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Non-smooth decomposition of homogeneous Triebel–Lizorkin–Morrey spaces

Keisuke Asami and Yoshihiro Sawano

Summary. The aim of this paper is to develop a theory of non-smooth decomposition in Triebel–Lizorkin–Morrey spaces. As a byproduct, we obtain the non-smooth decomposition results for Hardy spaces and Morrey spaces. The result extends what Frazier and Jawerth obtained in 1990 with the parameters subject to a condition. Unlike this foregoing work, the result in this paper is valid for all admissible parameters for Triebel–Lizorkin–Morrey spaces. As an application, an improvement of the Olsen inequality is obtained.

Keywords
Atomic decomposition;
Triebel–Lizorkin–Morrey spaces;
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1. Introduction

This paper is a follow-up to the paper [1]. We know that the Triebel–Lizorkin–Morrey space $\dot{\mathcal{E}}^s_{p,q,r}(\mathbb{R}^n)$ admits a smooth atomic decomposition if $0 < q \leqslant p < \infty$, $0 < r \leqslant \infty$ and $s \in \mathbb{R}$ (see [24,27,31]). The aim of this paper is to apply this decomposition and to study the non-smooth decomposition of $\dot{\mathcal{E}}^s_{p,q,r}(\mathbb{R}^n)$ for $0 < q \leqslant p < \infty$, $0 < r \leqslant \infty$ and $s \in \mathbb{R}$.

Keisuke Asami, Department of Mathematics and Information Sciences, Tokyo Metropolitan University, 1-1 Minamiosawa, Hachioji, Tokyo 192-0397, Japan (*e-mail*: not specified)

Yoshihiro Sawano, Department of Mathematics and Information Sciences, Tokyo Metropolitan University, 1-1 Minamiosawa, Hachioji, Tokyo 192-0397, Japan and Department of Mathematics Analysis and the Theory of functions, Peoples' Friendship University of Russian, Moscow, Russian (*e-mail*: yoshihiro-sawano@celery.ocn.ne.jp)

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Let us recall the definition of Morrey spaces. Let $0 < q \leqslant p < \infty$. Then the Morrey (quasi-)norm $\| \cdot \|_{\mathcal{M}^p_q}$ is given by

$$\|f\|_{\mathcal{M}^p_q} \equiv \sup_B |B|^{1/p-1/q} \left(\int_B |f(x)|^q \, dx \right)^{1/q},$$

where B runs over all open balls in $\mathbb{R}^n$. Using the monotone convergence theorem, we can show that $\mathcal{M}^p_p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ with coincidence of norms.

1.1. Definition. For $\nu \in \mathbb{Z}$ and $m = (m_1, m_2, \dots, m_n) \in \mathbb{Z}^n$, we define

$$Q_{\nu,m} \equiv \prod_{j=1}^n \left[\frac{m_j}{2^\nu}, \frac{m_j+1}{2^\nu} \right).$$

Denote by $\mathcal{D} = \mathcal{D}(\mathbb{R}^n)$ the set of such cubes. The elements in $\mathcal{D}(\mathbb{R}^n)$ are called *dyadic cubes*.

When we are given a collection $\{\lambda_Q\}_{Q\in\mathcal{D}}$ of complex coefficients, we adopt the definition of the function $g^s_r(\{\lambda_Q\}_{Q\in\mathcal{D}}; \cdot)$ by Grafakos; see [3, Definition 2.3.5].

1.2. Definition. Let $0 < q \leqslant p < \infty$, $0 < r \leqslant \infty$ and $s \in \mathbb{R}$. We consider the set of sequences $\lambda = \{\lambda_Q\}_{Q\in\mathcal{D}}$ such that the function

$$g^s_r(\{\lambda_Q\}_{Q\in\mathcal{D}}; x) \equiv \left(\sum_{Q\in\mathcal{D}} \left(|Q|^{-\frac{s}{n}} |\lambda_Q| \chi_Q(x) \right)^r \right)^{\frac{1}{r}}, \quad x \in \mathbb{R}^n,$$

is in $\mathcal{M}^p_q(\mathbb{R}^n)$. For such sequences $\lambda = \{\lambda_Q\}_{Q\in\mathcal{D}}$ we set

$$\|\lambda\|_{\dot{\mathbf{e}}^s_{p,q,r}} \equiv \left\| g^s_r(\lambda; \cdot) \right\|_{\mathcal{M}^p_q}.$$

A sequence $\lambda \equiv \{\lambda_Q\}_{Q\in\mathcal{D}}$ is said to belong to $\dot{\mathbf{e}}^s_{p,q,r}(\mathbb{R}^n)$ if $\|\lambda\|_{\dot{\mathbf{e}}^s_{p,q,r}} < \infty$. Sometimes, we identify $\lambda = \{\lambda_{\nu,m}\}_{\nu\in\mathbb{Z}, m\in\mathbb{Z}^n}$ with $\lambda = \{\lambda_Q\}_{Q\in\mathcal{D}}$ via $\lambda_{\nu,m} = \lambda_Q$ when $Q = Q_{\nu,m}$.

We define $\dot{\mathbf{f}}^s_{p,r}(\mathbb{R}^n) = \dot{\mathbf{e}}^s_{p,p,r}(\mathbb{R}^n)$, the sequence space for the homogeneous Triebel–Lizorkin space $\dot{F}^s_{p,q}(\mathbb{R}^n)$ whose definition we recall later.

1.3. Definition. Let $0 < q \leqslant p < \infty$, $0 < r \leqslant \infty$ and $s \in \mathbb{R}$. A sequence $\lambda \equiv \{\lambda_Q\}_{Q\in\mathcal{D}}$ is said to be an ∞-atom with a dyadic cube Q_0 or shortly an ∞-atom if $g^s_r(\lambda; \cdot) \leqslant \chi_{Q_0}$.

Our first theorem is as follows:

1.4. Theorem. *Suppose that we are given parameters p, q, r, s, u satisfying*

$$0 < q \leqslant p < \infty, \quad 0 < r \leqslant \infty, \quad s \in \mathbb{R}, \; 0 < u \leqslant \min(1, r).$$

(i) *For any* $t \in \dot{\mathbf{e}}^s_{p,q,r}(\mathbb{R}^n)$ *there exists a decomposition*

$$t = \sum_{j=1}^{\infty} \lambda_j r_j \tag{1}$$

in the sense of pointwise convergence, where each r_j *is an* ∞*-atom for* $\dot{\mathbf{e}}^s_{p,q,r}(\mathbb{R}^n)$ *with a cube* Q_j *and* $\{\lambda_j\}_{j=1}^{\infty} \subset \mathbb{C}$ *satisfies*

$$\left\| \left(\sum_{j=1}^{\infty} |\lambda_j|^u \chi_{Q_j} \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_q} \lesssim \|t\|_{\dot{\mathbf{e}}^s_{p,q,r}}.$$

(ii) *Let* r_j, $j = 1, 2, \ldots$, *be an* ∞*-atom for* $\dot{\mathbf{e}}^s_{p,q,r}(\mathbb{R}^n)$ *with a cube* Q_j. *If* $\{\lambda_j\}_{j=1}^{\infty} \subset \mathbb{C}$ *and* $\{Q_j\}_{j=1}^{\infty}$ *satisfy*

$$\left\| \left(\sum_{j=1}^{\infty} |\lambda_j|^u \chi_{Q_j} \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_q} < \infty,$$

then the series t given by (1) *belongs to* $\dot{\mathbf{e}}^s_{p,q,r}(\mathbb{R}^n)$.

When $p = q$, Theorem 1.4 is [1, Theorem 5].

The Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ has many nontrivial proper closed subspaces. For example, the tilde subspace $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ is the closure of $L^\infty_{\mathrm{c}}(\mathbb{R}^n)$ in $\mathcal{M}^p_q(\mathbb{R}^n)$, and if $0 < q < p < \infty$, then we can show that $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ is a proper subset of $\mathcal{M}^p_q(\mathbb{R}^n)$ [25]. In this connection, we define $\widetilde{\dot{\mathbf{e}}}^s_{p,q,r}(\mathbb{R}^n)$ to be the closure of the set of all finitely supported sequences in $\dot{\mathbf{e}}^s_{p,q,r}(\mathbb{R}^n)$, so that any element $r \in \widetilde{\dot{\mathbf{e}}}^s_{p,q,r}(\mathbb{R}^n)$ can be approximated by $r_j = \{r_{j,Q}\}_{Q\in\mathcal{D}} \in \dot{\mathbf{e}}^s_{p,q,r}(\mathbb{R}^n)$ satisfying $\sharp\{Q \in \mathcal{D} : r_{j,Q} \neq 0\} < \infty$. We note that the following property holds.

