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On Pell and Pell–Lucas Hybrid Numbers

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Summary. In this paper we introduce the Pell and Pell–Lucas hybrid numbers as special kinds of hybrid numbers. We describe some properties of Pell hybrid numbers and Pell–Lucas hybrid numbers among other we give the Binet formula, the character and the generating function for these numbers.

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Pell numbers;
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1. Introduction

The hybrid numbers are generalization of complex, hyperbolic and dual numbers and they were introduced by Özdemir in [7].

Let $\mathbb{K}$ be the set of hybrid numbers $\mathbf{Z}$ of the form

$$\mathbf{Z} = a + b\mathbf{i} + c\epsilon + d\mathbf{h},$$

where $a, b, c, d \in \mathbb{R}$ and $\mathbf{i}, \epsilon, \mathbf{h}$ are operators such that

$$\mathbf{i}^2 = -1, \ \epsilon^2 = 0, \ \mathbf{h}^2 = 1 \tag{1}$$

and

$$\mathbf{ih} = -\mathbf{hi} = \epsilon + \mathbf{i}. \tag{2}$$

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If $\mathbf{Z}_1 = a_1 + b_1\mathbf{i} + c_1\epsilon + d_1\mathbf{h}$, and $\mathbf{Z}_2 = a_2 + b_2\mathbf{i} + c_2\epsilon + d_2\mathbf{h}$, are any two hybrid numbers then equality, addition, substraction and multiplication by scalar $s \in \mathbb{R}$ are defined in the following way:

(i) $\mathbf{Z}_1 = \mathbf{Z}_2$ only if $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$,

(ii) $\mathbf{Z}_1 + \mathbf{Z}_2 = (a_1 + a_2) + (b_1 + b_2)\mathbf{i} + (c_1 + c_2)\epsilon + (d_1 + d_2)\mathbf{h}$,

(iii) $\mathbf{Z}_1 - \mathbf{Z}_2 = (a_1 - a_2) + (b_1 - b_2)\mathbf{i} + (c_1 - c_2)\epsilon + (d_1 - d_2)\mathbf{h}$,

(iv) $s\mathbf{Z}_1 = sa_1 + sb_1\mathbf{i} + sc_1\epsilon + sd_1\mathbf{h}$,

respectively.

The hybrid numbers multiplication is defined applying formulas (1) and (2). Then we can find the product of any two hybrid units. For example, to find $\epsilon\mathbf{h}$ we can multiply $\mathbf{ih} = \epsilon + \mathbf{i}$ by $\mathbf{h}$ from the right. We obtain $\mathbf{ih}^2 = \epsilon\mathbf{h} + \mathbf{ih}$ and after calculation $\epsilon\mathbf{h} = -\epsilon$. Products of $\mathbf{i}$, ϵ, and $\mathbf{h}$ are presented in the Table 1.

Table 1. The hybrid number multiplication

$\cdot$	$\mathbf{i}$	ϵ	$\mathbf{h}$
$\mathbf{i}$	-1	$1 - \mathbf{h}$	$\epsilon + \mathbf{i}$
ϵ	$\mathbf{h} + 1$	0	$-\epsilon$
$\mathbf{h}$	$-\epsilon - \mathbf{i}$	ϵ	1

Using given above rules the multiplication of hybrid numbers can be made in the same way as multiplications of algebraic expressions. Note that multiplication operation in the hybrid numbers is associative, but not commutative.

The conjugate of a hybrid number $\mathbf{Z}$ is defined as follows

$$\overline{\mathbf{Z}} = \overline{a + b\mathbf{i} + c\epsilon + d\mathbf{h}} = a - b\mathbf{i} - c\epsilon - d\mathbf{h}.$$

Moreover the real number

$$C(\mathbf{Z}) = \mathbf{Z}\overline{\mathbf{Z}} = \overline{\mathbf{Z}}\mathbf{Z} = a^2 + (b - c)^2 - c^2 - d^2 = a^2 + b^2 - 2bc - d^2 \tag{3}$$

is named as the character of the hybrid number $\mathbf{Z}$.

