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Forms and mappings. III. Regular m-applications

Any mapping obtained from a form of degree m satisfies two known relations: it is homogeneous and its m-th defect is m-linear. The paper yields the third relation for such mappings, which is independent from the choice of the base ring R. This gives us, for $n \leqslant 1$ and $n \geqslant m-1$, a presentation by generators and relations of the modules $\bar{\Gamma}^{m,n}(R)$ defined in [3], or, in the terminology of [5], a strong system of n-covering relations of the functor Hom^m constituted by mappings mentioned above.

Introduction. Let R be a commutative ring and let M and N be R-modules. For any mapping $f\colon M \to N$ define the *n-th defect* $\Delta^n f\colon \underbrace{M \times \ldots \times M}_{n} \to N$ of f $(n = 0, 1, \ldots)$ by the formula

$$(\Delta^n f)(x_1, \ldots, x_n) = \sum_{H \subset [1,n]} (-1)^{n-|H|} f\Big(\sum_{i \in H} x_i\Big).$$

This is, obviously, a symmetric function. Moreover, $\Delta^0 f = f(0)$, $\Delta^1 f = f - f(0)$ and

$$(!)\quad (\Delta^{n+1} f)(x_0, \ldots, x_n) = (\Delta^n f)(x_0+x_1, x_2, \ldots, x_n) - (\Delta^n f)(x_0, x_2, \ldots, x_n) - (\Delta^n f)(x_1, \ldots, x_n).$$

Consequently, $\Delta^n f$ can be defined inductively, and it is easy to see that $(\Delta^n f)(0, -) = 0$ for $n > 0$. The mapping f is called an *m-application* if it satisfies the following conditions:

(A1) $\qquad f(rx) = r^m f(x)$ for any $r \in R$ and $x \in M$,

(A2) $\qquad \Delta^m f$ is m-linear.

In the natural way we obtain the functor of m-applications $\mathrm{Appl}^m\colon R-\mathrm{Mod}^0 \times R-\mathrm{Mod} \to R-\mathrm{Mod}$. Any m-application on M factorizes by the standard m-application $\delta^m\colon M \to \Delta^m(M) = R\{\delta^m(x);\, x \in M\}$, and hence Appl^m is represented by Δ^m (see [3] or [1]). Since Appl^0 and Appl^1 are well-known functors of constants and linear mappings, we will assume in the sequel that $m \geqslant 2$. Consequently, $f(0) = 0$ for any m-application f in question.

Let $\Gamma^m(M)$ denote the m-th divided power of M. Consider the mapping $\gamma^m\colon M \to \bar{\Gamma}^m(M) := R\{x^{(m)};\ x \in M\} \subset \Gamma^m(M)$ defined by $\gamma^m(x) = x^{(m)}$. Observe that ([3], (4.5))

$$(!!) \qquad (\Delta^n \gamma^m)(x_1, \ldots, x_n) = \sum_{\substack{m_1+\ldots+m_n=m \\ m_1,\ldots,m_n>0}} x_1^{(m_1)} \ldots x_n^{(m_n)} \in \bar{\Gamma}^m(M),$$

and hence γ^m is an m-application. This gives us an epimorphism

$$h^m = h^m(M)\colon \Delta^m(M) \to \bar{\Gamma}^m(M), \qquad h^m(\delta^m(x)) = \gamma^m(x).$$

Any mapping f obtained from a form of degree m in the usual sense or (more generally) in the sense of N. Roby (see [6]) factorizes by γ^m, and hence f is an m-application. Consequently, in the notation of [3], $\mathrm{Hom}^m(M, -) \subset \overline{\mathrm{Hom}}^m(M, -) \subset \mathrm{Appl}^m(M, -)$, where $\overline{\mathrm{Hom}}^m(M, -)$ is the subfunctor of $\mathrm{Appl}^m(M, -)$ represented by $\bar{\Gamma}^m(M)$. The last inclusion is an equality iff $h^m(M)$ is an isomorphism.

It follows from [3], Section 4, that the investigation of h^m on free R-modules reduces to the study of restrictions $h^{m,n}\colon \Delta^{m,n}(R) \to \bar{\Gamma}^{m,n}(R)$, where

$$\Delta^{m,n}(R) = R\{(\Delta^n \delta^m)(r_1 e_1, \ldots, r_n e_n);\ r_1, \ldots, r_n \in R\} \subset \Delta^m(Re_1 \oplus \ldots \oplus Re_n),$$

$$\bar{\Gamma}^{m,n}(R) = R\{(\Delta^n \gamma^m)(r_1 e_1, \ldots, r_n e_n);\ r_1, \ldots, r_n \in R\} \subset \bar{\Gamma}^m(Re_1 \oplus \ldots \oplus Re_n)$$

(see also Section 3). In the present paper we study $\Delta^{m,m-1}(R)$ and $h^{m,m-1}$. Section 1 yields some relations in $\Delta^{m,m-1}(R)$ and a sufficient condition for $h^{m,m-1}$ to be injective (Theorem 1.7). In Section 2 we introduce an additional regularity condition (A) and its consequences (B)–(F) (Proposition 2.5). We prove in Section 3 that (A) is the generating relation of $\mathrm{Ker}(h^{m,m-1})$ (Theorem 3.11). Finally, in Section 4 we give examples of irregular m-applications. In particular, Example 4.5 answers negatively the concluding question of [2]: there is a 4-application over a field which is not obtained from a form of degree 4.

1. Relations. For the study of m-applications, we need a number of conditions following from the defining ones (see introduction).

LEMMA 1.1. *Any m-application f satisfies relations*

$$\text{(i)} \qquad \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} f(x+iy) = m!\, f(y),$$

$$\text{(ii)} \qquad \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} (\Delta^n f)(ix, -) = 0 \quad \textit{for } n > 1.$$

Proof. It follows from [6] or from Remark 1.4 of [1] that $(\Delta^m f)(y, \ldots, y) = m!\,f(y)$. Moreover,

$$(\Delta^m f)(x, y, \ldots, y) = \sum_{i=0}^{m-1} (-1)^{m-i}\binom{m-1}{i}\big(f(iy) - f(x+iy)\big).$$

Consequently,

$$\begin{aligned} m!\,f(y) &= (\Delta^m f)(y, \ldots, y) = (\Delta^m f)(x+y, y, \ldots, y) - (\Delta^m f)(x, y, \ldots, y) \\ &= \sum_{i=0}^{m-1} (-1)^{m-i}\binom{m-1}{i}\big(f(x+iy) - f(x+(i+1)y)\big) \\ &= \sum_{i=0}^{m} (-1)^{m-i}\left(\binom{m-1}{i} + \binom{m-1}{i-1}\right) f(x+iy) \\ &= \sum_{i=0}^{m} (-1)^{m-i}\binom{m}{i} f(x+iy). \end{aligned}$$

Using (i), we compute that

$$\begin{aligned} &\sum_{i=0}^{m} (-1)^{m-i}\binom{m}{i}(\Delta^n f)(ix_1, x_2, \ldots, x_n) \\ &= \sum_{1\in H\subset[1,n]} (-1)^{n-|H|}\left(\sum_{i=0}^{m} (-1)^{m-i}\binom{m}{i} f(ix_1 + \sum_{j\in H\setminus\{1\}} x_j)\right) \\ &\qquad + \sum_{1\notin H\subset[1,n]} (-1)^{n-|H|}\left(\sum_{i=0}^{m} (-1)^{m-i}\binom{m}{i} f(\sum_{j\in H} x_j)\right) \\ &= \sum_{1\in H\subset[1,n]} (-1)^{n-|H|}\, m!\,f(x_1) + \sum_{1\notin H\subset[1,n]} (-1)^{n-|H|}\, m!\,f(0) \\ &= \Big(\sum_{H'\subset[2,n]} (-1)^{(n-1)-|H'|}\Big) m!\,f(x_1) = 0 \end{aligned}$$

since $|[2, n]| = n-1 > 0$. This proves (ii).

Remark 1.2. Relation (i) for $m = 2$ and $x = z - y$ gives us the 'law of parallelogram':

$$f(z-y) + f(z+y) = 2f(z) + 2f(y).$$

Similarly, for $m = 4$ and $x = z - 2y$, we obtain the formula

$$f(z+2y) + f(z-2y) = 4f(z+y) + 4f(z-y) - 6f(z) + 24f(y).$$

Substituting $f(a+b)$ by $(\Delta^2 f)(a, b) + f(a) + f(b)$ we get

$$(\Delta^2 f)(z, 2y) + (\Delta^2 f)(z, -2y) = 4\big((\Delta^2 f)(z, y) + (\Delta^2 f)(z, -y)\big).$$

A more general relation

$$(\Delta^2 f)(z, ry)+(\Delta^2 f)(z, -ry) = r^2\big((\Delta^2 f)(z, y)+(\Delta^2 f)(z, -y)\big), \qquad r \in R,$$

is fulfilled by γ^4 (see (!!)) and hence by any $f \in \overline{\mathrm{Hom}^4}(M, N)$.

Let f be an m-application (for instance, let $f = \delta^m$) and let $(x_1, \ldots, x_n)$ denote $(\Delta^n f)(x_1, \ldots, x_n)$ for $n \geqslant 1$. Then condition (A1) gives us

LEMMA 1.3.

(a) $$(-x_1, \ldots, -x_n) = (-1)^m (x_1, \ldots, x_n),$$

(b) $$(x_1, \ldots, x_n)+(-x_1, x_2, \ldots, x_n) = -(-x_1, x_1, \ldots, x_n),$$

(c) $$\sum_{i=1}^{s} (-1)^i (-x_1, \ldots, -x_i, x_i, \ldots, x_n) = (x_1, \ldots, x_n)+(-1)^{s-1}(-x_1, \ldots, -x_s, x_{s+1}, \ldots, x_n),$$

(d) $$\sum_{i=1}^{n} (-1)^i (-x_1, \ldots, -x_i, x_i, \ldots, x_n) = \begin{cases} 2(x_1, \ldots, x_n), & m \not\equiv n \pmod 2, \\ 0 & m \equiv n \pmod 2, \end{cases}$$

(e) $$\sum_{i=2}^{n} (-1)^{i-1} (x_1, -x_2, \ldots, -x_i, x_i, \ldots, x_n) = \begin{cases} (x_1, \ldots, x_n)-(-x_1, x_2, \ldots, x_n), & m \not\equiv n \pmod 2, \\ (x_1, \ldots, x_n)+(-x_1, x_2, \ldots, x_n), & m \equiv n \pmod 2. \end{cases}$$

Proof. (a) and (b) are obvious (see (!)), (c) follows from (b), because

$$\begin{aligned}\sum_{i=1}^{s} (-1)^i (-x_1, \ldots, -x_i, x_i, \ldots, x_n) &= \sum_{i=1}^{s} (-1)^{i-1} \big((-x_1, \ldots, -x_{i-1}, x_i, \ldots, x_n) \\ &\qquad +(-x_1, \ldots, -x_i, x_{i+1}, \ldots, x_n)\big) \\ &= (x_1, \ldots, x_n)+(-1)^{s-1}(-x_1, \ldots, -x_s, x_{s+1}, \ldots, x_n).\end{aligned}$$

Finally, (d) and (e) follow from (c) and (a).