1.5. Proposition. *Suppose that we are given parameters* p, q, r, s, u *satisfying*

$$0 < q \leqslant p < \infty, \quad 0 < r \leqslant \infty, \quad s \in \mathbb{R}.$$

Let $D_1 \subset D_2 \subset \cdots$ *be an exhaustion of* $\mathcal{D}$, *that is,* $\sharp\, D_N$ *is finite for all N and*

$$\mathcal{D} = \bigcup_{N=1}^{\infty} D_N.$$

Define $Q_N(\{r_Q\}_{Q\in\mathcal{D}}) \equiv \{\chi_{D_N}(Q) r_Q\}_{Q\in\mathcal{D}}$. *Then a sequence* $t \in \dot{\mathbf{e}}^s_{p,q,r}(\mathbb{R}^n)$ *belongs to* $\widetilde{\dot{\mathbf{e}}}^{\mathbf{s}}_{\mathbf{p},\mathbf{q},\mathbf{r}}(\mathbb{R}^{\mathbf{n}})$ *if and only if* $Q_N(t) \to t$ *as* $N \to \infty$ *in* $\dot{\mathbf{e}}^s_{p,q,r}(\mathbb{R}^n)$.

We do not prove Proposition 1.5, since it is clear from the uniform boundedness of Q_N.

Having set down the definition of Morrey spaces and their sequence spaces, now we move on to the (homogeneous) Triebel–Lizorkin–Morrey space $\dot{\mathcal{E}}^s_{p,q,r}(\mathbb{R}^n)$. Here and below we write $\psi(D)f \equiv \mathcal{F}^{-1}[\psi \cdot \mathcal{F}f]$ for $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\chi_{B(4)\setminus B(2)} \leqslant \varphi \leqslant \chi_{B(8)\setminus B(1)}$ and write $\varphi_j = \varphi(2^{-j}\cdot)$ for $j \in \mathbb{Z}$, where $B(r) = \{x \in \mathbb{R}^n : |x| < r\}$ for $r > 0$. Let $0 < q \leqslant p < \infty$, $0 < r \leqslant \infty$ and $s \in \mathbb{R}$. We define the (homogeneous) Triebel–Lizorkin–Morrey norm $\|\cdot\|_{\dot{\mathcal{E}}^s_{p,q,r}}$ by

$$\|f\|_{\dot{\mathcal{E}}^s_{p,q,r}} \equiv \left\| \left(\sum_{j=-\infty}^{\infty} 2^{jrs} |\varphi_j(D)f|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}^p_q}$$

for $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$, where the symbol $\mathcal{P}(\mathbb{R}^n)$ stands for the subspace of all polynomials. The space $\dot{\mathcal{E}}^s_{p,q,r}(\mathbb{R}^n)$ is made up of all $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ for which the (quasi-)norm $\|f\|_{\dot{\mathcal{E}}^s_{p,q,r}}$ is finite. We have $\dot{F}^s_{p,r}(\mathbb{R}^n) = \dot{\mathcal{E}}^s_{p,p,r}(\mathbb{R}^n)$, the homogeneous Triebel–Lizorkin space. As a special case of $p = q$, Triebel–Lizorkin–Morrey spaces cover Triebel–Lizorkin spaces. One of the fundamental facts in the theory of Triebel–Lizorkin–Morrey spaces is the Morrey type Sobolev embedding

$$\dot{\mathcal{E}}^s_{p,q,\infty}(\mathbb{R}^n) \hookrightarrow \dot{B}^{s-n/p}_{\infty,\infty}(\mathbb{R}^n), \tag{2}$$

where $\dot{B}^{\alpha}_{\infty,\infty}(\mathbb{R}^n)$, $\alpha \in \mathbb{R}$ stands for the homogeneous Besov spaces consisting of all elements $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ for which

$$\|f\|_{\dot{B}^{\alpha}_{\infty,\infty}} \equiv \sup_{j\in\mathbb{Z}} 2^{j\alpha} \left\|\varphi_j(D)f\right\|_{\infty}$$

is finite. See [20], for example. We also note that Triebel–Lizorkin–Morrey spaces inherit the nesting structure from the underlying sequence space ℓ^r:

$$\dot{\mathcal{E}}^s_{p,q,r}(\mathbb{R}^n) \hookrightarrow \dot{\mathcal{E}}^s_{p,q,\infty}(\mathbb{R}^n) \tag{3}$$

when $0 < q \leqslant p < \infty$, $0 < r \leqslant \infty$ and $s \in \mathbb{R}$. To handle $\dot{\mathcal{E}}^s_{p,q,r}(\mathbb{R}^n)$, it may be convenient to work on the corresponding sequence space $\dot{\mathbf{e}}^s_{p,q,r}(\mathbb{R}^n)$. It is simpler to handle sequences than to handle distributions. We now transform the results to the one of the sequences. Let $L \in \mathbb{N}_0 \cup \{-1\} = \{-1, 0, 1, \ldots\}$. The set $\mathcal{P}_L(\mathbb{R}^n)^{\perp}$ denotes the set of all measurable functions f for which

$$\int_{\mathbb{R}^n} (1+|x|)^L |f(x)|\, dx < \infty$$

and $\int_{\mathbb{R}^n} x^{\alpha} f(x) dx = 0$ for all $\alpha \in \mathbb{R}^n$ with $|\alpha| \leqslant L$. Such a function f is said to satisfy the *moment condition of order* L. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies the moment condition of order L for any $L \in \mathbb{N}$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is said to satisfy the *moment condition of order* ∞. Denote by $\mathcal{S}_{\infty}(\mathbb{R}^n)$ the set of all such functions. The space $\mathcal{S}'_{\infty}(\mathbb{R}^n)$ is the topological dual space $\mathcal{S}_{\infty}(\mathbb{R}^n)$. It is known that $\mathcal{S}'_{\infty}(\mathbb{R}^n) \approx \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ isometrically; see [28, Propositions 35.4 and 35.5] and in [7, Theorem 24.0.4] as well as [21] and [14, §6].

1.6. Definition (Smooth atoms for Triebel–Lizorkin–Morrey spaces). Let $\nu \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$. Suppose that the integers $K, L \in \mathbb{Z}$ satisfy $K \geqslant 0$ and $L \geqslant -1$. A function $a \in C^K(\mathbb{R}^n) \cap$

$\mathcal{P}_L(\mathbb{R}^n)^\perp$ is said to be a *smooth L-atom with a cube $Q_{\nu,m}$* if it is supported on $3\,Q_{\nu,m}$, the triple of $Q_{\nu,m}$, and if it satisfies $\|\partial^\alpha a\|_{L^\infty} \leqslant 2^{\nu|\alpha|}$ for $|\alpha| \leqslant K$.

To state our main result, we present the following definition:

1.7. Definition (Non-smooth atoms for Triebel–Lizorkin–Morrey spaces). Assume $K \geqslant [1+s]_+$ and $L \geqslant \max(-1, [\sigma_{q,r} - s])$, where $[\cdot]$ denotes the Gauss sign. One says that A is a non-smooth atom for $\dot{\mathcal{E}}^s_{p,q,r}(\mathbb{R}^n)$ if $A = \sum_{Q \subset \tilde{Q}} r_Q a_Q$, where $r = \{r_Q\}_{Q \in \mathcal{D}}$ is an atom with the dyadic cube $\tilde{Q}$ for $\dot{\mathbf{e}}^s_{p,q,r}(\mathbb{R}^n)$ and each a_Q is a smooth L-atom with a cube Q.

Define $\sigma_r \equiv \frac{n}{\min(1,r)} - n$ and $\sigma_{q,r} \equiv \max(\sigma_q, \sigma_r)$.

The following theorem extends [3, Corollary 2.3.9] and is the main result in this paper.

