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A coupled system of fractional order integral equations in reflexive Banach spaces

Abstract. We present an existence theorem for at least one weak solution for a coupled system of integral equations of fractional order in reflexive Banach spaces relative to the weak topology.

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1. Introduction and Preliminaries. Systems occur in various problems of applied nature, for instance, see ([1]-[3], [11], [12] and [13]). Recently, Su [23] discussed a two-point boundary value problem for a coupled system of fractional differential equations. Gafiychuk et al. [24] analyzed the solutions of coupled nonlinear fractional reaction-diffusion equations.

Let $L_1(I)$ be the space of Lebesgue integrable functions defined on the interval $I = [0,1]$. Let E be a reflexive Banach space with the norm $||.||$ and its dual E^* and denote by $C[I,E]$ the Banach space of strongly continuous functions $x : I \to E$ with sup-norm.

The existence of weak solutions of the integral equations and differential equations has been studied by many authors such as [4], [18], [17] ,[5], [19] and [10].
Recently, the existence of weak solution of the nonlinear fractional-order integral equation

$$x(t) \;=\; g(t) \;+\; \lambda\, I^{\alpha}\, f(t, x(t)), \quad t \in I, \quad 0 < \alpha < 1 \tag{1}$$

was proved in [19] where x takes values in reflexive Banach spaces and f is weakly measurable in t and weakly sequentially continuous in x .
An existence result for (1), in the case $E = R$ found in [9] where the real-valued

function f satisfies Carathéodory condition.

In this paper, we shall extend the results obtained in [19] and prove the existence of a weak solution to the coupled system of fractional order integral equations

$$
\begin{aligned}
x(t) &= g_1(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, y(s))\, ds, \quad t \in I, \\
y(t) &= g_2(t) + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2(s, x(s))\, ds, \quad t \in I,
\end{aligned}
\tag{2}
$$

where $0 < \alpha,\ \beta < 1$.
Now, we shall present some auxiliary results that will be need in this work. Let E be a Banach space (need not be reflexive) and let $x:\ I \to E$, then

(1) $x(.)$ is said to be weakly continuous (measurable) at $t_0 \in I$ if for every $\phi \in E^*$, $\phi(x(.))$ is continuous (measurable) at t_0.

(2) A function $h : E \to E$ is said to be weakly sequentially continuous if h maps weakly convergent sequences in E to weakly convergent sequences in E.

If x is weakly continuous on I, then x is strongly measurable and hence weakly measurable (see[8] and [7]). Note that in reflexive Banach space weakly measurable functions are Pettis integrable if and only if $\phi(x(.))$ is Lebesgue integrable on I for every $\phi \in E^*$ (see[8] pp. 78).
While it is not always possible to show that a given mapping between Banach spaces is weakly continuous, quite often its weak sequential continuity and weakly sequentially continuous offers no problem. A śequentialćoncept of continuity is more general than the continuity and moreover, more useful (for example the Lebesgue's dominated convergence theorem is valid for sequence but not for nets) so we shall state a fixed point theorem and some propositions which will be used in the sequel (see[18]).

Theorem 1.1 *Let E be a Banach space and let Q be a nonempty, bounded, closed and convex subset of the space E and let $T : Q \to Q$ be a weakly sequentially continuous and assume that $TQ(t)$ is relatively weakly compact in E for each $t \in [0,1]$. Then, T has a fixed point in the set Q.*

Proposition 1.2 *A subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology.*

Proposition 1.3 *Let E be a normed space with $y \in E$ and $y \neq 0$. Then there exists a $\phi \in E^*$ with $||\phi|| = 1$ and $||y|| = \phi(y)$.*

The fractional order integral operator in reflexive Banach spaces is defined as [19]:

DEFINITION 1.4 Let $x : I \to E$ be weakly measurable function such that $\phi(x(.)) \in L_1$, and let $\alpha > 0$. Then the fractional (arbitrary) order Pettis-integral $I^\alpha x(t)$
is defined by [19]

$$I^\alpha \, x(t) \; = \; \int_0^t \frac{(t \; - \; s)^{\alpha-1}}{\Gamma(\alpha)} \; x(s) \; ds.$$

The above definition is an extension of fractional order integral operator for real valued functions [14], [15], [20]-[22] and in the above definition the sign " $\int$ " denotes the Pettis-integral. Such an integral is well defined.
THEOREM 1.5 ([19]) *Let $x : I \to E$ be weakly measurable function $\phi(x(.)) \in L_1$, and let $\alpha > 0$. The fractional (arbitrary) order Pettis-integral*

$$I^\alpha \, x(t) \; = \; \int_0^t \frac{(t \; - \; s)^{\alpha-1}}{\Gamma(\alpha)} \; x(s) \; ds.$$

exists for almost every $t \in I$ as a function from I into E and $\phi(I^\alpha \, x(t)) = I^\alpha \, \phi(x(t))$.