Let now $(x, -) = (\Delta^{m-1} f)(x, -)$ and $(x, y, -) = (\Delta^m f)(x, y, -)$. The above lemma (for $n = m-1$) and condition (A2) give the following corollary.

COROLLARY 1.4. *For any* $r \in R$

(1) $$(rx, -)+(-rx, -) = r^2(x, x, -) = r^2\big((x, -)+(-x, -)\big),$$

(2) $$(rx, -)-(-rx, -) = r\big((x, -)-(-x, -)\big),$$

(3) $$2(rx, -) = r(r+1)(x, -)+r(r-1)(-x, -) = 2r(x, -)+r(r-1)(x, x, -),$$

$$(4)\qquad 2(x_1, \ldots, x_{m-1}) = \sum_{i=1}^{m-1} (x_1, \ldots, x_i, x_i, \ldots, x_{m-1})$$
$$= (x_1, \ldots, x_{m-1}, x_1 + \ldots + x_{m-1}),$$

$$(5)\qquad (-x_1, \ldots, -x_s, x_{s+1}, \ldots, x_{m-1})$$
$$= (-1)^{s-1}\Big((s-1)(x_1, \ldots, x_{m-1}) + \sum_{i=1}^{s} (x_1, \ldots, -x_i, \ldots, x_{m-1})\Big),$$

$$(6)\qquad \sum_{i=1}^{m-1} (x_1, \ldots, -x_i, \ldots, x_{m-1}) + (m-3)(x_1, \ldots, x_{m-1}) = 0.$$

Proof. (1) follows from (b), (2) follows from (e), (4) follows from (b) and (e), and (3) follows from (1) and (2) or from (4). (5) follows from (c) and (1) since

$$(x_1, \ldots, x_{m-1}) + (-1)^{s-1}(-x_1, \ldots, -x_s, x_{s+1}, \ldots, x_{m-1})$$
$$= \sum_{i=1}^{s} (x_1, \ldots, x_i, x_i, \ldots, x_{m-1})$$
$$= s(x_1, \ldots, x_{m-1}) + \sum_{i=1}^{s} (x_1, \ldots, -x_i, \ldots, x_{m-1}).$$

Finally, (6) follows from (5) and (a).

Let $f = \delta^m \colon R^{m-1} \to \Delta^m(R^{m-1})$. Write

$$\delta = R\,\{(e_1, \ldots, e_{m-1}), (-e_1, e_2, \ldots, e_{m-1}), \ldots, (e_1, \ldots, -e_{m-2}, e_{m-1})\}$$
$$\subset \Delta^{m,m-1}(R).$$

The above formulas give the following proposition.

Proposition 1.5.

$$(1)\qquad 2\Delta^{m,m-1}(R) \subset \delta.$$

$$(2)\qquad 2\,\mathrm{Ker}(h^{m,m-1}) = 2\Delta^{m,m-1}(R) \cap \mathrm{Ker}(h^{m,m-1}) = \delta \cap \mathrm{Ker}(h^{m,m-1}) = 0.$$

Proof. It follows from conditions (4), (1) and (6) of Corollary 1.4 that

$$2(r_1 e_1, \ldots, r_{m-1} e_{m-1}) = r_1 \ldots r_{m-1}\Big(\sum_{i=1}^{m-1} r_i (e_1, \ldots, e_i, e_i, \ldots, e_{m-1})\Big)$$
$$= r_1 \ldots r_{m-1}\Big(\sum_{i=1}^{m-1} r_i\big((e_1, \ldots, e_{m-1}) + (e_1, \ldots, -e_i, \ldots, e_{m-1})\big)\Big)$$
$$= r_1 \ldots r_{m-1}\Big((r_1 + \ldots + r_{m-2} - (m-4) r_{m-1})(e_1, \ldots, e_{m-1}) +$$
$$+ \Big(\sum_{i=1}^{m-2} (r_i - r_{m-1})(e_1, \ldots, -e_i, \ldots, e_{m-1})\Big) \in \delta.$$

This proves (1). It remains to check that $\delta \cap \mathrm{Ker}(h^{m,m-1}) = 0$. By formula (!!),

$$h^m((x_1, \ldots, x_{m-1})) = (\Delta^{m-1}\gamma^m)(x_1, \ldots, x_{m-1}) = \sum_{i=1}^{m-1} x_1 \ldots x_i^{(2)} \ldots x_{m-1}.$$

Write $a_i = e_1 \ldots e_i^{(2)} \ldots e_{m-1}$ and $\sigma = a_1 + \ldots + a_{m-1}$. Then $h^{m,m-1}(e_1, \ldots, e_{m-1}) = \sigma$ and $h^{m,m-1}(e_1, \ldots, -e_i, \ldots, e_{m-1}) = 2a_i - \sigma$. Suppose that

$$z = r(e_1, \ldots, e_{m-1}) + \sum_{i=1}^{m-2} r_i(e_1, \ldots, -e_i, \ldots, e_{m-1}) \in \mathrm{Ker}(h^{m,m-1}).$$

Then $r\sigma + \sum_{i=1}^{m-2} r_i(2a_i - \sigma) = 0$ and hence $r = r_1 + \ldots + r_{m-2}$ and $2r_1 = \ldots$ $\ldots = 2r_{m-2} = 0$, because $\sigma, a_1, \ldots, a_{m-2}$ are linearly independent in $\Gamma^m(Re_1 \oplus \ldots \oplus Re_{m-1})$. Consequently,

$$z = \sum_{i=1}^{m-2} r_i((e_1, \ldots, e_{m-1}) + (e_1, \ldots, -e_i, \ldots, e_{m-1})) = 0$$

because of the following lemma:

LEMMA 1.6. *If* $2r = 0$ *then* $r(x, x, -) = r((x, -) + (-x, -)) = 0$.

Proof. It follows from Corollary 1.4(2) that $r((x, -) - (-x, -)) = (rx, -) - (-rx, -) = 0$.

If 2 is invertible in R then Corollary 1.4(3) gives the following formula:

$$(rx, -) = \frac{r(r+1)}{2}(x, -) + \frac{r(r-1)}{2}(-x, -) = r(x, -) + \frac{r(r-1)}{2}(x, x, -).$$

We will prove that the same holds in the more general case when 2 divides $r(r-1)$ for $r \in R$, that is, in the notation of [3] and [4], if $I(R) = (r^2 - r;\, r \in R)$ is equal to $2R$. Observe that $r(r-1) = 2a = 2b$ implies that $a(x, x, -) = b(x, x, -)$, by Lemma 1.6.

THEOREM 1.7. *If* $m \geq 3$ *then the following conditions are equivalent*:

(1) $$I(R) = 2R,$$

(2) $$\Delta^{m,m-1}(R) = \delta,$$

(3) $$\Gamma^{m,m-1}(R) = h^{m,m-1}(\delta),$$

(4) $R/2R$ *is a Boolean ring.*

If the above are satisfied then

(5) $$(rx, -) = r(x, -) + a(x, x, -) = (r+a)(x, -) + a(-x, -)$$

for $2a = r(r-1)$

and

$$\text{(6)} \qquad \mathrm{Ker}(h^{m,m-1}) = 0.$$

Proof. (1) $\Rightarrow$ (2), (5), (6). Because of [3] (Lemma 5.1 and Corollary 3.6) we can assume that (R, M) is a local ring. It follows from Proposition 5.5 of [3] that $I(R) = M$ if $R/M \approx Z_2$ and $I(R) = R$ otherwise. In the first case, any $r \in R$ is of the form $2b$ or $2b+1$ for some $b \in R$. By Corollary 1.4(3),

$$(2bx, -) = 2(bx, -)+(bx, bx, -) = 2b(x, -)+b(2b-1)(x, x, -),$$

and hence

$$\begin{aligned}((2b+1)x, -) &= (2bx, -)+(x, -)+(2bx, x, -)\\ &= (2b+1)(x, -)+b(2b+1)(x, x, -).\end{aligned}$$

Consequently, $(rx, -) = r(x, -)+a(x, x, -) = (r+a)(x, -)+a(-x, -)$, where $2a = r(r-1)$. The same formula holds, obviously, if $2R = I(R) = R$. Hence $\Delta^{m,m-1}(R) = \delta$ by Corollary 1.4(5) (6), and finally, $\mathrm{Ker}(h^{m,m-1}) = 0$, by Proposition 1.5 (2).

(2) $\Rightarrow$ (3) is evident.

(3) $\Rightarrow$ (1). It follows from the proof of Proposition 1.5 (2) that

$$h^{m,m-1}(\delta) = R\sigma \oplus \bigoplus_{i=1}^{m-2} 2Ra_i,$$

where $a_i = e_1 \ldots e_i^{(2)} \ldots e_{m-1}$ and $\sigma = a_1 + \ldots + a_{m-1}$. On the other hand,

$$\bar{\Gamma}^{m,m-1}(R) = R\sigma \oplus \bigoplus_{i=1}^{m-2} I(R)a_i$$

by [3], Theorem 5.9 (1). Since $m \geqslant 3$ it follows that $I(R) = 2R$.

(1) $\Leftrightarrow$ (4). Since $I(R/2R) = I(R)/2R$ (see [3], Lemma 5.1) it follows that $I(R) = 2R$ iff $I(R/2R) = 0$ iff $R/2R$ is a Boolean ring.

It is proved in [4], Proposition 2.7 (3), that $I(R)\mathrm{Ker}(h^{m,m-1}) = 0$ for $m = 3$, and hence $\mathrm{Ker}(h^{3,2}) = 0$ provided that $I(R) = R$. Example 4.5 below shows that this is not true in general. Namely, for $R = Z_2(T)$ we have $I(R) = R$ and $\mathrm{Ker}(h^{m,m-1}) \neq 0$ for all $m \geqslant 4$.

2. Regular m-applications. Let us assume in the rest of the paper that $m \geqslant 3$. An m-application $f\colon M \to N$ over R will be called *regular* if $(\ ) = \Delta^{m-1} f$ satisfies the following condition

$$\text{(A)} \quad (rx, sy, -)-r(x, sy, -)-s(rx, y, -)+rs(x, y, -) = 0$$

for $r, s \in R$ and $x, y \in M$.

In the natural way, we obtain the functor $\underline{\mathrm{Appl}}^m$: $R-\mathrm{Mod}^0 \times R-\mathrm{Mod} \to R-\mathrm{Mod}$ of regular m-applications, which is an equationally definable functor in the sense of [5]. Then [5] gives us the following corollary.