1.8. Theorem. *Let $s \in \mathbb{R}$, $0 < q \leqslant p < \infty$, $0 < r \leqslant \infty$ and $K, L \in \mathbb{Z}$. Assume $K \geqslant [1+s]_+$ and $L \geqslant \max(-1, [\sigma_{q,r} - s])$.*

(i) *Let $0 < u \leqslant \min(1, r)$. Suppose that each A_j is a smooth L-atom with a cube Q_j with the following structure*:

- *For each j there exists an ∞-atom $r_j = \{r_{j,Q}\}_{Q \in \mathcal{D}}$ with the cube Q_j.*
- *There exists a complex sequence $\{\lambda_j\}_{j=1}^\infty$ satisfying*

$$\left\| \left(\sum_{j=1}^\infty |\lambda_j|^u \chi_{Q_j} \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_q} < \infty.$$

- *For each $Q \in \mathcal{D}$ there exists a smooth L-atom a_Q with a cube Q such that*

$$A_j = \sum_{Q \in \mathcal{D}} r_{j,Q} a_Q = \sum_{\nu=-\infty}^\infty \left(\sum_{Q \in \mathcal{D}_\nu} r_{j,Q} a_Q \right)$$

in $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$, where $\mathcal{D}_\nu = \{Q \in \mathcal{D} : |Q| = 2^{-\nu n}\}$.

Then by letting $f \equiv \sum_{j=1}^\infty \lambda_j A_j$, the sum defining f converges in $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ and satisfies

$$\|f\|_{\dot{\mathcal{E}}^s_{p,q,r}} \lesssim \left\| \left(\sum_{j=1}^\infty |\lambda_j|^u \chi_{Q_j} \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_q}.$$

(ii) *Let $f \in \dot{\mathcal{E}}^s_{p,q,r}(\mathbb{R}^n)$. Then there exists $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$, a collection $\{A_j\}_{j=1}^\infty$ of non-smooth atoms and a collection $\{Q_j\}_{j=1}^\infty$ of dyadic cubes such that each A_j is a non-smooth atom with the cube Q_j and*

$$f = \sum_{j=1}^\infty \lambda_j A_j \tag{4}$$

in $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$, where for each $u > 0$

$$\left\| \left(\sum_{j=1}^{\infty} |\lambda_j|^u \chi_{Q_j} \right)^{\frac{1}{u}} \right\|_{\mathcal{M}_q^p} \lesssim \|f\|_{\dot{\mathcal{E}}^s_{p,q,r}}$$

with the implicit constant depending on p, q, r, s, u.

The case of $s \in \mathbb{R}$, $0 < p = q = u \leqslant 1$ and $0 < p \leqslant r \leqslant \infty$ recaptures [2, Theorem 7.4 (ii)]. The case of $s \in \mathbb{R}$, $0 < p = q = u \leqslant 1$ and $0 < p \leqslant r \leqslant \infty$ recaptures [2, Theorem 7.2], while the case of $0 < p = q$, $u \leqslant \min(1, r)$ goes back to the result in [1]. The case when $p \geqslant q > 1$ and $r = 2$ is especially interesting. As we will see, this, together with $\mathcal{M}_q^p(\mathbb{R}^n) \approx \dot{\mathcal{E}}^0_{p,q,2}(\mathbb{R}^n)$ established by Mazzucato [11], will yield and reinforce the non-smooth decomposition for $\mathcal{M}_q^p(\mathbb{R}^n)$ [8].

Theorems 1.4 and 1.8 are the main theorems in this paper. Theorem 1.8 is a corollary of Theorem 1.4. We make a brief remark on the method of the proof of Theorem 1.4. The proof of Theorem 1.4 is essentially made up of two tools. The first one is a method to decompose sequences and the second tool serves to describe the condition on coefficients. In [2, Section 6], Frazier and Jarwerth employed the $\frac{1}{8}$ median and the stopping time argument together with L^0, the set of all measurable functions f for which $\{f \neq 0\}$ has finite measure. Grafakos refined this method in [3, §6.6.4]. Since our proof hinges on the decomposition in [3, §6.6.4], we essentially use the technique of the paper [2] and the textbook [3]. What is different from these sources is the second tool, which comes from the main idea in the paper [13]. As described in (7.4) of [2] and in (7.7) of [2], we have

$$\|\lambda_1 + \lambda_2\|^p_{\dot{\mathbf{f}}^s_{p,q}} \leqslant \|\lambda_1\|^p_{\dot{\mathbf{f}}^s_{p,q}} + \|\lambda_2\|^p_{\dot{\mathbf{f}}^s_{p,q}}, \lambda_1, \lambda_2 \in \dot{\mathbf{f}}^s_{p,q}(\mathbb{R}^n) \tag{5}$$

and

$$\|f_1 + f_2\|^p_{\dot{F}^s_{p,q}} \leqslant \|f_1\|^p_{\dot{F}^s_{p,q}} + \|f_2\|^p_{\dot{F}^s_{p,q}}, \quad f_1, f_2 \in \dot{F}^s_{p,q}(\mathbb{R}^n) \tag{6}$$

for $0 < p \leqslant 1$, $0 < p \leqslant q \leqslant \infty$ and $s \in \mathbb{R}$. Frazier and Jawerth used (5) and (6) to decompose the sum into small units. One of the important facts on the decomposition of Frazier and Jawerth is that the condition on the position of the cubes Q_j does not come into play when $0 < p \leqslant r < \infty$ and $0 < p = q = u \leqslant 1$. Since (5) and (6) are no longer available for the general case, we need a trick. To accommodate all admissible parameters, we take into account the position of the cubes Q_j. This is just like we moved from ℓ^p to a special sequence norm in [13].

In addition to the new technique employed in [1], we need to take care of some problems in convergence arising from Morrey spaces. As is seen from the difference between $\widetilde{\dot{e}}^s_{p,q,r}(\mathbb{R}^n)$ and $\dot{e}^s_{p,q,r}(\mathbb{R}^n)$, the argument in [1] does not carry over to our current setting. So we go back to the definition of the convergence of the sequence in $\mathcal{S}'(\mathbb{R}^n)$ as we do in (13).

Our results carry over to the non-homogeneous spaces.

Let $\psi, \varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\chi_{B(4)} \leqslant \psi \leqslant \chi_{B(8)}$ and $\chi_{B(4)\setminus B(2)} \leqslant \varphi \leqslant \chi_{B(8)\setminus B(1)}$ and write $\varphi_j = \varphi(2^{-j}\cdot)$ for $j \in \mathbb{Z}$. Let $0 < q \leqslant p < \infty$, $0 < r \leqslant \infty$ and $s \in \mathbb{R}$. We define the (non-homogeneous) Triebel–Lizorkin–Morrey norm $\|\cdot\|_{\mathcal{E}^s_{p,q,r}}$ by

$$\|f\|_{\mathcal{E}^s_{p,q,r}} \equiv \left\|\psi(D)f\right\|_{\mathcal{M}^p_q} + \left\|\left(\sum_{j=0}^{\infty} 2^{jrs}|\varphi_j(D)f|^r\right)^{\frac{1}{r}}\right\|_{\mathcal{M}^p_q}$$

for $f \in \mathcal{S}'(\mathbb{R}^n)$. The (non-homogeneous) Triebel–Lizorkin–Morrey space $\mathcal{E}^s_{p,q,r}(\mathbb{R}^n)$ is made up of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which the (quasi-)norm $\|f\|_{\mathcal{E}^s_{p,q,r}}$ is finite.

Denote by $\mathcal{D}_+ = \mathcal{D}_+(\mathbb{R}^n)$ the set of all dyadic cubes with volume less than or equal to 1.

Let $0 < q \leqslant p < \infty$, $0 < r \leqslant \infty$ and $s \in \mathbb{R}$. Given a sequence $\lambda = \{\lambda_Q\}_{Q\in\mathcal{D}_+}$ we define $\lambda^\dagger = \{\lambda^\dagger_Q\}_{Q\in\mathcal{D}}$ by $\lambda^\dagger_Q = \lambda_Q$ if $Q \in \mathcal{D}_+$ and $\lambda^\dagger_Q = 0$ otherwise. For $\lambda = \{\lambda_Q\}_{Q\in\mathcal{D}_+}$, we define $g^s_r(\lambda;\cdot) = g^s_r(\lambda^\dagger;\cdot)$ and $\|\lambda\|_{\mathbf{e}^s_{p,q,r}} \equiv \|g^s_r(\lambda;\cdot)\|_{\mathcal{M}^p_q}$. A sequence $\lambda \equiv \{\lambda_Q\}_{Q\in\mathcal{D}_+}$ is said to belong to $\mathbf{e}^s_{p,q,r}(\mathbb{R}^n)$ if $\|\lambda\|_{\mathbf{e}^s_{p,q,r}} < \infty$. In this way, via $\lambda \mapsto \lambda^\dagger$, we have an embedding $\mathbf{e}^s_{p,q,r}(\mathbb{R}^n) \hookrightarrow \dot{\mathbf{e}}^s_{p,q,r}(\mathbb{R}^n)$. Using this embedding, we define ∞-atoms in $\mathbf{e}^s_{p,q,r}(\mathbb{R}^n)$ and the subspace $\tilde{\mathbf{e}}^s_{p,q,r}(\mathbb{R}^n)$. A sequence $\lambda \equiv \{\lambda_Q\}_{Q\in\mathcal{D}}$ is said to be an ∞-atom with a cube Q_0 if $\lambda^\dagger = \{\lambda^\dagger_Q\}_{Q\in\mathcal{D}}$ is an ∞-atom with a cube Q_0. We define $\widetilde{\mathbf{e}}^s_{p,q,r}(\mathbb{R}^n) = \widetilde{\dot{\mathbf{e}}}{}^{\mathbf{s}}_{\mathbf{p},\mathbf{q},\mathbf{r}}(\mathbb{R}^{\mathbf{n}}) \cap \mathbf{e}^{\mathbf{s}}_{\mathbf{p},\mathbf{q},\mathbf{r}}(\mathbb{R}^{\mathbf{n}})$.

