain $|y| \wedge |a| = 0$. Hence $U_y = \emptyset$ contrary to assumption.

(5.5) *If a is a positive, archimedean element of L, b and c are elements of $\{a\}^{\perp\perp}$ such that one of b and c is archimedean and $(b/a, \mathfrak{p}) \geqslant (c/a, \mathfrak{p})$ for all $\mathfrak{p} \in U_a$, then $b \geqslant c$.*

Proof. We suppose that c is archimedean. Then by (5.4) there is an open set $A \subset U_a$ such that $A^- = U_a$ and $(c/a, \mathfrak{p})$ is finite in A. Thus $(b - c/a, \mathfrak{p}) \geqslant 0$ for $\mathfrak{p} \in A$ by (4.6). Since $\big((b-c)/a, \mathfrak{p}\big)$ is continuous by (5.3), we have $\big((b-c)/a, \mathfrak{p}\big) \geqslant 0$ for $\mathfrak{p} \in U_a$, and we obtain $b - c \geqslant 0$ by (5.2).

(5.6) *For any pair of archimedean elements a and b of L, if $(b/a, \mathfrak{p}) = (c/a, \mathfrak{p})$ for all $\mathfrak{p} \in U_a$ and $b, c \in \{a\}^{\perp\perp}$, then $b = c$.*

Proof. By (3.9), (3.10), and (4.10) we obtain immediately that $(b^{\pm}/|a|, \mathfrak{p}) = (c^{\pm}/|a|, \mathfrak{p})$. The desired conclusion follows from (5.5).

(5.7) *For a positive, archimedean $a \in L$, $U_{(\lambda a - b)^+ \wedge a} = \{\mathfrak{p} \colon (b/a, \mathfrak{p}) < \lambda\}^-$ and $U_{(\lambda a - b)^- \wedge a} = \{\mathfrak{p} \colon (b/a, \mathfrak{p}) > \lambda\}^-$.*

Proof. If $(b/a, \mathfrak{p}) < \lambda$, then since $a \geqslant 0$, we have $\mathfrak{p} \in U_a$ and $(\lambda a - b, \mathfrak{p}) = (a, \mathfrak{p}) = +\infty$. Thus $\mathfrak{p} \in U_{(\lambda a-b)^+} \cap U_a = U_{(\lambda a-b)^+ \wedge a}$ by (2.3). This proves $\{\mathfrak{p}: (b/a, \mathfrak{p}) < \lambda\}^- \subset U_{(\lambda a-b)^+ \wedge a}$. Next suppose $\mathfrak{p} \in U_{(\lambda a-b)^+ \wedge a}$. We evidently have $(\lambda a - b, \mathfrak{p}) = +\infty = (a, \mathfrak{p})$. By (4.1) $(b/a, \mathfrak{p}) \leqslant \lambda$. In other words for any real λ we have $U_{(\lambda a-b)^+ \wedge a} \subset \{p : (b/a, \mathfrak{p}) \leqslant \lambda\}$. Now

$$\{\mathfrak{p} : (b/a, \mathfrak{p}) < \lambda\} = \bigcup_{\nu=1}^{\infty} \left\{\mathfrak{p} : (b/a, \mathfrak{p}) \leqslant \lambda - \frac{1}{\nu}\right\} \supset \bigcup_{\nu=1}^{\infty} U_{\left[\left(\lambda - \frac{1}{\nu}\right)a-b\right]^+ \wedge a}.$$

Since a is archimedean by assumption we have $\bigvee_{\nu=1}^{\infty} \left[\left(\lambda - \frac{1}{\nu}\right)a - b\right]^+ \wedge a =$

$= (\lambda a - b)^+ \wedge a$. Therefore by (2.6)

$$U_{(\lambda a-b)^+ \wedge a} = \left(\bigcup_{\nu=1}^{\infty} U_{\left[\left(\lambda - \frac{1}{\nu}\right)a-b\right]^+ \wedge a}\right)^- \subset \{\mathfrak{p} : (b/a, \mathfrak{p}) < \lambda\}^-.$$

Clearly we also have

$$U_{(\lambda a-b)^- \wedge a} = U_{(-\lambda a+b)^+ \wedge a} = \{\mathfrak{p} : (-b/a, \mathfrak{p}) < -\lambda\}^- = \{\mathfrak{p} : (b/a, \mathfrak{p}) > \lambda\}^-$$

by (4.5).

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