For hybrid numbers details, see [7].

In [8] it was introduced a special kind of hybrid numbers namely the Horadam hybrid numbers. For integer $n \geqslant 0$ the nth Horadam number W_n is defined by the recurrence relation of the form $W_n = p \cdot W_{n-1} - q \cdot W_{n-2}$ with initial conditions W_0, W_1 where $p, q \in \mathbb{Z}$ and $W_0, W_1 \in \mathbb{R}$. If $W_0 = 0$, $W_1 = 2$, $p = 2$ and $q = -1$ then we obtain the nth Pell number P_n. For $W_0 = 2$, $W_1 = 2$, $p = 2$ and $q = -1$ we have the nth Pell–Lucas number Q_n.

2. The Pell and Pell–Lucas numbers

Let $n \geqslant 0$ be an integer. The nth Pell number P_n is defined in the following way $P_n = 2P_{n-1} + P_{n-2}$, for $n \geqslant 2$ with $P_0 = 0,\ P_1 = 1$.

Solving the above recurrence equation we obtain the direct formula of the form

$$P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}},$$

named also as the Binet formula for Pell numbers.

A special version of the Pell numbers is the the Pell–Lucas numbers Q_n (also named as the companion Pell numbers). Then $Q_n = 2Q_{n-1} + Q_{n-2}$, for $n \geqslant 2$ with $Q_0 = Q_1 = 2$.

The Binet formula for Pell–Lucas numbers has the form

$$Q_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n. \tag{4}$$

For $n = 0, 1, 2, \ldots$ the Pell numbers and the Pell–Lucas numbers are $0, 1, 2, 5, 12, 29, \ldots$ and $2, 2, 6, 14, 34, 82, \ldots$, respectively.

We recall some well-known properties of Pell and Pell–Lucas numbers which can be found in[5, 6]

$$P_{n+1} + P_{n-1} = Q_n \tag{5}$$

$$Q_{n+1} + Q_{n-1} = 8P_n \tag{6}$$

$$P_n + P_{n-1} = \frac{Q_n}{2} \tag{7}$$

$$Q_n + Q_{n-1} = 4P_n \tag{8}$$

$$\sum_{l=0}^{n} P_l = \frac{Q_{n+1} - 2}{4} \tag{9}$$

$$\sum_{l=0}^{n} Q_l = 2P_{n+1}. \tag{10}$$

3. The Pell and Pell–Lucas hybrid numbers

The nth Pell hybrid number PH_n and nth Pell–Lucas hybrid number QH_n are numbers defined as follows

$$PH_n = P_n + P_{n+1}\mathbf{i} + P_{n+2}\epsilon + P_{n+3}\mathbf{h}, \tag{11}$$

$$QH_n = Q_n + Q_{n+1}\mathbf{i} + Q_{n+2}\epsilon + Q_{n+3}\mathbf{h}, \tag{12}$$

respectively.

Using the above definitions we can give initial Pell and Pell–Lucas hybrid numbers, i.e.

$$\begin{aligned}
PH_0 &= \mathbf{i} + 2\epsilon + 5\mathbf{h},\\
PH_1 &= 1 + 2\mathbf{i} + 5\epsilon + 12\mathbf{h},\\
PH_2 &= 2 + 5\mathbf{i} + 12\epsilon + 29\mathbf{h},\\
&\dots\\
QH_0 &= 2 + 2\mathbf{i} + 6\epsilon + 14\mathbf{h},\\
QH_1 &= 2 + 6\mathbf{i} + 14\epsilon + 34\mathbf{h},\\
QH_2 &= 6 + 14\mathbf{i} + 34\epsilon + 82\mathbf{h},\\
&\dots
\end{aligned}$$

Applying the formula (3) we will calculate the character of the Pell hybrid number and the Pell–Lucas hybrid number.