Similar to Proposition 1.5, we have the following characterization.

1.9. Proposition. *Suppose that we are given parameters p, q, r, s, u satisfying*

$$0 < q \leqslant p < \infty, \quad 0 < r \leqslant \infty, \quad s \in \mathbb{R}.$$

Let $D_1 \subset D_2 \subset \cdots$ be an exhaustion of $\mathcal{D}_+$, that is, $\sharp\, D_N$ is finite for all N and

$$\mathcal{D}_+ = \bigcup_{N=1}^{\infty} D_N.$$

Define $Q_N(\{r_Q\}_{Q\in\mathcal{D}_+}) \equiv \{\chi_{D_N}(Q)r_Q\}_{Q\in\mathcal{D}_+}$. Then a sequence $t \in \mathbf{e}^s_{p,q,r}(\mathbb{R}^n)$ belongs to $\widetilde{\mathbf{e}}^s_{p,q,r}(\mathbb{R}^n)$ if and only if $Q_N(t) \to t$ as $N \to \infty$ in $\mathbf{e}^s_{p,q,r}(\mathbb{R}^n)$.

In analogy to Theorem 1.4 and Theorem 1.8 we can prove the following theorems whose proofs we omit due to similarity.

1.10. Theorem. *Suppose that we are given parameters p, q, r, s, u satisfying*

$$0 < q \leqslant p < \infty, \quad 0 < r \leqslant \infty, \quad s \in \mathbb{R}, \quad 0 < u \leqslant \min(1, r).$$

(i) *For any* $t \in \mathbf{e}^s_{p,q,r}(\mathbb{R}^n)$ *there exists a decomposition*

$$t = \sum_{j=1}^{\infty} \lambda_j r_j \tag{7}$$

in the sense of pointwise convergence, where each r_j *is an* ∞*-atom for* $\mathbf{e}^s_{p,q,r}(\mathbb{R}^n)$ *with a cube* $Q_j \in \mathcal{D}_+$ *and* $\{\lambda_j\}_{j=1}^{\infty} \subset \mathbb{C}$ *satisfies*

$$\left\| \left(\sum_{j=1}^{\infty} |\lambda_j|^u \chi_{Q_j} \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_q} \lesssim \|t\|_{\mathbf{e}^s_{p,q,r}}.$$

(ii) *Let* $r_j, j = 1, 2, \ldots$ *be an* ∞*-atom for* $\mathbf{e}^s_{p,q,r}(\mathbb{R}^n)$ *with a cube* $Q_j \in \mathcal{D}_+$*. If* $\{Q_j\}_{j=1}^{\infty}$ *and* $\{\lambda_j\}_{j=1}^{\infty} \subset \mathbb{C}$ *satisfy*

$$\left\| \left(\sum_{j=1}^{\infty} |\lambda_j|^u \chi_{Q_j} \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_q} < \infty,$$

then the series t *given by* (7) *belongs to* $\mathbf{e}^s_{p,q,r}(\mathbb{R}^n)$*.*

We can transplant Theorem 1.8 to non-homogeneous Triebel–Lizorkin–Morrey spaces via the embedding $\mathbf{e}^s_{p,q,r}(\mathbb{R}^n) \hookrightarrow \dot{\mathbf{e}}^s_{p,q,r}(\mathbb{R}^n)$ after we modify slightly the definition of non-smooth atoms for Triebel–Lizorkin–Morrey spaces. In Definitions 1.6 and 1.7, we do not assume

$$\int_{\mathbb{R}^n} x^{\beta} a_{0m}(x)\, dx = 0, \quad (|\beta| \leqslant L, \quad m \in \mathbb{Z}^n).$$

After this modification, we can state the counterpart of Theorem 1.8 for non-homogeneous Triebel–Lizorkin–Morrey spaces, whose proof we omit again due to similarity.

1.11. Theorem. *Let* $s \in \mathbb{R}$, $0 < q \leqslant p < \infty$, $0 < r \leqslant \infty$ *and* $K, L \in \mathbb{Z}$*. Assume* $K \geqslant [1+s]_+$ *and* $L \geqslant \max(-1, [\sigma_{q,r} - s])$*. Then we have the following.*

(i) *Let* $0 < u \leqslant \min(1, r)$*. Suppose that* A_j *is a smooth* L*-atom with a cube* $Q_j \in \mathcal{D}_+$ *with the following structure*:

- *For each* j *there exists an* ∞*-atom* $r_j = \{r_{j,Q}\}_{Q \in \mathcal{D}}$ *with the cube* Q_j*.*
- *There exists a complex sequence* $\{\lambda_j\}_{j=1}^{\infty}$ *satisfying*

$$\left\| \left(\sum_{j=1}^{\infty} |\lambda_j|^u \chi_{Q_j} \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_q} < \infty.$$

- *For each* $Q \in \mathcal{D}_+$ *there exists a smooth* L*-atom* a_Q *with a cube* Q *such that in* $\mathcal{S}'(\mathbb{R}^n)$

$$A_j = \sum_{\nu=0}^{\infty} \left(\sum_{Q \in \mathcal{D}_{\nu}} r_{j,Q} a_Q \right).$$

Then by letting $f \equiv \sum_{j=1}^{\infty} \lambda_j A_j$, *the sum defining* f *converges in* $\mathcal{S}'(\mathbb{R}^n)$ *and satisfies*

$$\|f\|_{\mathcal{E}^s_{p,q,r}} \lesssim \left\| \left(\sum_{j=1}^{\infty} |\lambda_j|^u \chi_{Q_j} \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_q}.$$

(ii) *Let* $f \in \mathcal{E}^s_{p,q,r}(\mathbb{R}^n)$. *Then there exists* $\{\lambda_j\}_{j=1}^{\infty} \subset \mathbb{C}$, *a collection* $\{A_j\}_{j=1}^{\infty}$ *of non-smooth atoms and a collection* $\{Q_j\}_{j=1}^{\infty} \subset \mathcal{D}_+$ *such that each* A_j *is a non-smooth atom with the cube* Q_j, *that*

$$f = \sum_{j=1}^{\infty} \lambda_j A_j$$

in $\mathcal{S}'(\mathbb{R}^n)$ *and that for all* $0 < u < \infty$

$$\left\| \left(\sum_{j=1}^{\infty} |\lambda_j|^u \chi_{Q_j} \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_q} \lesssim \|f\|_{\mathcal{E}^s_{p,q,r}}.$$

Here the implicit constant depends on p, q, r, s, u.

As an application, we refine the Olsen inequality obtained in [22]. Let $0 < \alpha < n$. Let I_α be a fractional maximal operator given by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, dy$$

as long as $I_\alpha f$ makes sense. Another definition of $I_\alpha f$ is to use the Fourier transform:

$$\mathcal{F}^{-1}[I_\alpha f] = C|\xi|^{n-\alpha} \mathcal{F} f,$$

where C is chosen so that the two definitions of $I_\alpha f$ above concide when $f \in \mathcal{S}(\mathbb{R}^n)$. Concerning the second definition of I_α, we remark that $I_\alpha : \dot{\mathcal{E}}^s_{p,q,r}(\mathbb{R}^n) \to \dot{\mathcal{E}}^{s+\alpha}_{p,q,r}(\mathbb{R}^n)$ is an isomorphism [20, Theorem 1.1]. Thus, if $0 < \alpha < n$, $1 < q \leqslant p < \infty$ and $1 < t \leqslant s < \infty$ satisfy

$$\frac{q}{p} = \frac{t}{s}, \quad \frac{1}{p} - \frac{\alpha}{n} = \frac{1}{s},$$

then we get $I_\alpha : \dot{\mathcal{E}}^0_{p,q,\infty}(\mathbb{R}^n) \to \mathcal{M}^s_t(\mathbb{R}^n)$ using the embedding $\dot{\mathcal{E}}^\alpha_{p,q,\infty}(\mathbb{R}^n) \hookrightarrow \dot{\mathcal{E}}^0_{s,t,1}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^s_t(\mathbb{R}^n)$. Concerning this operator, we obtain the following inequality.