COROLLARY 2.1. (1) $\underline{\mathrm{Appl}}^m$ *is represented by the functor* $\bar{\Delta}^m$: $R-\mathrm{Mod} \to R-\mathrm{Mod}$ *defined by*

$$\bar{\Delta}^m(M) = \Delta^m(M)/A^m(M) = R\{\bar{\delta}^m(x);\ x \in M\}, \quad \bar{\Delta}^m(f)(\bar{\delta}^m(x)) = \bar{\delta}^m(f(x)),$$

where

$$A^m(M) = R\{(rx, sy, -) - r(x, sy, -) - s(rx, y, -) + rs(x, y, -);$$
$$r, s \in R,\ x, y, \ldots \in M\},$$

$(\ ) = \Delta^{m-1}\delta^m$, *and* $\bar{\delta}^m(x)$ *denotes the class of* $\delta^m(x)$ *in* $\bar{\Delta}^m(M)$.

(2) $\bar{\Delta}^m$ *preserves direct limits and Grothendieck sequences.*

Moreover, formula (!!) gives us the following corollary.

COROLLARY 2.2. (1) $\mathrm{Hom}^m \subset \overline{\mathrm{Hom}}^m \subset \underline{\mathrm{Appl}}^m \subset \mathrm{Appl}^m$.

(2) h^m: $\Delta^m \xrightarrow{\mathrm{nat}} \bar{\Delta}^m \xrightarrow{\bar{h}^m} \bar{\Gamma}^m$, *where* $\bar{h}^m(\bar{\delta}^m(x)) = \gamma^m(x) = x^{(m)}$.

Proof. It suffices to check that $\overline{\mathrm{Hom}}^m \subset \underline{\mathrm{Appl}}^m$, or, equivalently, that γ^m: $M \to \bar{\Gamma}^m(M)$ is a regular m-application for every R-module M. This can be proved directly using the formula

$$(\Delta^{m-1}\gamma^m)(x_1, \ldots, x_{m-1}) = \sum_{i=1}^{m-1} x_1 \ldots x_i^{(2)} \ldots x_{m-1}.$$

We prove in Section 3 that $\bar{h}^{m,m-1}$: $\bar{\Delta}^{m,m-1}(R) \to \bar{\Gamma}^{m,m-1}(R)$ is an isomorphism, or, in the terminology of [5], that $\underline{\mathrm{Appl}}^m$ is an $(m-1)$-covering functor of Hom^m. In particular, $\bar{\Delta}^3$ and $\bar{\Gamma}^3$ coincide on the category of flat R-modules (cf. [4], Section 3). It follows from [5] that (A1), (A2) and (A) are the only covering relations of Hom^m which are strong (i.e., independent from the choice of R) for any $m \geq 3$. This explains the expression 'regular'.

It follows from [3], Proposition 3.5, that any m-application f: $M \to N$ can be localized to an m-application over R_S

$$f_S\colon M_S \to N_S, \quad f_S(x/s) = f(x)/s^m$$

for any multiplicative subset S of R. We prove the following result.

LEMMA 2.3. *Any localization of a regular m-application is also regular.*

Proof. Relation (A) allows us to compute that

$$s^2(ax, by, -) - as(sx, by, -) - bs(ax, sy, -) + ab(sx, sy, -) = 0.$$

Hence

$$\left(\frac{a}{s}\cdot\frac{x_1}{t}, \frac{b}{s}\cdot\frac{x_2}{t}, \frac{x_3}{st}, \ldots, \frac{x_{m-1}}{st}\right) - \frac{a}{s}\left(\frac{x_1}{t}, \frac{b}{s}\cdot\frac{x_2}{t}, \frac{x_3}{st}, \ldots, \frac{x_{m-1}}{st}\right)$$
$$-\frac{b}{s}\left(\frac{a}{s}\cdot\frac{x_1}{t}, \frac{x_2}{t}, \frac{x_3}{st}, \ldots, \frac{x_{m-1}}{st}\right) + \frac{a}{s}\cdot\frac{b}{s}\left(\frac{x_1}{t}, \frac{x_2}{t}, \frac{x_3}{st}, \ldots, \frac{x_{m-1}}{st}\right)$$

$$= \frac{1}{s^2(st)^m}\big(s^2(ax_1, bx_2, x_3, \ldots, x_{m-1}) - as(sx_1, bx_2, x_3, \ldots, x_{m-1})$$

$$- bs(ax_1, sx_2, x_3, \ldots, x_{m-1}) + ab(sx_1, sx_2, x_3, \ldots, x_{m-1})\big) = 0,$$

as desired.

As in [3], Section 3, we obtain the following corollary.

COROLLARY 2.4.

(1) $$\bar{\Delta}^m_R(M)_S \simeq \bar{\Delta}^m_{R_S}(M_S),$$

(2) $$\bar{\Delta}^{m,n}(R)_S \simeq \bar{\Delta}^{m,n}(R_S),$$

(3) $$\operatorname{Ker}(h^{m,n}_R)_S \simeq \operatorname{Ker}(h^{m,n}_{R_S}),$$

(4) $$\operatorname{Ker}(h^{m,n}_R) = 0 \quad \textit{iff} \quad \operatorname{Ker}(h^{m,n}_{R_P}) = 0$$

for any prime (maximal) ideal P in R.

For the study of regular m-applications f: $M \to N$ we need another relations, mentioned in the proposition below, which are satisfied by $(\ ) = \Delta^{m-1} f$. Each of these relations gives us a strong equationally definable functor contained in Appl^m.

PROPOSITION 2.5. *Any regular m-application satisfies the following relations (r_i, r, $s \in R$, x_i, $x \in M$)*

(B) $$(r_1 x_1, \ldots, r_{m-1} x_{m-1})$$
$$= \sum_{i=1}^{m-1} r_1 \ldots \hat{r}_i \ldots r_{m-1}(x_1, \ldots, x_{i-1}, r_i x_i, x_{i+1}, \ldots, x_{m-1})$$
$$-(m-2) r_1 \ldots r_{m-1}(x_1, \ldots, x_{m-1}),$$

(C) $$C(r) := \sum_{i=1}^{m-1} (x_1, \ldots, x_{i-1}, rx_i, x_{i+1}, \ldots, x_{m-1})$$
$$-\big(r^2 + (m-2)r\big)(x_1, \ldots, x_{m-1}) = 0,$$

(D) $$D(r, s) := (rsx, -) - r^2(sx, -) - s(rx, -) + r^2 s(x, -) = 0,$$

(E) $$E(r) := (r^2 x, -) - (r + r^2)(rx, -) + r^3(x, -) = 0,$$

(F) $$F(r, s) := (r - r^2)(sx, -) - (s - s^2)(rx, -) - (rs^2 - r^2 s)(x, -) = 0.$$

More precisely, for any m-application,

$$\begin{array}{c} (\mathrm{A}) \Leftrightarrow (\mathrm{B}) \Rightarrow (\mathrm{D}) \Rightarrow (\mathrm{E}) \Rightarrow (\mathrm{F}) \\ \Downarrow \\ (\mathrm{C}) \end{array}$$

Proof. (A) $\Leftrightarrow$ (B). It is easy to see that (B) $\Rightarrow$ (A). Assuming (A), we prove by induction on k that

$$(r_1 x_1, \ldots, r_k x_k, -) = \sum_{i=1}^{k} r_1 \ldots \hat{r}_i \ldots r_k (x_1, \ldots, x_{i-1}, r_i x_i, x_{i+1}, \ldots, x_k, -) \\ -(k-1) r_1 \ldots r_k (x_1, \ldots, x_k, -).$$

The formula holds evidently for $k = 1, 2$. If it is true for some $k < m-1$ then

$$\begin{aligned}(r_1 x_1, \ldots, r_{k+1} x_{k+1}, -) &= \sum_{i=1}^{k} r_1 \ldots \hat{r}_i \ldots r_k (x_1, \ldots, r_i x_i, \ldots, x_k, r_{k+1} x_{k+1}, -) \\ &\quad -(k-1) r_1 \ldots r_k (x_1, \ldots, x_k, r_{k+1} x_{k+1}, -) \\ &= \sum_{i=1}^{k} r_1 \ldots \hat{r}_i \ldots r_k r_{k+1} (x_1, \ldots, r_i x_i, \ldots, x_k, x_{k+1}, -) \\ &\quad + \sum_{i=1}^{k} r_1 \ldots r_k (x_1, \ldots, x_k, r_{k+1} x_{k+1}, -) \\ &\quad - \sum_{i=1}^{k} r_1 \ldots r_{k+1} (x_1, \ldots, x_{k+1}, -) \\ &\quad -(k-1) r_1 \ldots r_k (x_1, \ldots, x_k, r_{k+1} x_{k+1}, -) \\ &= \sum_{i=1}^{k+1} r_1 \ldots \hat{r}_i \ldots r_{k+1} (x_1, \ldots, r_i x_i, \ldots, x_{k+1}, -) \\ &\quad - k r_1 \ldots r_{k+1} (x_1, \ldots, x_{k+1}, -),\end{aligned}$$

as desired.

(B) $\Rightarrow$ (C). It follows from (B) that

$$\begin{aligned} r^m (x_1, \ldots, x_{m-1}) &= (r x_1, \ldots, r x_{m-1}) \\ &= r^{m-2} \sum_{i=1}^{m-1} (x_1, \ldots, r x_i, \ldots, x_{m-1}) - (m-2) r^{m-1} (x_1, \ldots, x_{m-1}),\end{aligned}$$

and hence $r^{m-2} C(r) = 0$. Observe that

$$\begin{aligned} C(r+s) &= \sum_{i=1}^{m-1} \big((x_1, \ldots, r x_i, \ldots, x_{m-1}) + (x_1, \ldots, s x_i, \ldots, x_{m-1}) \\ &\qquad + rs (x_1, \ldots, x_i, x_i, \ldots, x_{m-1})\big) \\ &\quad - \big(r^2 + 2rs + s^2 + (m-2) r + (m-2) s\big)(x_1, \ldots, x_{m-1}) \\ &= C(r) + C(s) + rs\big((x_1, \ldots, x_{m-1}, x_1 + \ldots + x_{m-1}) \\ &\qquad - 2(x_1, \ldots, x_{m-1})\big) = C(r) + C(s),\end{aligned}$$

by Corollary 1.4 (4). Since $C(1)=0$, it follows that $C(r)=C(r+1)$, and hence r^{m-2}, $(r+1)^{m-2}\in\operatorname{Ann}(C(r))$. Consequently, no maximal ideal contains $\operatorname{Ann}(C(r))$, that is, $C(r)=0$.