3.1. Theorem ([8]). *Let $n \geqslant 0$ be an integer. Then*

$$C(PH_n) = -3P_n^2 - 28P_{n+1}^2 - 22P_nP_{n+1}.$$

3.2. Theorem. *Let $n \geqslant 0$ be an integer. Then*

$$C(QH_n) = -3Q_n^2 - 28Q_{n+1}^2 - 22Q_nQ_{n+1}.$$

Proof. Let $Q_{n+2} = 2Q_{n+1} + Q_n$, $Q_{n+3} = 2Q_{n+2} + Q_{n+1} = 5Q_{n+1} + 2Q_n$ and $C(QH_n) = Q_n^2 + Q_{n+1}^2 - 2Q_{n+1}Q_{n+2} - Q_{n+3}^2$. Then $C(QH_n) = Q_n^2 + Q_{n+1}^2 - 2Q_{n+1}(2Q_{n+1} + Q_n) - (5Q_{n+1} + 2Q_n)^2$ and by simple calculations the result follows. □

Next we give the Binet formula for the Pell hybrid numbers and the Pell–Lucas hybrid numbers.

3.3. Theorem ([8]). *Let $n \geqslant 0$ be an integer. Then*

$$\begin{aligned}
PH_n = {} & \frac{1}{2\sqrt{2}}\left(1+\sqrt{2}\right)^n\left(1 + (1+\sqrt{2})\mathbf{i} + (3+2\sqrt{2})\epsilon + (7+5\sqrt{2})\mathbf{h}\right)\\
& - \frac{1}{2\sqrt{2}}\left(1-\sqrt{2}\right)^n\left(1 + (1-\sqrt{2})\mathbf{i} + (3-2\sqrt{2})\epsilon + (7-5\sqrt{2})\mathbf{h}\right).
\end{aligned}$$

3.4. Theorem. *Let $n \geqslant 0$ be an integer. Then*

$$\begin{aligned}
QH_n = {} & \left(1+\sqrt{2}\right)^n\left(1 + (1+\sqrt{2})\mathbf{i} + (3+2\sqrt{2})\epsilon + (7+5\sqrt{2})\mathbf{h}\right)\\
& + \left(1-\sqrt{2}\right)^n\left(1 + (1-\sqrt{2})\mathbf{i} + (3-2\sqrt{2})\epsilon + (7-5\sqrt{2})\mathbf{h}\right).
\end{aligned}$$

Proof. By the definition of Pell–Lucas hybrid number (12) and the Binet formula for the Pell–Lucas numbers (4) we obtain

$$\begin{aligned} QH_n &= (1+\sqrt{2})^n + (1-\sqrt{2})^n + \left((1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right)\mathbf{i} \\ &\quad + \left((1+\sqrt{2})^{n+2} + (1-\sqrt{2})^{n+2}\right)\epsilon + \left((1+\sqrt{2})^{n+3} + (1-\sqrt{2})^{n+3}\right)\mathbf{h} \end{aligned}$$

and the Theorem follows by simple calculations. □

For the Pell hybrid numbers the and Pell–Lucas hybrid numbers we can determine the ordinary generating functions.

3.5. Theorem ([8]). *The generating function for the Pell hybrid number sequence* $\{PH_n\}$ *is*

$$\sum_{n=0}^{\infty} PH_n t^n = \frac{PH_0 + t(PH_1 - 2PH_0)}{1-2t-t^2} = \frac{(\mathbf{i} + 2\epsilon + 5\mathbf{h}) + t(1+\epsilon+2\mathbf{h})}{1-2t-t^2}.$$

3.6. Theorem. *The generating function for the Pell–Lucas hybrid number sequence* $\{QH_n\}$ *is*

$$\begin{aligned} \sum_{n=0}^{\infty} QH_n t^n &= \frac{QH_0 + t(QH_1 - 2QH_0)}{1-2t-t^2} \\ &= \frac{(2+2\mathbf{i}+6\epsilon+14\mathbf{h}) + t(-2+2\mathbf{i}+2\epsilon+6\mathbf{h})}{1-2t-t^2}. \end{aligned}$$