1.12. Theorem. *Let* $1 < q \leqslant p < \infty$, *and let* $0 < \alpha < n$. *Assume in addition that* u *satisfies*

$$q < u \leqslant \frac{n}{\alpha}.$$

Then

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^p_q} \lesssim \|g\|_{\mathcal{M}^{n/\alpha}_u} \|f\|_{\dot{\mathcal{E}}^0_{p,q,\infty}}$$

for all $f \in \mathcal{S}_\infty(\mathbb{R}^n) = \bigcap_{L \in \mathbb{N}}(\mathcal{S}(\mathbb{R}^n) \cap \mathcal{P}_L^\perp(\mathbb{R}^n))$ *and* $g \in \mathcal{M}_u^{n/\alpha}(\mathbb{R}^n)$.

In view of the embedding

$$\mathcal{M}_q^p(\mathbb{R}^n) \approx \dot{\mathcal{E}}_{p,q,2}^0(\mathbb{R}^n) \hookrightarrow \dot{\mathcal{E}}_{p,q,\infty}^0(\mathbb{R}^n) \quad (1 < q \leqslant p < \infty),$$

Theorem 1.12 refines the following result:

1.13. Theorem ([8, 22]). *Let* $0 < \alpha < n$, $0 < u < \infty$ *and* $1 < q \leqslant p < \infty$ *satisfy*

$$q < u \leqslant \frac{n}{\alpha}, \quad \frac{1}{p} > \frac{\alpha}{n}.$$

Then

$$\|g \cdot I_\alpha f\|_{\mathcal{M}_q^p} \leqslant C\|g\|_{\mathcal{M}_u^{n/\alpha}} \cdot \|f\|_{\mathcal{M}_q^p},$$

where the constant C *is independent of* $f \in \mathcal{S}_\infty(\mathbb{R}^n)$ *and* $g \in \mathcal{M}_u^{n/\alpha}(\mathbb{R}^n)$.

This type of estimates goes back to [16]. We will see in Section 5 that in general $\mathcal{M}_q^p(\mathbb{R}^n)$ is not embedded into the closure V of $\mathcal{S}_\infty(\mathbb{R}^n)$ in $\dot{\mathcal{E}}_{p,\infty,\infty}^0(\mathbb{R}^n)$ as long as $1 < q < p < \infty$.

We are dealing with one of the amalgam function spaces, Triebel–Lizorkin spaces. This is a huge recent branch of harmonic analysis. It goes back to Netrusov, who defined Besov–Morrey spaces [15]. Kozono and Yamazaki shed some light on Besov–Morrey spaces while investigating the Cauchy problem for the Navier–Stokes equation [10]. Later on, Najafov considered Besov–Morrey spaces of the mixed derivative in [12]. Motivated by this, Tang and Xu [27] defined non-homogeneous Triebel–Lizorkin–Morrey spaces. The wavelet characterization of Triebel–Lizorkin–Morrey spaces can be found in [17, 18]. This means that in this paper we can replace atoms by wavelets. According to [19, Theorem 4.2], Triebel–Lizorkin–Morrey spaces cover Hardy–Morrey spaces defined by Jia and Wang [9], so that our results contain the non-smooth decomposition of Hardy–Morrey spaces. We refer to [4–6, 20, 24, 27] for embedding relations of these function spaces.

Although we are considering Triebel–Lizorkin–Morrey spaces, in view of [26] a similar decomposition seems available for Triebel–Lizorkin-type spaces introduced by Yang and Yuan. See [21, 30, 32–34] for a systematic treatment of these function spaces. This is left as a future work.

The proofs of Theorems 1.4, 1.8 and 1.12 are given in Sections 2, 3 and 4, respectively. Section 5 considers the relation between V and $\dot{\mathcal{E}}_{p,q,\infty}^0(\mathbb{R}^n)$ with $1 < q < p < \infty$.

2. Proof of Theorem 1.4

We recall the following facts from [3, § 2.3.4] (see also [21]). Let $\nu \in \mathbb{Z}$. The set of dyadic cubes of the ν-th generation is given by

$$\mathcal{D}_\nu = \mathcal{D}_\nu(\mathbb{R}^n) \equiv \left\{Q_{\nu k} : k \in \mathbb{Z}^n\right\} = \left\{Q \in \mathcal{D} : \ell(Q) = 2^{-\nu}\right\}.$$

2.1. Lemma. *Let $t = \{t_Q\}_{Q\in\mathcal{D}}$ be a sequence. Let $R \in \mathcal{D}$. Define*

$$g^s_{r,R}\left(\{t_Q\}_{Q\in\mathcal{D}}; x\right) \equiv \left(\sum_{Q\in\mathcal{D},\, R\subset Q} \left(|Q|^{-\frac{s}{n}}|t_Q|\chi_Q(x)\right)^r\right)^{\frac{1}{r}}, \quad x \in \mathbb{R}^n.$$

(i) *If $R_1 \subset R_2$ and $x \in \mathbb{R}^n$, then $g^s_{r,R_2}(t;x) \leqslant g^s_{r,R_1}(t;x)$.*

(ii) *For any $x \in \mathbb{R}^n$,*

$$\lim_{\nu\to\infty} \sum_{Q\in\mathcal{D}_\nu} \chi_Q(x) g^s_{r,Q}(t;x) = 0.$$

(iii) *For any $x \in \mathbb{R}^n$,*

$$\lim_{\nu\to-\infty} \sum_{Q\in\mathcal{D}_\nu} \chi_Q(x) g^s_{r,Q}(t;x) = g^s_r(t;x).$$

2.2. Lemma. *Let $t = \{t_Q\}_{Q\in\mathcal{D}}$ be a sequence. Let $k \in \mathbb{Z}$. We set*

$$\mathcal{A}_k = \left\{R \in \mathcal{D} : g^s_{r,R}(t;x) > 2^k, \textit{ for } x \in R\right\}.$$

(i) [3, p. 116] *If $Q \in \mathcal{D}$ does not belong to any $\mathcal{A}_k$, $k \in \mathbb{Z}$, then $t_Q = 0$.*

(ii) [3, p. 115] $\mathcal{A}_{k+1} \subset \mathcal{A}_k$.

(iii) [3, p. 115 (2.3.16)]

$$\left\{x \in \mathbb{R}^n : g^s_r(t;x) > 2^k\right\} = \bigcup_{R\in\mathcal{A}_k} R.$$

(iv) [3, p. 115 (2.3.17)] *For almost every $x \in \mathbb{R}^n$,*

$$\left(\sum_{Q\in\mathcal{D}\setminus\mathcal{A}_k} \left(|Q|^{-\frac{s}{n}}|t_Q|\chi_Q(x)\right)^r\right)^{\frac{1}{r}} \leqslant 2^k.$$

2.3. Lemma. *Let $k \in \mathbb{Z}$, and let $t = \{t_Q\}_{Q\in\mathcal{D}}$ be a sequence. Define $\mathcal{A}_k$ as in Lemma 2.2.*

(i) *Let $v = \{v_Q\}_{Q\in\mathcal{D}}$ be a sequence indexed by $Q \in \mathcal{D}$. Assume*

$$g^s_r(v;x) < \infty$$

for a.e. $x \in \mathbb{R}^n$. *We set*

$$\mathcal{B}_k \equiv \left\{ J \in \mathcal{D} : J \text{ is a maximal dyadic cube in } \mathcal{A}_k \setminus \mathcal{A}_{k+1} \right\}.$$

For $J \in \mathcal{B}_k$, *we define*

$$t(k,J) \equiv \left\{ t(k,J)_Q \right\}_{Q \in \mathcal{D}} \equiv \left\{ v_Q \chi_{\mathcal{A}_k \setminus \mathcal{A}_{k+1}}(Q) \chi_{\{S \in \mathcal{D} : S \subset J\}}(Q) \right\}_{Q \in \mathcal{D}},$$
$$r(k,J) \equiv 2^{-k-1} t(k,J).$$

(ii) [3, p. 116 (2.3.18)] and [3, p. 116 (2.3.21)] *We have*

$$v = \sum_{k=-\infty}^{\infty} \sum_{J \in \mathcal{B}_k} t(k,J) = \sum_{k=-\infty}^{\infty} \sum_{J \in \mathcal{B}_k} 2^{k+1} r(k,J) \tag{8}$$

in the sense of pointwise convergence.