(B) $\Rightarrow$ (D). Using (B) and its consequence (C), we compute

$$\begin{aligned} r^m(sx_1, x_2, \ldots, x_{m-1}) &= (rsx_1, rx_2, \ldots, rx_{m-1}) \\ &= r^{m-2}(rsx_1, x_2, \ldots, x_{m-1}) \\ &\quad + r^{m-2}s\sum_{i=2}^{m-1}(x_1, \ldots, rx_i, \ldots, x_{m-1}) - (m-2)r^{m-1}s(x_1, \ldots, x_{m-1}) \\ &= r^{m-2}(rsx_1, x_2, \ldots, x_{m-1}) \\ &\quad - r^{m-2}s(rx_1, x_2, \ldots, x_{m-1}) + r^m s(x_1, \ldots, x_{m-1}). \end{aligned}$$

This means that $r^{m-2}D(r, s)=0$. Moreover, by Corollary 1.4 (3),

$$\begin{aligned} D(r+r', s) &= D(r, s)+D(r', s)+rr's^2(x, x, -)-2rr'(sx, -) \\ &\quad - srr'(x, x, -)+2rr's(x, -) \\ &= D(r, s)+D(r', s). \end{aligned}$$

Since $D(1, s)=0$ it follows that $D(r+1, s)=D(r, s)$, and hence r^{m-2}, $(r+1)^{m-2}\in\operatorname{Ann}(D(r, s))$. As above, we obtain $D(r, s)=0$.

(D) $\Rightarrow$ (E). Observe that $E(r)=D(r, r)=0$ by (D). (The more general formula

$$(r^k x, -) = r^{k-1}(1+r+\ldots+r^{k-1})(rx, -) - r^{k+1}(1+r+\ldots+r^{k-2})(x, -)$$

can be also proved by induction on k with the aid of (D).)

(E) $\Rightarrow$ (F). Observe that $F(r, s)=D(r, s)-D(s, r)=D(r, s)+D(s, r)$ since $2D(s, r)=0$ by Corollary 1.4 (3) or Proposition 1.5 (2). Moreover, D is biadditive, as follows from above and from the symmetric consideration. Consequently, $F(r, s)=D(r+s, r+s)-D(r, r)-D(s, s)=E(r+s)-E(r)-E(s)=0$.

Remark 2.6. If $m=3$ then (A) $\Leftrightarrow$ (B) $\Leftrightarrow$ (D) and the remaining relations hold for any 3-application (see [4]). Section 4 shows that for $m\geqslant 4$ the implications in Proposition 2.5 are proper in general. Moreover, (C) & (D) $\not\Rightarrow$ (A), (C) & (E) $\not\Rightarrow$ (D), and (C) & (F) $\not\Rightarrow$ (E). However, it follows from (F) that $I(R)D(r, s)=0$, and hence (D) $\Leftrightarrow$ (E) $\Leftrightarrow$ (F) in the case of $I(R)=R$. (This is satisfied, for example, if R is a field with more than two elements.)

3. Determination of $\operatorname{Ker}(h^{m,m-1})$. As follows from [3], Section 4,

$$\Delta^m(M_1\oplus\ldots\oplus M_k) = \bigoplus_{n=1}^{k}\ \bigoplus_{1\leqslant j_1<\ldots<j_n\leqslant k} \Delta^{m,n}(M_{j_1}, \ldots, M_{j_n}),$$

$$A^m(M_1\oplus\ldots\oplus M_k) = \bigoplus_{n=1}^{k}\ \bigoplus_{1\leqslant j_1<\ldots<j_n\leqslant k} A^{m,n}(M_{j_1}, \ldots, M_{j_n}),$$

$$\bar{\Delta}^m(M_1 \oplus \ldots \oplus M_k) = \bigoplus_{n=1}^{k} \bigoplus_{1 \leqslant j_1 < \ldots < j_n \leqslant k} \bar{\Delta}^{m,n}(M_{j_1}, \ldots, M_{j_n}),$$

where

$$\Delta^{m,n}(M_1, \ldots, M_n) = R\{(x_1, \ldots, x_n);\ x_i \in M_i\} \subset \Delta^m(M_1 \oplus \ldots \oplus M_n),$$

$$A^{m,n}(M_1, \ldots, M_n) = \Delta^{m,n}(M_1, \ldots, M_n) \cap A^m(M_1 \oplus \ldots \oplus M_n),$$

$$\bar{\Delta}^{m,n}(M_1, \ldots, M_n) = \Delta^{m,n}(M_1, \ldots, M_n)/A^{m,n}(M_1, \ldots, M_n).$$

Since $h^{m,m}\colon \Delta^{m,m} \xrightarrow{\approx} \bar{\Gamma}^{m,m}$ (see [3], Corollary 4.2) it follows that $A^{m,m} = 0$ and $\bar{\Delta}^{m,m} = \Delta^{m,m}$.

LEMMA 3.1. *$A^{m,m-1}(M_1, \ldots, M_{m-1})$ is generated by elements*

$$(x_1, \ldots, rx_i, \ldots, sx_j, \ldots, x_{m-1}) - r(x_1, \ldots, sx_j, \ldots, x_{m-1})$$
$$- s(x_1, \ldots, rx_i, \ldots, x_{m-1}) + rs(x_1, \ldots, x_{m-1}),$$
$$1 \leqslant i < j \leqslant m-1,\ r, s \in R,\ x_k \in M_k.$$

Proof. Let us denote the above element by $A(x_1, \ldots, x_{m-1})$. Then $A^m(M_1 \oplus \ldots \oplus M_{m-1})$ is generated by all elements of the form $A(\sum_{j=1}^{m-1} x_{1j}, \ldots, \sum_{j=1}^{m-1} x_{m-1,j})$, where $x_{ij} \in M_j$. Since $A^{m,m} = 0$ it follows that A is multiadditive (this can be also computed directly). Then the generator is of the form $\sum_{j_1, \ldots, j_{m-1}=1}^{m-1} A(x_{1j_1}, \ldots, x_{m-1,j_{m-1}})$. It follows from (!) that all summands with $j_s = j_t$ (for some $s \neq t$) belong to $\Delta^{m,k}(M_{i_1}, \ldots, M_{i_k})$ for some $k < m-1$ and some $1 \leqslant i_1 < \ldots < i_k \leqslant m-1$. The remaining summands belong to $A^{m,m-1}(M_1, \ldots, M_{m-1})$ and have the desired form.

As in introduction, we restrict our consideration to the case of $M_1 = \ldots = M_{m-1} = R$, and we denote the base elements by $e_1, \ldots, e_{m-1}$, respectively. In the preceding notation, $\Delta^{m,m-1}(R, \ldots, R) = \Delta^{m,m-1}(R)$, $A^{m,m-1}(R, \ldots, R) = A^{m,m-1}(R)$, etc. Then Lemma 3.1 gives us the following corollary.

COROLLARY 3.2. *$A^{m,m-1}(R)$ is generated by elements*

$$A_{ij}(r_1, \ldots, r_{m-1}) = (r_1 e_1, \ldots, r_{m-1} e_{m-1})$$
$$- r_i(r_1 e_1, \ldots, e_i, \ldots, r_{m-1} e_{m-1})$$
$$- r_j(r_1 e_1, \ldots, e_j, \ldots, r_{m-1} e_{m-1})$$
$$+ r_i r_j(r_1 e_1, \ldots, e_i, \ldots, e_j, \ldots, r_{m-1} e_{m-1})$$

for $1 \leqslant i < j \leqslant m-1$ and $r_1, \ldots, r_{m-1} \in R$.

Proof. It suffices to compute that

$$\begin{aligned}
&(r_1 e_1, \ldots, rr_i e_i, \ldots, sr_j e_j, \ldots, r_{m-1} e_{m-1})\\
&\quad -r(r_1 e_1, \ldots, sr_j e_j, \ldots, r_{m-1} e_{m-1})\\
&\quad -s(r_1 e_1, \ldots, rr_i e_i, \ldots, r_{m-1} e_{m-1})+rs(r_1 e_1, \ldots, r_{m-1} e_{m-1})\\
&= A_{ij}(r_1, \ldots, rr_i, \ldots, sr_j, \ldots, r_{m-1})-rA_{ij}(r_1, \ldots, sr_j, \ldots, r_{m-1})\\
&\quad -sA_{ij}(r_1, \ldots, rr_i, \ldots, r_{m-1})+rsA_{ij}(r_1, \ldots, r_{m-1}).
\end{aligned}$$

We will prove that the kernel of the homomorphism $h^{m,m-1}\colon \Delta^{m,m-1}(R) \to \bar{\Gamma}^{m,m-1}(R)$ is equal to $A^{m,m-1}(R)$, or, equivalently, that $\bar{\Delta}^{m,m-1}(R) \simeq \bar{\Gamma}^{m,m-1}(R)$ by $\bar{h}^{m,m-1}$. For this goal some auxiliary constructions will be needed.

Recall that $\bar{\Gamma}^{m,m-1}(R) = R\sigma \oplus \bigoplus_{i=1}^{m-2} I(R)a_i$, where $a_i = e_1 \ldots e_i^{(2)} \ldots e_{m-1}$ and $\sigma = a_1 + \ldots + a_{m-1}$. Let $p_i\colon \bar{\Gamma}^{m,m-1}(R) \to I(R)$ $(i = 1, \ldots, m-2)$ denote the projections. Consider the compositions

$$P_i\colon \Delta^{m,m-1}(R) \xrightarrow{h^{m,m-1}} \bar{\Gamma}^{m,m-1}(R) \xrightarrow{p_i} I(R), \qquad i = 1, \ldots, m-2.$$

It follows from (!!) that

$$\begin{aligned}
&h^{m,m-1}(r_1 e_1, \ldots, r_{m-1} e_{m-1})\\
&\qquad = \sum_{i=1}^{m-1} (r_1 e_1)\ldots(r_i e_i)^{(2)}\ldots(r_{m-1} e_{m-1}) = \sum_{i=1}^{m-1} r_1 \ldots r_i^2 \ldots r_{m-1} a_i\\
&\qquad = \sum_{i=1}^{m-2} (r_1 \ldots r_i^2 \ldots r_{m-1} - r_1 \ldots r_{m-1}^2) a_i + r_1 \ldots r_{m-1}^2 \sigma
\end{aligned}$$

and hence

$$P_i(r_1 e_1, \ldots, r_{m-1} e_{m-1}) = r_1 \ldots r_i^2 \ldots r_{m-1} - r_1 \ldots r_{m-1}^2.$$

Let us consider also the submodules

$$\Delta_i = R\{(e_1, \ldots, e_{i-1}, re_i, e_{i+1}, \ldots, e_{m-1});\ r \in R\}, \qquad i = 1, \ldots, m-2$$

of $\Delta^{m,m-1}(R)$. Note that $P_i(e_1, \ldots, re_i, \ldots, e_{m-1}) = r^2 - r$ and $P_i|_{\Delta_j} = 0$ for $i \neq j$.