Proof. Assuming that the generating function of the Pell–Lucas hybrid number sequence $\{QH_n\}$ has the form $G(t) = \sum_{n=0}^{\infty} QH_n t^n$, we obtain that

$$\begin{aligned} (1-2t-t^2)G(t) &= (1-2t-t^2)\cdot(QH_0 + QH_1 t + QH_2 t^2 + \dots) \\ &= QH_0 + QH_1 t + QH_2 t^2 + \dots \\ &\quad - 2QH_0 t - 2QH_1 t^2 - 2QH_2 t^3 - \dots \\ &\quad - QH_0 t^2 - QH_1 t^3 - QH_2 t^4 + \dots \\ &= QH_0 + t(QH_1 - 2QH_0), \end{aligned}$$

since $QH_n = 2QH_{n-1} + QH_{n-2}$ and the coefficients of t^n for $n \geqslant 2$ are equal to zero.

Moreover, $QH_0 = 2 + 2\mathbf{i} + 6\epsilon + 14\mathbf{h}$ and $QH_1 - 2QH_0 = -2 + 2\mathbf{i} + 2\epsilon + 6\mathbf{h}$. □

4. Properties of Pell and Pell–Lucas hybrid numbers

In this section we give some identities for Pell hybrid numbers and Pell–Lucas hybrid numbers.

Using (5–8), (11), and (12) it immediately follows.

4.1. Theorem. *Let* $n \geqslant 1$. *Then*

(i) $PH_{n+1} + PH_{n-1} = QH_n$,

(ii) $QH_{n+1} + QH_{n-1} = 8 \cdot PH_n$,

(iii) $PH_n + PH_{n-1} = \frac{QH_n}{2}$,

(iv) $QH_n + QH_{n-1} = 4 \cdot PH_n$.

Now we give formulas for the sum of Pell hybrid numbers and Pell–Lucas hybrid numbers.

4.2. Theorem. *Let* $n \geqslant 0$. *Then*

(i) $\sum\limits_{l=0}^{n} PH_l = \frac{1}{4}\left(QH_{n+1} - 2QH_0\right)$,

(ii) $\sum\limits_{l=0}^{n} QH_l = 2PH_{n+1} - 2PH_0$.

Proof. (i). Using (9) and (11) we have

$$\begin{aligned}
\sum_{l=0}^{n} PH_l &= PH_0 + PH_1 + \ldots + PH_n \\
&= (P_0 + P_1\mathbf{i} + P_2\epsilon + P_3\mathbf{h}) + (P_1 + P_2\mathbf{i} + P_3\epsilon + P_4\mathbf{h}) \\
&\quad + \ldots + (P_n + P_{n+1}\mathbf{i} + P_{n+2}\epsilon + P_{n+3}\mathbf{h}) \\
&= (P_0 + P_1 + \ldots + P_n) + (P_1 + P_2 + \ldots + P_{n+1} + P_0 - P_0)\mathbf{i} \\
&\quad + (P_2 + P_3 + \ldots + P_{n+2} + P_0 + P_1 - P_0 - P_1)\epsilon \\
&\quad + (P_3 + P_4 + \ldots + P_{n+3} + P_0 + P_1 + P_2 - P_0 - P_1 - P_2)\mathbf{h} \\
&= \frac{Q_{n+1} - 2}{4} + \frac{Q_{n+2} - 2}{4}\mathbf{i} + \frac{Q_{n+3} - 6}{4}\epsilon + \frac{Q_{n+4} - 14}{4}\mathbf{h} \\
&= \frac{QH_{n+1} - (2 + 2\mathbf{i} + 6\epsilon + 14\mathbf{h})}{4}
\end{aligned}$$

which ends the proof.

(ii). Prooving analogously as above, using (10) and (12) the formula (ii) follows, which ends the proof. □

Apart the classical Pell sequence different generalizations of it were introduced and studied, see [1–4]. Using these generalizations we can introduce and study new types of hybrid numbers.

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