(iii) [3, p. 116 (2.3.19)] *For all* $J \in \mathcal{B}_k$, $g_r^s(t(k,J); \cdot) \leqslant 2^{k+1}$.

Now we prove Theorem 1.4.

(i) Let $t \in \dot{\mathbf{e}}^s_{p,q,r}(\mathbb{R}^n)$ be given so that $g_r^s(t;\cdot) < \infty$ almost everywhere. By (8), we can write

$$t = \sum_{k \in \mathbb{Z}} \sum_{J \in \mathcal{B}_k} 2^{k+1} r(k,J).$$

As we did in [1], we enumerate this sum as follows:

$$t = \sum_{j=1}^{\infty} \lambda_j r_j,$$

where each r_j is an ∞-atom.
We observe that

$$\sum_{J \in \mathcal{B}_k} \chi_J \leqslant \chi_{\cup_{Q \in \mathcal{A}_k} Q} \leqslant \chi_{\{g_r^s(t;\cdot) > 2^k\}}.$$

We calculate

$$\begin{aligned}
\left\| \left(\sum_{j=1}^{\infty} |\lambda_j|^u \chi_{Q_j} \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_q} &\lesssim \left\| \left(\sum_{k=-\infty}^{\infty} \sum_{J \in \mathcal{B}_k} (2^{k+1} \chi_J)^u \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_q} \\
&\lesssim \left\| \left(\sum_{k=-\infty}^{\infty} 2^{ku} \chi_{\{g_r^s(t;\cdot) > 2^k\}} \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_q} \\
&\leqslant \left\| \left(\sum_{k=-\infty}^{[1+\log_2 g_r^s(t;\cdot)]} 2^{ku} \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_q}.
\end{aligned}$$

Recall that $u > 0$. If we calculate the geometric series, then we obtain

$$\left\| \left(\sum_{j=1}^{\infty} |\lambda_j|^u \chi_{Q_j} \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_q} \lesssim \| 2^{[1+\log_2 g^s_r(t;\cdot)]} \|_{\mathcal{M}^p_q} \lesssim \| g^s_r(t;\cdot) \|_{\mathcal{M}^p_q} = \|t\|_{\dot{\mathbf{e}}^s_{p,q,r}}.$$

(ii) Conversely suppose we are given a sequence $r_j = \{r_{j,Q}\}_{Q\in\mathcal{D}}$ of ∞-atoms. Denote by Q_j the cube for r_j in the definition of atoms. Then, setting

$$t = \sum_{j=1}^{\infty} \lambda_j r_j,$$

we have

$$\|t\|_{\dot{\mathbf{e}}^s_{p,q,r}} \leqslant \left\{ \left\| \left(g^s_r \left(\sum_{j=1}^{\infty} \lambda_j r_j; \cdot \right) \right)^u \right\|_{\mathcal{M}^{p/u}_{q/u}} \right\}^{\frac{1}{u}} \leqslant \left\{ \left\| \sum_{j=1}^{\infty} |\lambda_j|^u g^s_r(r_j;\cdot)^u \right\|_{\mathcal{M}^{p/u}_{q/u}} \right\}^{\frac{1}{u}}.$$

Here we have used $u \leqslant \min(1,r)$ to obtain the last inequality.
If we use $g^s_r(r_j;\cdot) \leqslant \chi_{Q_j}$, then we obtain

$$\|t\|_{\dot{\mathbf{e}}^s_{p,q,r}} \leqslant \left\{ \left\| \sum_{j=1}^{\infty} |\lambda_j|^u \chi_{Q_j} \right\|_{\mathcal{M}^{p/u}_{q/u}} \right\}^{\frac{1}{u}} = \left\| \left(\sum_{j=1}^{\infty} |\lambda_j|^u \chi_{Q_j} \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_q} < \infty.$$

Thus, the proof is complete.

3. Proof of Theorem 1.8

We use the following decomposition result for $\dot{\mathcal{E}}^s_{p,q,r}(\mathbb{R}^n)$ from [24, 27, 34].

3.1. Theorem. *Let $0 < q \leqslant p < \infty$, $0 < r \leqslant \infty$ and $s \in \mathbb{R}$. Let K be an integer satisfying $K \geqslant [1+s]_+ \equiv \max(0,[1+s])$. Furthermore, let $L \in \mathbb{Z}$ satisfy*

$$L \geqslant \max(-1, [\sigma_{q,r} - s]).$$

(i) *Let $\kappa = \{\kappa_{\nu,m}\}_{\nu\in\mathbb{Z},\, m\in\mathbb{Z}^n} \in \dot{\mathbf{e}}^s_{p,q,r}(\mathbb{R}^n)$ and each $a_{\nu,m}$ is a smooth L-atom centered at $Q_{\nu,m}$ for each ν, m. Then*

$$f \equiv \sum_{\nu=-\infty}^{\infty} \sum_{m\in\mathbb{Z}^n} \kappa_{\nu,m} a_{\nu,m}$$

converges in $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ and

$$\|f\|_{\dot{\mathcal{E}}^s_{p,q,r}} \lesssim \|\kappa\|_{\dot{\mathbf{e}}^s_{p,q,r}}.$$

(ii) *Any* $f \in \dot{\mathcal{E}}^s_{p,q,r}(\mathbb{R}^n)$ *admits a decomposition*:

$$f = \sum_{\nu=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} \kappa_{\nu,m} a_{\nu,m}. \tag{9}$$

Here, the convergence takes place in $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ *and each* $a_{\nu,m}$ *is a smooth L-atom with a cube* $Q_{\nu,m}$ *and the coefficient* $\kappa = \{\kappa_{\nu,m}\}_{\nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n}$ *satisfies*

$$\|\kappa\|_{\dot{\mathbf{e}}^s_{p,q,r}} \lesssim \|f\|_{\dot{\mathcal{E}}^s_{p,q,r}}. \tag{10}$$

We prove Theorem 1.8. First we start with the first half of Theorem 1.8.

Let $f \in \mathcal{S}'(\mathbb{R}^n)$ be such that $f = \sum_{j=1}^{\infty} \lambda_j A_j$ in $\mathcal{S}'(\mathbb{R}^n)$ for some A_j described in Theorem 1.8. We set

$$f^T = \sum_{j=1}^{T} \lambda_j A_j = \sum_{j=1}^{T} \sum_{\nu=-\infty}^{\infty} \lambda_j \left(\sum_{Q \in \mathcal{D}_\nu} r_{j,Q} a_Q \right) = \sum_{\nu=-\infty}^{\infty} \sum_{Q \in \mathcal{D}_\nu} \left(\sum_{j=1}^{T} \lambda_j r_{j,Q} \right) a_Q$$

for $T \in \mathbb{N}$. We know that $f^T \to f$ as $T \to \infty$. We set

$$\kappa^T = \left\{ \sum_{j=1}^{T} \lambda_j r_{j,Q} \right\}_{Q \in \mathcal{D}}, \quad T \in \mathbb{N}.$$

Then we have

$$\|\kappa^T\|_{\dot{\mathbf{e}}^s_{p,q,r}} = \left\| g^s_r \left(\left\{ \sum_{j=1}^{T} \lambda_j r_{j,Q} \right\}_{Q \in \mathcal{D}} ; \cdot \right) \right\|_{\mathcal{M}^p_q} = \left(\left\| g^s_r \left(\left\{ \sum_{j=1}^{T} \lambda_j r_{j,Q} \right\}_{Q \in \mathcal{D}} ; \cdot \right)^u \right\|_{\mathcal{M}^{p/u}_{q/u}} \right)^{\frac{1}{u}}.$$