PROPOSITION 3.3. (1) $\Delta^{m,m-1}(R) = \Delta_1 + \ldots + \Delta_{m-2} + A^{m,m-1}(R)$.

(2) *The formula* $K = (\Delta_1 \cap K) + \ldots + (\Delta_{m-2} \cap K) + A^{m,m-1}(R)$ *holds for* $K = \operatorname{Ker}(P_1) \cap \ldots \cap \operatorname{Ker}(P_{m-2})$ *and for* $K = \operatorname{Ker}(h^{m,m-1})$.

Proof. Let $\bar{\Delta}_i$ denote the image of Δ_i in $\bar{\Delta}^{m,m-1}(R)$ for $i = 1, \ldots, m-1$. Since $\bar{\delta}^m$ is regular, it satisfies relations (B) and (C) of Proposition 2.5. Consequently, $\bar{\Delta}^{m,m-1}(R) = \bar{\Delta}_1 + \ldots + \bar{\Delta}_{m-1}$ and $\bar{\Delta}_{m-1} \subset \bar{\Delta}_1 + \ldots + \bar{\Delta}_{m-2}$. This gives us (1). The formula in (2) holds for $K = \mathrm{Ker}(P_1) \cap \ldots \cap \mathrm{Ker}(P_{m-2})$ because $\Delta^{m,m-1}(R) \subset \mathrm{Ker}(h^{m,m-1}) \subset K$ and $\Delta_i \subset \bigcap_{j \neq i} \mathrm{Ker}(P_j)$. Since $\mathrm{Ker}(p_1) \cap \ldots \cap \mathrm{Ker}(p_{m-2}) = R\sigma = Rh^{m,m-1}(e_1, \ldots, e_{m-1})$ and σ is linearly independent, it follows that $K = R(e_1, \ldots, e_{m-1}) \oplus \mathrm{Ker}(h^{m,m-1})$. Hence the second version of the formula can be easily deduced from the first one.

The above proposition reduces our consideration to the study of restrictions $h^{m,m-1}|_{\Delta_i}$. By symmetry, we can consider one of them only, say, for $i = 1$. Let us denote $\Delta(R) = \Delta_1$, $\bar{\Gamma}(R) = R\sigma \oplus I(R)a_1$, $h = h(R) = h^{m,m-1}|_{\Delta_1}$, $p = -p_1|_{\bar{\Gamma}(R)}$, $P = -P_1|_{\Delta_1}$, $e = e_1$ and $(x, -) = (x, e_2, \ldots, e_{m-1})$, so that

$$\Delta(R) = R\{(re, -);\ r \in R\} \subset \Delta^{m,m-1}(R),$$

$$P\colon \Delta(R) \xrightarrow{h(R)} \bar{\Gamma}(R) \xrightarrow{p} I(R), \qquad (re, -) \mapsto r\sigma + (r^2 - r)a_i \mapsto r - r^2.$$

Moreover, let us denote by $D(R)$ the submodule of $\Delta(R)$ generated by elements

$$D(r, s) = (rse, -) - r^2(se, -) - s(re, -) + r^2 s(e, -), \qquad r, s \in R.$$

It follows from Proposition 2.5 that

$$D(R) \subset \Delta^{m,m-1}(R) \cap \Delta(R) \subset \mathrm{Ker}(h^{m,m-1}) \cap \Delta(R) = \mathrm{Ker}(h(R)).$$

We prove that in fact $D(R) = \mathrm{Ker}(h(R))$, and the arguments are similar to those of [4], Section 3.

Lemma 3.4. (1) $\mathrm{Ker}(P) = R(e, -) \oplus \mathrm{Ker}(h)$, $\mathrm{Im}(P) = I(R)$.

(2) $P(u)v - P(v)u \in R(e, -) + D(R)$ *for any* $u, v \in \Delta(R)$.

(3) $P(M)N + R(e, -) + D(R) = P(N)M + R(e, -) + D(R)$ *for any submodules* M, N *of* $\Delta(R)$.

(4) $I(R)\mathrm{Ker}(P) \subset R(e, -) + D(R)$, $I(R)\mathrm{Ker}(h) \subset D(R)$.

Proof. To prove (1), it suffices to observe that $\mathrm{Ker}(p) = R\sigma = Rh(e, -)$ and that σ is linearly independent. In the proof of (2), we can assume that $u = (re, -)$ and $v = (se, -)$ for some $r, s \in R$. In this case,

$$\begin{aligned} P(u)v - P(v)u &= (r - r^2)(se, -) - (s - s^2)(re, -) \\ &= (rs^2 - r^2 s)(e, -) + D(r, s) - D(s, r), \end{aligned}$$

as desired. Finally, (3) follows directly from (2), and (4) follows from (3) and (1).

Lemma 3.5. *For any* $u \in \Delta(R)$,

$$u \equiv (P(u)e, -) \bmod (I(R)\Delta(R) + R(e, -) + D(R)).$$

Proof. First observe that $(re, -) = ((r-r^2)e, -)+(r^2 e, -)+(r-r^2)\times$ $\times r^2(e, e, -) \equiv ((r-r^2)e, -)$, because $(r^2 e, -) = (r+r^2)(re, -)-r^3(e, -)+$ $+D(r, r)$ and $(e, e, -)\in \Delta(R)$. Consequently,

$$s(re, -) \equiv s((r-r^2)e, -) \equiv s((r-r^2)e, -)+(r-r^2)^2(se, -)$$
$$-s(r-r^2)^2(e, -)+D(r-r^2, s)$$
$$= (s(r-r^2)e, -).$$

Finally, let $u = \sum_i s_i(r_i e, -)$. The above computation gives us

$$u \equiv \sum_i (s_i(r_i-r_i^2)e, -) \equiv (\sum_i s_i(r_i-r_i^2)e, -) = (P(u)e, -),$$

since $\Delta^m\delta^m$ is m-linear and the coefficients belong to $I(R)$.

COROLLARY 3.6. *Let N be a submodule of $\Delta(R)$. The following conditions are equivalent*:

(1) $\Delta(R) = N+R(e, -)+D(R)$,

(2) $\Delta(R) = N+\mathrm{Ker}(P)$,

(3) $P(N) = I(R)$.

Proof. Obviously, (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Let us assume (3) and suppose that $u\in\Delta(R)$. Since $P(u) = P(v)$ for some $v\in N$ it follows that $u \equiv (P(u)e, -)$ $= (P(v)e, -) \equiv v \bmod (I(R)\Delta(R)+R(e, -)+D(R))$, by Lemma 3.5. Moreover,

$$I(R)\Delta(R) = P(N)\Delta(R) \subset N+R(e, -)+D(R),$$

by Lemma 3.4 (3). Hence $u\in N+R(e, -)+D(R)$, as required in (1).

LEMMA 3.7. *Suppose that N is a submodule of $\Delta(R)$, $u_1, \ldots, u_n \in\Delta(R)$ and $P(u_1), \ldots, P(u_n)$ form a regular sequence on $R/P(N)$. Then*

$$(N+R\{u_1, \ldots, u_n\})\cap \mathrm{Ker}(P) \subset N+R(e, -)+D(R).$$

Proof. Induction on n. Let $n = 1$ and $u_1 = u$. Suppose that $v\in N$, $r\in R$ and $v-ru\in\mathrm{Ker}(P)$. Then $rP(u) = P(v)\in P(N)$ and hence, by the regularity assumption, $r = P(w)$ for some $w\in N$. Consequently,

$$ru = P(w)u \equiv P(u)w \mod (R(e, -)+D(R)),$$

and finally $v-ru\in N+R(e, -)+D(R)$. Let now $n > 1$. Since $P(u_n)$ is regular on $R/P(N')$, where $N' = N+R\{u_1, \ldots, u_{n-1}\}$, the preceding case and the inductive assumption show that

$$\begin{aligned}(N+R\{u_1, \ldots, u_n\}) \cap \operatorname{Ker}(P) &= (N'+Ru_n) \cap \operatorname{Ker}(P) \\ &\subset \big(N'+R(e, -)+D(R)\big) \cap \operatorname{Ker}(P) \\ &= \big(N' \cap \operatorname{Ker}(P)\big)+R(e, -)+D(R) \\ &\subset N+R(e, -)+D(R).\end{aligned}$$

LEMMA 3.8. *Let* $S = R[T_j;\ j \in J]$. *Then*

(1) *In the natural way* $\Delta(R)$ *is contained in* $\Delta(S)$ *as a direct summand* (*over* R) *and* $h(R)$ *is the restriction of* $h(S)$.

(2) *If* N *is a submodule of* $\Delta(R)$ *then* $SN \cap \operatorname{Ker}(h(S)) = S\big(N \cap \operatorname{Ker}(h(R))\big)$.

Proof. (1) It suffices to observe that R is a retract of S and that $h\colon \Delta \to \bar{\Gamma}$ is a natural transformation of functors on the category of commutative rings (cf. [3], Section 3).

(2) Let $u = \sum_i f_i u_i \in \operatorname{Ker}(h(S))$ for some $f_i \in S$ and $u_i \in N \subset \Delta(R)$, and let r_i be the coefficient of f_i at a fixed monomial. Since $\sum_i f_i h(u_i) = 0$ in $S\bar{\Gamma}(R) \subset \bar{\Gamma}(S)$ it follows that $\sum_i r_i h(u_i) = 0$ in $\bar{\Gamma}(R)$. Consequently, $\sum_i r_i u_i \in N \cap \operatorname{Ker}(h(R))$ and hence $\sum_i f_i u_i \in S\big(N \cap \operatorname{Ker}(h(R))\big)$. The second inclusion is obvious.

COROLLARY 3.9. *Let* $S = R[T_j; j \in J]$.

(1) *If* $\Delta(R) = N+\operatorname{Ker}(h(R))$ *and* $(e, -) \in N$, *then* $\Delta(S) = N(S)+D(S)$, *where* $N(S) = SN+S\{(T_j e, -); j \in J\}$.

(2) *If* $\operatorname{Ker}(h(R)) = D(R)$, *then* $\operatorname{Ker}(h(S)) = D(S)$.

Proof. (1) Observe that

$$P(N(S)) = SP(N)+S\{T_j-T_j^2; j \in J\} = S\big(I(R),\ T_j-T_j^2; j \in J\big) = I(S)$$

(cf. Lemma 1.4 of [4]). Hence $\Delta(S) = N(S)+D(S)$, by Corollary 3.6.

(2) Let N be as in (1) (e.g. $N = \Delta(R)$). Observe that any finite sequence of different elements $T_j-T_j^2 = P(T_j e, -)$ is regular on $S/P(SN) = S/SI(R) \simeq (R/I(R))[T_j; j \in J]$. Then Lemma 3.7 shows that $N(S) \cap \operatorname{Ker}(P) \subset SN+D(S)$. Consequently, by Lemma 3.8 (2),

$$\begin{aligned}N(S) \cap \operatorname{Ker}(h(S)) &\subset SN \cap \operatorname{Ker}(h(S))+D(S) \\ &= S\big(N \cap \operatorname{Ker}(h(R))\big)+D(S) \subset SD(R)+D(S) = D(S).\end{aligned}$$

By (1), $\operatorname{Ker}(h(S)) = N(S) \cap \operatorname{Ker}(h(S))+D(S) = D(S)$, as required.

LEMMA 3.10. *If* $\operatorname{Ker}(h(R)) = D(R)$, *then* $\operatorname{Ker}(h(S)) = D(S)$ *for* $S = R/J$.

Proof. Consider $\Delta'(R) = \Delta(R)/(R(e, -) \oplus D(R))$ and $P'(R)\colon \Delta'(R) \to I(R)$ induced by $P(R)$. In virtue of Lemma 3.4 (1) it should be proved that $P'(S)$ is an isomorphism provided that so is $P'(R)$. Let us consider

$$f\colon \dot{I}(R) \xrightarrow{P'(R)^{-1}} \Delta'(R) \to \Delta'(S), \qquad r - r^2 \mapsto [(re, -)] \mapsto [(\bar{r}e, -)].$$

If $r \in J \cap I(R)$ then $f(r) = f(r-r^2) + rf(r) = [(\bar{r}e, -)] + \bar{r}f(r) = 0$. Hence f induces a homomorphism $g\colon I(S) = I(R)/J \cap I(R) \to \Delta'(S)$, which is evidently inverse to $P'(S)$.

We are ready to prove the following theorem.

THEOREM 3.11. *For any commutative ring* R,

(1) $\operatorname{Ker}(h^{m,m-1}(R)) \cap \Delta(R) = \operatorname{Ker}(h(R)) = D(R)$,

(2) $\operatorname{Ker}(h^{m,m-1}(R)) = A^{m,m-1}(R)$.

Proof. (1) It follows from Theorem 1.7 that $\operatorname{Ker}(h^{m,m-1}(\boldsymbol{Z})) = 0$, and hence the equality holds for $R = \boldsymbol{Z}$. Then Corollary 3.9 (2) and Lemma 3.10 give us the result for any commutative ring R.

(2) By Proposition 2.5, $D(R)$ is contained in $A^{m,m-1}(R)$, and hence the result follows from (1) and Proposition 3.3.

COROLLARY 3.12. (1) $A^{m,m-1}(R) \cap \Delta(R) = D(R)$.

(2) $\bar{\Delta}^{m,m-1}(R) \simeq \bar{\Gamma}^{m,m-1}(R)$ *and hence* ([5]) $\underline{\mathrm{Appl}}^m$ *is an* $(m-1)$*-covering functor of* Hom^m.

(3) $\operatorname{Ker}(h^{m,m-1}(R)) \to \operatorname{Ker}(h^{m,m-1}(R/I))$ *is an epimorphism.* (Cf. also [5], the Main Theorem 6.2.)

4. Irregularity in examples. In this section we show that there is in general (for all $m \geqslant 4$) no implication between conditions (A)–(F) except of those indicated in Proposition 2.5 (cf. Remark 2.6). Examples 4.1 and 4.3 show also that in general $\Delta_1 + \ldots + \Delta_{m-2} \subsetneq \Delta_1 + \ldots + \Delta_{m-1} \subsetneq \Delta^{m,m-1}(R)$ (cf. Proposition 3.3). Moreover, Example 4.5 shows that Hom^4 and Appl^4 are different over fields $k(T)$ with $\operatorname{char}(k) = 2$, what answers negatively the concluding question of [2].

In all the following examples we consider m-applications $f\colon R^{m-1} \to N$ satisfying the following property:

$$f(r_1 e_1 + \ldots + r_{m-1} e_{m-1}) = 0 \quad \text{if some } r_i = 0.$$

Consequently,

$$\begin{aligned} f(r_1 e_1 + \ldots + r_{m-1} e_{m-1}) &= (\Delta^{m-1} f)(r_1 e_1, \ldots, r_{m-1} e_{m-1}) \\ &= \bar{f}(r_1 e_1, \ldots, r_{m-1} e_{m-1}), \end{aligned}$$

where $\bar{f}\colon \Delta^{m,m-1}(R) \to N$ is the induced homomorphism. Moreover,

$$(\Delta^n f)(r_1 e_{i_1}, \ldots, r_n e_{i_n}) = 0 \quad \text{if} \quad \{i_1, \ldots, i_n\} \not\subseteq \{1, \ldots, m-1\}.$$

Except of Example 4.4, we have also $\Delta^m f = 0$. In this case, $\Delta^{m-1} f$ is multiadditive, and hence it is completely determined by the values $(r_1 e_1, \ldots, r_{m-1} e_{m-1}) = (\Delta^{m-1} f)\ (r_1 e_1, \ldots, r_{m-1} e_{m-1})$. This allows us to check formulas (C)–(F) for $x_i \in Re_i$ only. It can be proved, however, that the left-hand sides of those relations are multiadditive (in variables $x_1, \ldots, x_{m-1}$), and hence in fact the condition $\Delta^m f = 0$ is not needed. This remark is used for relation (C) in Example 4.4.

The last useful remark is following: a product of an m-application and an n-application (with values in some R-algebra) is an $(m+n)$-application. This follows from the formula

$$(\Delta^s fg)(x_1, \ldots, x_s) = \sum (\Delta^k f)(x_{i_1}, \ldots, x_{i_k})(\Delta^l g)(x_{j_1}, \ldots, x_{j_l}),$$

where the sum runs over all systems $(i_1, \ldots, i_k, j_1, \ldots, j_l)$ satisfying the following conditions: $k, l \geqslant 0$, $1 \leqslant i_1 < \ldots < i_k \leqslant s$, $1 \leqslant j_1 < \ldots < j_l \leqslant s$, and $\{i_1, \ldots, i_k, j_1, \ldots, j_l\} = \{1, \ldots, s\}$. In our case, $s = m+n$ implies, obviously, $k = m$ and $l = n$, because the remaining summands are zero.

Example 4.1. (C) & (D) $\not\Rightarrow$ (A) and $\Delta_1 + \ldots + \Delta_{m-1} \subsetneq \Delta^{m,m-1}(R)$ for $m \geqslant 4$. Let $R = S[T_2, \ldots, T_{m-1}]$, where $I(S) \neq S$, and let $N = R/(I(S), T_2, \ldots, T_{m-1}) = S/I(S)$.

Define $f\colon R^{m-1} \to N$ as follows:

$$f(A_1 e_1 + \ldots + A_{m-1} e_{m-1}) = \det(a_{ij}) \bmod I(S),$$

where $A_i = a_{i1} + a_{i2} T_2 + \ldots + a_{i,m-1} T_{m-1} + \ldots$, $i = 1, \ldots, m-1$. (This convention is also assumed in the sequel.) The mapping f satisfies (A1) since

$$\begin{aligned} f\big(A(B_1 e_1 + \ldots + B_{m-1} e_{m-1})\big) &= f(AB_1 e_1 + \ldots + AB_{m-1} e_{m-1}) \\ &= \det\big((AB_i)_j\big) = \det(a_1 b_{*1}, a_1 b_{*2} + a_2 b_{*1}, \ldots, a_1 b_{*m-1} + a_{m-1} b_{*1}) \\ &= a_1 \det(b_{*1}, a_1 b_{*2}, \ldots, a_1 b_{*m-1}) = a_1^{m-1} f(B_1 e_1 + \ldots + B_{m-1} e_{m-1}) \\ &\equiv A^m f(B_1 e_1 + \ldots + B_{m-1} e_{m-1}) \bmod \big(I(S), T_2, \ldots, T_{m-1}\big). \end{aligned}$$

Moreover, observe that f is obtained from a form of degree $m-1$ over S, and hence $\Delta^m f = 0$. This proves that f is an m-application such that $\Delta^{m-1} f$ is multiadditive, and the induced homomorphism $\bar{f}\colon \Delta^{m,m-1}(R) \to N$ is given by

$$\bar{f}(A_1 e_1, \ldots, A_{m-1} e_{m-1}) = \det(a_{ij}) \bmod I(S).$$

The above value is zero if $A_k = A_l$ for some $k \neq l$. Consequently, since $m-1 \geqslant 3$, $\bar{f}$ vanishes on $\Delta_1 + \ldots + \Delta_{m-1}$, although $\bar{f}(e_1, T_2 e_2, \ldots, T_{m-1} e_{m-1}) = 1 \neq 0$. This proves that $\Delta_1 + \ldots + \Delta_{m-1} \subsetneq \Delta^{m,m-1}(R)$. Moreover, in notation of Corollary 3.2,

$$\bar{f}(A_{ij}(1, T_2, \ldots, T_{m-1})) = \bar{f}(e_1, T_2 e_2, \ldots, T_{m-1} e_{m-1}) = 1 \neq 0$$

for $2 \leqslant i < j \leqslant m-1$, since the remaining summands have coefficients 1 at both e_1 and e_i or e_j. This proves that f is irregular.

It remains to prove that $(\ ) = \Delta^{m-1} f$ satisfies conditions

$$\text{(C)} \qquad \sum_{i=1}^{m-1} (B_1 e_1, \ldots, AB_i e_i, \ldots, B_{m-1} e_{m-1}) = \big(A^2 + (m-2) A\big)(B_1 e_1, \ldots, B_{m-1} e_{m-1})$$

and (because of the skew-symmetry of f)

$$\text{(D)} \qquad (ABC_1 e_1, C_2 e_2, \ldots, C_{m-1} e_{m-1}) - A^2 (BC_1 e_1, C_2 e_2, \ldots, C_{m-1} e_{m-1}) - B(AC_1 e_1, C_2 e_2, \ldots, C_{m-1} e_{m-1}) + A^2 B(C_1 e_1, \ldots, C_{m-1} e_{m-1}) = 0.$$