If we use $u \leqslant \min(1, r)$, then we have

$$\begin{aligned}
\|\kappa^T\|_{\dot{\mathbf{e}}^s_{p,q,r}} &\leqslant \left(\left\| \sum_{j=1}^{T} g^s_r \left(\left\{ \lambda_j r_{j,Q} \right\}_{Q \in \mathcal{D}} ; \cdot \right)^u \right\|_{\mathcal{M}^{p/u}_{q/u}} \right)^{\frac{1}{u}} \\
&= \left(\left\| \sum_{j=1}^{T} |\lambda_j|^u g^s_r \left(\{ r_{j,Q} \}_{Q \in \mathcal{D}} ; \cdot \right)^u \right\|_{\mathcal{M}^{p/u}_{q/u}} \right)^{\frac{1}{u}} \\
&\leqslant \left(\left\| \sum_{j=1}^{T} |\lambda_j|^u \chi_{Q_j} \right\|_{\mathcal{M}^{p/u}_{q/u}} \right)^{\frac{1}{u}} \\
&= \left\| \left(\sum_{j=1}^{T} |\lambda_j|^u \chi_{Q_j} \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_q};
\end{aligned}$$

i.e.,

$$\|\kappa^T\|_{\dot{\mathbf{e}}^s_{p,q,r}} \leqslant \left\| \left(\sum_{j=1}^{T} |\lambda_j|^u \chi_{Q_j} \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_q}. \tag{11}$$

Thus, by Theorem 3.1, we have

$$\|f^T\|_{\dot{\mathcal{E}}^s_{p,q,r}} \lesssim \|\kappa^T\|_{\dot{\mathrm{e}}^s_{p,q,r}} \leqslant \left\| \left(\sum_{j=1}^{T} |\lambda_j|^u \chi_{Q_j} \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_q}.$$

From the Fatou property of $\dot{\mathcal{E}}^s_{p,q,r}(\mathbb{R}^n)$ or from the classical Fatou lemma, we conclude

$$\|f\|_{\dot{\mathcal{E}}^s_{p,q,r}} \lesssim \left\| \left(\sum_{j=1}^{\infty} |\lambda_j|^u \chi_{Q_j} \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_q}.$$

Next we prove the latter half of Theorem 1.8. Let $f \in \dot{\mathcal{E}}^s_{p,q,r}(\mathbb{R}^n)$. Decompose f according to Theorem 3.1, so that (9) and (10) hold. If $Q = Q_{\nu,m}$, we write $\lambda_Q \equiv \kappa_{\nu,m}$ and $a_Q \equiv a_{\nu,m}$. We let $\mathcal{B} \equiv \{(k,J) : k \in \mathbb{Z}, J \in \mathcal{B}_k\}$. Let $N\colon j \in \mathbb{N} \mapsto (k_j, J_j) \in \mathcal{B}$ be its enumeration. Let $\lambda = \{\lambda_Q\}_{Q\in\mathcal{D}}$. Since $g^s_q(\lambda;x) < \infty$ for almost all $x \in \mathbb{R}^n$ and for all $R \in \mathcal{D}$, we have a decomposition:

$$\lambda = \sum_{k=-\infty}^{\infty} \sum_{J\in\mathcal{B}_k} 2^{k+1} r(k,J) = \sum_{j=1}^{\infty} 2^{k_j+1} r(k_j, J_j),$$

where each $r(k,J)$ is an ∞-atom with a cube J. According to Theorem 1.4,

$$\left\| \left(\sum_{j=1}^{\infty} 2^{(k_j+1)u} \chi_{J_j} \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_q} = \left\| \left(\sum_{k=-\infty}^{\infty} \sum_{J\in\mathcal{B}_k} 2^{(k+1)u} \chi_J \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_q} \lesssim \|\lambda\|_{\dot{\mathrm{e}}^s_{p,q,r}}. \tag{12}$$

If we combine (10) and (12), then we obtain

$$\|f\|_{\dot{\mathcal{E}}^s_{p,q,r}} \gtrsim \left\| \left(\sum_{j=1}^{\infty} 2^{(k_j+1)u} \chi_{J_j} \right)^{\frac{1}{u}} \right\|_{\mathcal{M}^p_q}.$$

We claim that

$$\sum_{j=1}^{\infty} \sum_{\nu=-\infty}^{\infty} \sum_{Q\in\mathcal{D}_\nu} 2^{k_j+1} r(k_j,J_j)_Q a_Q = \sum_{\nu=-\infty}^{\infty} \sum_{Q\in\mathcal{D}_\nu} \sum_{j=1}^{\infty} 2^{k_j+1} r(k_j,J_j)_Q a_Q \tag{13}$$

in the topology of $\mathcal{S}'_\infty(\mathbb{R}^n)$. To this end, it suffices to check that

$$\sum_{j=1}^{\infty} \sum_{\nu=-\infty}^{\infty} \sum_{Q\in\mathcal{D}_\nu} 2^{k_j+1} |r(k_j,J_j)_Q \langle a_Q, \varphi\rangle| < \infty$$

for all $\varphi \in \mathcal{S}_\infty(\mathbb{R}^n)$. Let $J \in \mathbb{N}$ be fixed. Then letting

$$\tilde{r}(k_j,J_j) \equiv \{r(k_j,J_j)_Q \overline{\mathrm{sign}(\langle a_Q,\varphi\rangle)}\}_{Q\in\mathcal{D}},$$

we have

$$\sum_{j=1}^{T}\sum_{\nu=-\infty}^{\infty}\sum_{Q\in\mathcal{D}_\nu}2^{k_j+1}|r(k_j,J_j)_Q\langle a_Q,\varphi\rangle| = \sum_{j=1}^{T}\sum_{\nu=-\infty}^{\infty}\sum_{Q\in\mathcal{D}_\nu}2^{k_j+1}\tilde{r}(k_j,J_j)_Q\langle a_Q,\varphi\rangle$$
$$= \left\langle \sum_{\nu=-\infty}^{\infty}\sum_{Q\in\mathcal{D}_\nu}\sum_{j=1}^{T}2^{k_j+1}\tilde{r}(k_j,J_j)_Q a_Q,\varphi\right\rangle.$$

Meanwhile, from (2) and (3), we deduce $\dot{\mathcal{E}}^s_{p,q,r}(\mathbb{R}^n) \hookrightarrow \dot{\mathcal{E}}^s_{p,q,\infty}(\mathbb{R}^n) \hookrightarrow \dot{B}^{s-n/p}_{\infty,\infty}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'_\infty(\mathbb{R}^n)$. Thus, there exists a seminorm $p\colon \mathcal{S}_\infty(\mathbb{R}^n) \to [0,\infty)$ such that

$$\sum_{j=1}^{T}\sum_{\nu=-\infty}^{\infty}\sum_{Q\in\mathcal{D}_\nu}2^{k_j+1}|r(k_j,J_j)_Q\langle a_Q,\varphi\rangle| \lesssim p(\varphi)\left\|\sum_{\nu=-\infty}^{\infty}\sum_{Q\in\mathcal{D}_\nu}\sum_{j=1}^{T}2^{k_j+1}\tilde{r}(k_j,J_j)_Q a_Q\right\|_{\dot{\mathcal{E}}^s_{p,q,r}}.$$

Note that $|\tilde{r}(k_j,J_j)| = |r(k_j,J_j)_Q|$. Thus, if we go through the same argument as (11), then we obtain

$$\left\|g^s_r\left(\left\{\left(\sum_{j=1}^{T}2^{k_j+1}\tilde{r}(k_j,J_j)_Q\right)\right\}_{Q\in\mathcal{D}}\right)\right\|_{\mathcal{M}^p_q} \leqslant \left\|\left(\sum_{j=1}^{T}|\lambda_j|^u\chi_{Q_j}\right)^{\frac{1}{u}}\right\|_{\mathcal{M}^p_q}$$
$$\leqslant \left\|\left(\sum_{j=1}^{\infty}|\lambda_j|^u\chi_{Q_j}\right)^{\frac{1}{u}}\right\|_{\mathcal{M}^p_q}.$$

Using Theorem 3.1, we obtain

$$\left\|\sum_{\nu=-\infty}^{\infty}\sum_{Q\in\mathcal{D}_\nu}\sum_{j=1}^{T}2^{k_j+1}\tilde{r}(k_j,J_j)_Q a_Q\right\|_{\dot{\mathcal{E}}^s_{p,q,r}} \lesssim \left\|\left(\sum_{j=1}^{\infty}|\lambda_j|^u\chi_{Q_j}\right)^{\frac{1}{u}}\right\|_{\mathcal{M}^p_q}.$$

Hence,

$$\sum_{j=1}^{T}\sum_{\nu=-\infty}^{\infty}\sum_{Q\in\mathcal{D}_\nu}2^{k_j+1}|r(k_j,J_j)_Q\langle a_Q,\varphi\rangle| \lesssim p(\varphi)\left\|\left(\sum_{j=1}^{\infty}|\lambda_j|^u\chi_{Q_j}\right)^{\frac{1}{u}}\right\|_{\mathcal{M}^p_q}.$$

The right-hand side being independent of T, we obtain (13).