The i-th summand in (C) is a determinant with i-th row

$$(AB_i)_* = (a_1 b_{i1}, a_1 b_{i2} + a_2 b_{i1}, \ldots, a_1 b_{i,m-1} + a_{m-1} b_{i1}) = a_1 b_{i*} + \bar{a}_* b_{i1},$$

where $\bar{a}_* = (0, a_2, \ldots, a_{m-1})$. Changing rows to columns we obtain that the left-hand side of (C) is equal to

$$\begin{aligned}
&\sum_{i=1}^{m-1} \det(b_{1*}, \ldots, a_1 b_{i*} + \bar{a}_* b_{i1}, \ldots, b_{m-1*}) \\
&\quad = (m-1) a_1 \det(b_{1*}, \ldots, b_{m-1*}) \\
&\qquad\qquad + \sum_{i=1}^{m-1} b_{i1} \det(b_{1*}, \ldots, b_{i-1*}, \bar{a}_*, b_{i+1*}, \ldots, b_{m-1*}) \\
&\quad = (m-1) a_1 (B_1 e_1, \ldots, B_{m-1} e_{m-1}) + \sum_{i=1}^{m-1} b_{i1} \sum_{j=2}^{m-1} a_j M_{ij},
\end{aligned}$$

where M_{ij} denote respective cofactors of the matrix (b_{ij}). It suffices to observe that

$$\sum_{i=1}^{m-1} b_{i1} M_{ij} = 0 \qquad \text{for } j \geqslant 2$$

and

$$(m-1) a_1 \equiv A^2 + (m-2) A \bmod \big(I(S), T_2, \ldots, T_{m-1}\big).$$

To prove (D), observe that the left-hand side is a determinant with the first row equal to

$$\begin{aligned}(ABC_1)_* - A^2(BC_1)_* - B(AC_1)_* + A^2B(C_1)_*\\ &= \big(a_1b_1c_{11},\ a_1b_1c_{12}+(a_1b_2+a_2b_1)c_{11},\ \ldots,\ a_1b_1c_{1,m-1}\\ &\qquad\qquad +(a_1b_{m-1}+a_{m-1}b_1)c_{11}\big)\\ &\quad -a_1(b_1c_{11},\ b_1c_{12}+b_2c_{11},\ \ldots,\ b_1c_{1,m-1}+b_{m-1}c_{11})\\ &\quad -b_1(a_1c_{11},\ a_1c_{12}+a_2c_{11},\ \ldots,\ a_1c_{1,m-1}+a_{m-1}c_{11})\\ &\quad +a_1b_1(c_{11},\ c_{12},\ \ldots,\ c_{1,m-1}) = (0,\ \ldots,\ 0).\end{aligned}$$

This completes the proof.

EXAMPLE 4.2. (C) & (E) $\not\Rightarrow$ (D) for $m \geqslant 3$ (cf. [4], Theorem 2.10).

Let $R = S[X, Y]$, where $I(S) \neq S$. Define $f\colon R^{m-1} \to R/I(R)$ by the formula

$$f(F_1e_1+\ldots+F_{m-1}e_{m-1}) = \frac{\partial^2(F_1F_2)}{\partial X\,\partial Y}F_3\ldots F_{m-1}+I(R),$$

and observe that

$$\begin{aligned}f\big(G(F_1e_1+\ldots+F_{m-1}e_{m-1})\big) &= (G^2F_1F_2)_{XY}G^{m-3}F_3\ldots F_{m-1}+I(R)\\ &= G^2(F_1F_2)_{XY}G^{m-3}F_3\ldots F_{m-1}+I(R)\\ &= G^m f(F_1e_1+\ldots+F_{m-1}e_{m-1}),\end{aligned}$$

since $(G^2)_X = 2GG_X \in I(R)$, $(G^2)_Y = 2GG_Y \in I(R)$ and $G^{m-1} \equiv G^m \bmod I(R)$. Moreover, f is obtained from a form of degree $m-1$ over S, and hence $\Delta^m f = 0$. This proves that f is an m-application such that $\Delta^{m-1} f$ is multiadditive. Since $\bar{f}(F_1e_1, \ldots, F_{m-1}e_{m-1}) = (F_1F_2)_{XY}F_3\ldots F_{m-1}+I(R)$, it follows that $\bar{f}(D(X, Y)) = \bar{f}(XYe_1, e_2, \ldots, e_{m-1}) = 1 \neq 0$ and hence (D) is not satisfied by f. It remains to prove that $(\ ) = \Delta^{m-1} f$ satisfies conditions

$$\text{(C)}\quad \sum_{i=1}^{m-1}(F_1e_1, \ldots, GF_ie_i, \ldots, F_{m-1}e_{m-1}) = \big(G^2+(m-2)G\big)(F_1e_1, \ldots, F_{m-1}e_{m-1})$$

and

$$\begin{aligned}\text{(E)}\quad (F_1e_1, \ldots, G^2F_ie_i, \ldots, F_{m-1}e_{m-1})\\ -(G+G^2)(F_1e_1, \ldots, GF_ie_i, \ldots, F_{m-1}e_{m-1})\\ +G^3(F_1e_1, \ldots, F_{m-1}e_{m-1}) = 0, \quad i = 1, \ldots, m-1.\end{aligned}$$

The left-hand side of (C) equals to

$$2(GF_1F_2)_{XY}F_3\dots F_{m-1}+(m-3)(F_1F_2)_{XY}GF_3\dots F_{m-1}+I(R)$$
$$=(m-3)G(F_1e_1,\dots,F_{m-1}e_{m-1})$$
$$=\big(G^2+(m-2)G\big)(F_1e_1,\dots,F_{m-1}e_{m-1})$$

since $G^2+G\in I(R)$. To prove (E), it suffices to consider the cases $i=1$ and $i=m-1$ only. Since $G+G^2\in I(R)$ and $G^3\equiv -G^2 \bmod I(R)$, we must prove that

$$(G^2F_1e_1,F_2e_2,\dots,F_{m-1}e_{m-1})=G^2(F_1e_1,\dots,F_{m-1}e_{m-1})$$
$$=(F_1e_1,\dots,G^2F_{m-1}e_{m-1}),$$

but these equalities follow directly from definition of f.

EXAMPLE 4.3. (D) $\not\Rightarrow$ (C) and $\Delta_{m-1}\not\subset\Delta_1+\dots+\Delta_{m-2}$ for $m\geqslant 4$.

Let $R=S[T]/(T^2-d)=S[t]$, where $I(S)\neq S$ and $d\equiv 1 \bmod I(S)$ (e.g. $S=\mathbf{Z}$ and d is odd). Note that $R[T_1,\dots,T_n]=S[T_1,\dots,T_n][t]$ is also of this kind, and hence we can take the same ring in Examples 4.1 and 4.3. Since $I(R)=\big(I(S),t-t^2\big)=\big(I(S),t-1\big)$ it follows that $R/I(R)=S/I(S)\neq 0$. Let us define $f\colon R^{m-1}\to R/I(R)$ by the formula

$$f(A_1e_1+\dots+A_{m-1}e_{m-1})=(a_1a_2'a_3'+a_1'a_2a_3)A_4\dots A_{m-1}+I(R),$$

where A_i denotes (in the standard convention) $a_i+a_i't$ for $a_i, a_i'\in S$. Since $d\equiv 1\equiv t \bmod I(R)$ we compute that

$$f\big(B(A_1e_1+\dots+A_{m-1}e_{m-1})\big)=f(BA_1e_1+\dots+BA_{m-1}e_{m-1})$$
$$=\big((ba_1+b'a_1'd)(ba_2'+b'a_2)(ba_3'+b'a_3)$$
$$+(ba_1'+b'a_1)(ba_2+b'a_2'd)(ba_3+b'a_3'd)\big)B^{m-4}A_4\dots A_{m-1}+I(R)$$
$$=\big((b^2b'+bb'^2)(a_1a_2a_3+a_1a_2a_3'+a_1a_2'a_3$$
$$+a_1'a_2a_3'+a_1'a_2'a_3+a_1'a_2'a_3')$$
$$+(b^3+b'^3)(a_1a_2'a_3'+a_1'a_2a_3)\big)B^{m-4}A_4\dots A_{m-1}+I(R)$$
$$=(b+b')B^{m-4}(a_1a_2'a_3'+a_1'a_2a_3)A_4\dots A_{m-1}+I(R)$$
$$=B^mf(A_1e_1+\dots+A_{m-1}e_{m-1}).$$

Since f is obtained from a form of degree $m-1$ over S, it follows that $\Delta^m f=0$. Consequently, f is an m-application and $\Delta^{m-1}f$ is multiadditive. Observe that

$$\bar{f}(A_1e_1,\dots,A_{m-1}e_{m-1})=(a_1a_2'a_3'+a_1'a_2a_3)A_4\dots A_{m-1}+I(R)=0$$

for $A_1 = A_2 = 1$ (since $a'_1 = a'_2 = 0$), and similarly for $A_1 = A_3 = 1$. Therefore $\bar{f}$ vanishes on $\Delta_2, \ldots, \Delta_{m-1}$, although $\bar{f}(Ae_1, e_2, \ldots, e_{m-1}) = a'+I(R)$ proves that $\bar{f}$ is nonzero on Δ_1. Consequently, Δ_1 is not contained in $\Delta_2+\ldots+\Delta_{m-1}$. By symmetry, combining with Example 4.1, we obtain that $\Delta_1+\ldots+\Delta_{m-2} \subsetneq \Delta_1+\ldots+\Delta_{m-1} \subsetneq \Delta^{m,m-1}(R)$. Moreover, f does not satisfy (C), e.g. for $r = t$. It remains to prove that $(\ ) = \Delta^{m-1} f$ satisfies (D), i.e.,

$$(A_1 e_1, \ldots, BCA_i e_i, \ldots, A_{m-1} e_{m-1}) - B^2 (A_1 e_1, \ldots, CA_i e_i, \ldots, A_{m-1} e_{m-1})$$
$$- C(A_1 e_1, \ldots, BA_i e_i, \ldots, A_{m-1} e_{m-1}) + B^2 C(A_1 e_1, \ldots, A_{m-1} e_{m-1}) = 0.$$

The formula holds for $i \geqslant 4$. For $i = 1, 2, 3$, it suffices to prove that the analogous formula is satisfied by mappings g and g' carrying $(A_1 e_1, \ldots, A_{m-1} e_{m-1})$ to $a_1+I(R)$ and $a'_1+I(R)$, respectively. In the first case:

$$\begin{aligned} & g(BCA_1 e_1, -) - B^2 g(CA_1 e_1, -) - Cg(BA_1 e_1, -) + B^2 Cg(A_1 e_1, -) \\ &\quad = (bc+b'c')a_1 + (bc'+b'c)a'_1 - (b+b')(ca_1+c'a'_1) \\ &\qquad\qquad - (c+c')(ba_1+b'a'_1) + (b+b')(c+c')a_1 + I(R) \\ &\quad = 2b'c'(a_1-a'_1) + I(R) = 0, \end{aligned}$$

and the second computation is similar.

EXAMPLE 4.4. (C) & (F) $\not\Rightarrow$ (E) for $m \geqslant 4$.