Thus, we conclude from (13) that

$$f = \sum_{\nu=-\infty}^{\infty}\sum_{Q\in\mathcal{D}_\nu}\lambda_Q a_Q$$
$$= \sum_{\nu=-\infty}^{\infty}\sum_{Q\in\mathcal{D}_\nu}\sum_{j=1}^{\infty}2^{k_j+1}r(k_j,J_j)_Q a_Q$$
$$= \sum_{j=1}^{\infty}\sum_{\nu=-\infty}^{\infty}\sum_{Q\in\mathcal{D}_\nu}2^{k_j+1}r(k_j,J_j)_Q a_Q.$$

If we define

$$A_j \equiv \sum_{\nu=-\infty}^{\infty} \sum_{Q\in\mathcal{D}_\nu} r(k_j, J_j)_Q a_Q \quad (j \in \mathbb{N}),$$

then we have (4). Thus, we obtain the desired decomposition.

4. Proof of Theorem 1.12

We will need the following lemma:

4.1. Lemma ([8, Theorem 1.1]). *Suppose that the parameters p, q, s, t satisfy*

$$1 < q \leqslant p < \infty, \quad 1 < t \leqslant s < \infty, \quad q < t, \quad p < s.$$

Assume that $\{Q_j\}_{j=1}^{\infty} \subset \mathcal{D}(\mathbb{R}^n)$, $\{a_j\}_{j=1}^{\infty} \subset \mathcal{M}_t^s(\mathbb{R}^n)$ and $\{\lambda_j\}_{j=1}^{\infty} \subset [0,\infty)$ fulfill

$$\|a_j\|_{\mathcal{M}_t^s} \leqslant |Q_j|^{1/s}, \quad \operatorname{supp}(a_j) \subset Q_j, \quad \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_q^p} < \infty.$$

Then $f = \sum_{j=1}^{\infty} \lambda_j a_j$ converges in $\mathcal{S}'(\mathbb{R}^n) \cap L^q_{\mathrm{loc}}(\mathbb{R}^n)$ and satisfies

$$\|f\|_{\mathcal{M}_q^p} \leqslant C_{p,q,s,t} \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_q^p}.$$

We prove Theorem 1.12. Let $K, L \gg 1$ be fixed. Then we have a decomposition (4). Since $f \in L^\nu(\mathbb{R}^n)$ with $\nu < \frac{n}{\alpha}$, our proof of Theorem 1.11 with $p = q$ also shows that the convergence in (4) takes place in $L^\nu(\mathbb{R}^n)$ and we have

$$I_\alpha f(x) = \sum_{j=1}^{\infty} \lambda_j I_\alpha A_j(x)$$

almost everywhere. We observe

$$\begin{aligned} |g(x) I_\alpha f(x)| &\leqslant \sum_{j=1}^{\infty} |\lambda_j| \cdot |g(x) I_\alpha A_j(x)| \\ &= \sum_{j=1}^{\infty} |\lambda_j| \cdot |g(x) \chi_{3Q_j}(x) I_\alpha A_j(x)| + \sum_{j=1}^{\infty} |\lambda_j| \cdot |g(x) \chi_{\mathbb{R}^n \setminus 3Q_j}(x) I_\alpha A_j(x)|. \end{aligned}$$

As for the first term, if $q < \tilde{u} < u$, we observe

$$\left(\frac{1}{|3Q_j|} \int_{3Q_j} |g(x) \chi_{3Q_j}(x) I_\alpha A_j(x)|^{\tilde{u}} \, dx \right)^{\frac{1}{\tilde{u}}} \lesssim \|g\|_{\mathcal{M}_u^{n/\alpha}}.$$

In fact, according to [29, Theorem 6.1.1] we have the Sobolev embedding

$$\dot F^{s+\alpha}_{p_0,\infty}(\mathbb{R}^n) \hookrightarrow \dot F^{s}_{p_1,q}(\mathbb{R}^n)$$

for $0 < p_0 < p_1 < \infty$, $0 < q \leqslant \infty$ $0 < \alpha < \infty$ and $s \in \mathbb{R}$ satisfying $\frac{1}{p_1} = \frac{1}{p_0} - \frac{\alpha}{n}$. Thus,

$$I_\alpha A_j \in \bigcap_{u>0} \dot F^{\alpha}_{u,\infty}(\mathbb{R}^n) \subset \bigcap_{u>\frac{n}{\alpha}} \dot F^{0}_{u,1}(\mathbb{R}^n) \subset \bigcap_{u>\max\left(1,\frac{n}{\alpha}\right)} L^u(\mathbb{R}^n),$$

since $A_j \in \bigcap_{u>0} \dot F^0_{u,\infty}(\mathbb{R}^n)$. Hence we are in a position to use the Hölder inequality and Lemma 4.1 to control the first term.

As for the second term, we need

$$|\chi_{\mathbb{R}^n\setminus 3Q_j}(x) I_\alpha A_j(x)| \lesssim \frac{\ell(Q_j)^{n+L+1}}{(\ell(Q_j)+|x-c(Q_j)|)^{n+L+1-\alpha}}$$

(see [8, Lemma 4.2]). From this estimate, we see that

$$\sum_{j=1}^{\infty} |\lambda_j|\cdot|g(x)\chi_{\mathbb{R}^n\setminus 3Q_j}(x) I_\alpha A_j(x)| \lesssim \sum_{j=1}^{\infty} |\lambda_j|\cdot \ell(Q)^{\alpha}|g(x)| M\chi_{Q_j}(x)^{\frac{n+L+1-\alpha}{n}}.$$

Recall that L is at our disposal. If we use Lemma 4.1 and the Fefferman–Stein vector-valued inequality for Morrey spaces obtained in [23, 27], we arrive at the desired result.

5. Appendix: $\mathcal{M}^p_q(\mathbb{R}^n)$ and the closure of $\mathcal{S}_\infty(\mathbb{R}^n)$ in $\dot{\mathcal{E}}^0_{p,q,\infty}(\mathbb{R}^n)$

Let $1 < q < p < \infty$. Unfortunately, in general $\mathcal{M}^p_q(\mathbb{R}^n)$ is not embedded into the closure V of $\mathcal{S}_\infty(\mathbb{R}^n)$ in $\dot{\mathcal{E}}^0_{p,q,\infty}(\mathbb{R}^n)$. To see this, we let $E \equiv \{y + (R-1)(a_1 + Ra_2 + \cdots) : \{a_j\}_{j=1}^\infty \in \{0,1\}^n \cap \ell^1(\mathbb{N}), y \in [0,1]^n\}$ and $E_0 \equiv \{(R-1)(a_1 + Ra_2 + \cdots) : \{a_j\}_{j=1}^\infty \in \{0,1\}^n \cap \ell^1(\mathbb{N})\}$, where $R > 2$ solves $R^{\frac{n}{p}-\frac{n}{q}} 2^{\frac{n}{q}} = 1$. According to [22], $\chi_E \in \mathcal{M}^p_q(\mathbb{R}^n)$. We claim

$$f = \sum_{e\in E_0} \mathcal{F}^{-1}\varphi(\cdot - e) \in \mathcal{M}^p_q(\mathbb{R}^n) \setminus V.$$

By the boundedness of the Hardy–Littlewood maximal operator we have

$$|f| \lesssim \sum_{e\in E} (M\chi_E)^N$$

for any $N \in \mathbb{N}$. Thus, by the Fefferman–Stein vector-valued inequality for Morrey spaces [23, 27], $f \in \mathcal{M}^p_q(\mathbb{R}^n)$. Meanwhile, there exists a constant $c_0 > 0$ such that for any $\varphi_0 \in \mathcal{S}_\infty(\mathbb{R}^n)$,

$$\|\varphi(D)(f-\varphi_0)\|_{\mathcal{M}^p_q} \geqslant c_0,$$

because $\mathcal{F}^{-1}\varphi * \mathcal{F}^{-1}\varphi(0) > 0$ and

$$\varphi(D)f = c_n \sum_{e \in E_0} \mathcal{F}^{-1}\varphi * \mathcal{F}^{-1}\varphi(\cdot - e),$$

so $f \notin V$.

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