Let $R = \boldsymbol{Z}[T]/(T^2-d) = \boldsymbol{Z}[t]$, where $d \equiv 3 \pmod 4$. For any $A = a+a't \in R$ write $\bar{A} = a+a' \in \boldsymbol{Z}$. It is easy to see that $\overline{A+B} = \bar{A}+\bar{B}$, $\overline{AB} = \bar{A}\cdot\bar{B} + (d-1)a'b' \equiv \bar{A}\cdot\bar{B} \pmod 2$ and $A \equiv \bar{A} \bmod I(R)$. Consider mappings g, h: $R^2 = Rx \oplus Ry \to R/I(R) = \boldsymbol{Z}_2$ defined by

$$g(Ax+By) = \frac{\bar{A}\cdot\bar{B}\cdot(\bar{A}+\bar{B})}{2} + I(R),$$

$$h(Ax+By) = ab'+a'b+I(R) = (\text{the coefficient of } AB \text{ at } t)+I(R).$$

We prove that f: $R^3 = Rx \oplus Ry \oplus Rz \to R/I(R) = \boldsymbol{Z}_2$ given by the formula

$$\begin{aligned} f(Ax+By+Cz) &= g(Ax+By)C + \frac{d-1}{2} h(Ax+By)c' \\ &= \frac{\bar{A}\cdot\bar{B}\cdot(\bar{A}+\bar{B})}{2}\bar{C} + \frac{d-1}{2}(ab'+a'b)c' + I(R) \end{aligned}$$

is a 4-application. Observe that

$$\begin{aligned} & g(P(Ax+By)) - P^3 g(Ax+By) \\ &\quad = \frac{\overline{PA}\cdot\overline{PB}\cdot(\overline{PA}+\overline{PB}) - \bar{P}\cdot\bar{A}\cdot\bar{P}\cdot\bar{B}\cdot(\bar{P}\cdot\bar{A}+\bar{P}\cdot\bar{B})}{2} + I(R) \end{aligned}$$

$$= \frac{d-1}{2} p' \bar{P}^2(a' \bar{B}(\bar{A}+\bar{B}) + b' \bar{A}(\bar{A}+\bar{B}) + (a'+b') \bar{A} \cdot \bar{B}) + I(R)$$

$$= \frac{d-1}{2} p' P^2 (a' \bar{B} + b' \bar{A}) + I(R)$$

$$= \frac{d-1}{2} p' P^2 (ab' + a' b) + I(R)$$

and

$$h(P(Ax+By)) = (\text{the coefficient of } P^2 AB \text{ at } t) + I(R)$$
$$= (p^2 + p'^2 d)\, h(Ax+By) = P^3 h(Ax+By).$$

Consequently,

$$f(P(Ax+By+Cz)) - P^4 f(Ax+By+Cz)$$
$$= \big(g(P(Ax+By)) - P^3 g(Ax+By)\big) PC$$
$$+ \frac{d-1}{2}\big(h(P(Ax+By))(pc'+p'c) - P^4 h(Ax+By)\, c'\big)$$
$$= \frac{d-1}{2} P^3 (ab'+a'b)(p'C + pc' + p'c - Pc') + I(R) = 0,$$

and hence f satisfies (A1). To prove (A2), observe that $\Delta^3 g$ is 3-linear since

$$(\Delta^3 g)(A_1 x + B_1 y,\ A_2 x + B_2 y,\ A_3 x + B_3 y)$$
$$= \bar{A}_1 \bar{A}_2 \bar{B}_3 + \bar{A}_1 \bar{B}_2 \bar{A}_3 + \bar{B}_1 \bar{A}_2 \bar{A}_3 + \bar{B}_1 \bar{B}_2 \bar{A}_3 + \bar{B}_1 \bar{A}_2 \bar{B}_3 + \bar{A}_1 \bar{B}_2 \bar{B}_3 + I(R)$$
$$= A_1 A_2 B_3 + A_1 B_2 A_3 + B_1 A_2 A_3 + B_1 B_2 A_3 + B_1 A_2 B_3 + A_1 B_2 B_3 + I(R).$$

Let f' and f'' denote the summands in the definition of f. Then $\Delta^4 f'$ is 4-linear and $\Delta^4 f'' = 0$ since f'' is obtained from a form of degree 3 over $\boldsymbol{Z}$. This proves that f satisfies (A2).

Consequently, for any $m \geqslant 4$ we obtain an m-application

$$f_m\colon R^{m-1} \to R/I(R) = \boldsymbol{Z}_2,$$

$$f_m(A_1 e_1 + \ldots + A_{m-1} e_{m-1}) = f(A_1 x + A_2 y + A_3 z) A_4 \ldots A_{m-1}.$$

Since

$$\bar{f}_m(A_1 e_1, \ldots, A_{m-1} e_{m-1})$$
$$= \left(\frac{\bar{A}_1 \bar{A}_2 (\bar{A}_1 + \bar{A}_2)}{2} \bar{A}_3 + \frac{d-1}{2}(a_1 a_2' + a_1' a_2) a_3'\right) A_4 \ldots A_{m-1} + I(R),$$

it follows that

$$\bar{f}_m(E(t)) = \bar{f}_m((t^2 e_1, e_2, \ldots, e_{m-1}) \\ -(t+t^2)(te_1, e_2, \ldots, e_{m-1}) + t^3(e_1, \ldots, e_{m-1}))$$

$$= \bar{f}_m(de_1, e_2, \ldots, e_{m-1}) + t^3 \bar{f}_m(e_1, \ldots, e_{m-1})$$

$$= \frac{d(d+1)}{2} + 1 + I(R) = 1 + I(R) \neq 0,$$

because $d \equiv 3 \pmod 4$. Therefore, f_m does not satisfy (E). On the other hand, f_m satisfies (F) in the trivial way since all coefficients in (F) belong to $I(R)$. It remains to prove that f_m satisfies (C). Although $\Delta^{m-1} f$ is not multiadditive, it can easily be proved that so is the left-hand side of (C). Consequently, we can check (C), as before, for multiples of the base elements e_i. Moreover, it follows from the form of $\Delta^{m-1} f_m$ that the case $m = 4$ is sufficient. Therefore we must prove that

$$f(PAx+By+Cz) + f(Ax+PBy+Cz) + f(Ax+By+PCz) \\ = (P^2+2P) f(Ax+By+Cz).$$

A direct computation shows that the left-hand side is equal to

$$g(Ax+By)PC + (g(PAx+By) + g(Ax+PBy))\bar{C} \\ + \frac{d-1}{2}(h(PAx+By) + h(Ax+PBy))c' + \frac{d-1}{2} h(Ax+By)(pc'+p'c)$$

$$= g(Ax+By)PC + \frac{d-1}{2}(ab'+a'b)p'\bar{C} + \frac{d-1}{2}(ab'+a'b)(pc'+p'c) + I(R)$$

$$= P\left(g(Ax+By)C + \frac{d-1}{2}(ab'+a'b)c' + I(R)\right) = (P^2+2P) f(Ax+By+Cz),$$

as desired.

EXAMPLE 4.5. (C) $\not\Rightarrow$ (F) for $m \geqslant 4$.

Let $R = k[T]$ and $K = k(T) = R_{(0)}$, where k is a field of characteristic two. For any $F = \sum_i a_i T^i \in R$ define $F^+ = \sum_i a_{2i} T^{2i} \in R$ and $F^- = \sum_i a_{2i+1} T^{2i+1} \in R$ so that $F = F^+ + F^-$. Since $F^2 = (F^2)^+$ and $(FG)^+ = F^+G^+ + F^-G^-$, it follows that $(F^2G)^+ = F^2G^+$. Therefore, we can extend $(\ )^+$ to a k-endomorphism of K defined by

$$\left(\frac{F}{G}\right)^+ = \frac{(FG)^+}{G^2}$$

and preserving square multiples.

Let us define $f\colon K^{m-1} \to K$ by the formula

$$f(F_1 e_1 + \ldots + F_{m-1} e_{m-1}) = (F_1 F_2)^+ F_3^2 F_4 \ldots F_{m-1}.$$

It is an m-application. In fact, f satisfies (A1) since

$$\begin{aligned} f\big(G(F_1 e_1 + \ldots + F_{m-1} e_{m-1})\big) &= (G^2 F_1 F_2)^+ (GF_3)^2 (GF_4) \ldots (GF_{m-1}) \\ &= G^m (F_1 F_2)^+ F_3^2 F_4 \ldots F_{m-1} \\ &= G^m f(F_1 e_1 + \ldots + F_{m-1} e_{m-1}). \end{aligned}$$

To prove (A2) observe that f is obtained from a form of degree m over k, and hence $\Delta^m f$ is m-linear over k. We show that in fact $\Delta^m f = 0$. Because of m-additivity it suffices to prove that $(\Delta^m f)(F_1 e_{i_1}, \ldots, F_m e_{i_m}) = 0$. It is so if $\{i_1, \ldots, i_m\} \subsetneq \{1, \ldots, m-1\}$, by the definition of f. It remains to check that

$$(\Delta^m f)(F_1 e_1, \ldots, F_i e_i, F_i' e_i, \ldots, F_{m-1} e_{m-1}) = 0,$$

or, equivalently, that

$$\begin{aligned} &(\Delta^{m-1} f)\big(F_1 e_1, \ldots, (F_i + F_i') e_i, \ldots, F_{m-1} e_{m-1}\big) \\ &\quad = (\Delta^{m-1} f)(F_1 e_1, \ldots, F_i e_i, \ldots, F_{m-1} e_{m-1}) \\ &\qquad + (\Delta^{m-1} f)(F_1 e_1, \ldots, F_i' e_i, \ldots, F_{m-1} e_{m-1}). \end{aligned}$$

This is obvious, since

$$(\Delta^{m-1} f)(F_1 e_1, \ldots, F_{m-1} e_{m-1}) = (F_1 F_2)^+ F_3^2 F_4 \ldots F_{m-1}$$

and $(\ )^+$, $(\ )^2$ are additive.

Observe that f satisfies (C) since $(\ ) = \Delta^{m-1} f$ is multiadditive and

$$\begin{aligned} &\sum_{i=1}^{m-1} (F_1 e_1, \ldots, GF_i e_i, \ldots, F_{m-1} e_{m-1}) \\ &\quad = 2(GF_1 F_2)^+ F_3^2 F_4 \ldots F_{m-1} + (F_1 F_2)^+ (GF_3)^2 F_4 \ldots F_{m-1} \\ &\qquad + (m-4) G (F_1 F_2)^+ F_3^2 F_4 \ldots F_{m-1} \\ &\quad = \big(G^2 + (m-4) G\big)(F_1 F_2)^+ F_3^2 F_4 \ldots F_{m-1} \\ &\quad = \big(G^2 + (m-2) G\big)(F_1 e_1, \ldots, F_{m-1} e_{m-1}). \end{aligned}$$

On the other hand, (F) is not satisfied by f since

$$\begin{aligned} F(T, T^2) &= (T - T^2)(T^2 e_1, e_2, \ldots, e_{m-1}) \\ &\qquad - (T^2 - T^4)(T e_1, e_2, \ldots, e_{m-1}) - (T^5 - T^4)(e_1, \ldots, e_{m-1}) \\ &= (T - T^2) T^2 - (T^5 - T^4) = T^3 - T^5 \neq 0. \end{aligned}$$

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