For the properties of the integrals of fractional orders in reflexive spaces we have the following Lemma [19].

LEMMA 1.6 *Let $x : I \to E$ be weakly measurable function $\phi(x(.)) \in L_1$, and let $\alpha, \; \beta > 0$. Then we have:*

1. $I^\alpha I^\beta \; x(t) \; = \; I^{\alpha+\beta} \; x(t)$ *for a.e.* $t \in I$.
2. $\lim_{\alpha \to 1} I^\alpha x(t) = I^1 x(t)$ *weakly uniformly on I if only these integrals exist on I.*
3. $\lim_{\alpha \to 0} I^\alpha x(t) = x(t)$ *weakly in E for a.e. $t \in I$.*
4. *If for fixed $t \in I$, $\phi(x(t))$ is bounded for each $\phi \in E^*$, then* $\lim_{t \to 0} I^\alpha x(t) = 0$.

2. Weak solutions for a coupled system. This section deals with the existence of weak solutions for the coupled system (2) under the following assumptions:
Let $B_r = \{x \in E : \; ||x|| \leqslant r\}, \; r > 0$.

(1:) $g_i \in C[I, E], \quad i = 1, 2$.

(2:) $f_i : \; I \times B_r \to E, \quad i = 1, 2$ satisfy the following

- (i) For each $t \in I$, $f_{i_t} \; = \; f_i(t, .)$ are weakly sequentially continuous;
- (ii) For each $x \in B_r$, $f_i(., x(.))$ are weakly measurable on I ;
- (iii) For any $r > 0$, the weak closure of the range of $f_i(I \times B_r)$ are weakly compact in E
(or equivalently: there exist M_i such that $||f_i(t, x)|| \; \leqslant \; M_i, \;\; i = 1, 2$
$(t, x) \in I \times B_r$)

DEFINITION 2.1 By a weak solution of the coupled system (2) we mean that a pair of functions $(x,y) \in C[I,E] \times C[I,E]$ such that

$$\phi(x(t)) = \phi(g_1(t)) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(f_1(s,y(s)))\, ds, \qquad t \in I,$$

$$\phi(y(t)) = \phi(g_2(t)) + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \phi(f_2(s,x(s)))\, ds, \qquad t \in I$$

for all $\phi \in E^*$ and $0 < \alpha,\ \beta < 1$.

THEOREM 2.2 *Let the assumptions (1:) and (2:) be satisfied. Then the coupled system (2) has at least one weak solution* $(x,y) \in C[I,E] \times C[I,E]$.

PROOF Define the operators T_1, T_2 by

$$T_1 y(t) = g_1(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s,y(s))\, ds, \quad t \in I$$

$$T_2 x(t) = g_2(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_2(s,x(s))\, ds, \quad t \in I,$$

where $0 < \alpha,\ \beta < 1$.
Then the coupled (2) may be written as:

$$x(t) = T_1 y(t)$$

$$y(t) = T_2 x(t).$$

Define the operator T by

$$T(x,y)(t) = (T_1 y(t),\ T_2 x(t)).$$

For any $y \in C[I,E]$ and since $f_1(.,y(.))$ is weakly measurable on I and $||f_1(t,y)|| \leqslant M_1$, then $\phi(f_1(.,y(.)))$ is Lebesgue integrable on I $(\forall \phi \in E^*)$ and since $\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}$ is Lebesgue integrable on I, then we have

$$\phi(\frac{(t-.)^{\alpha-1}}{\Gamma(\alpha)} f_1(.,y(.))) = \frac{(t-.)^{\alpha-1}}{\Gamma(\alpha)} \phi(f_1(.,y(.)))$$

is Lebesgue integrable on I $(\forall \phi \in E^*)$, then $\frac{(t-.)^{\alpha-1}}{\Gamma(\alpha)} f_1(.,y(.))$ is Pettis integrable on I. Thus T_1 is well defined.
Now, we shall prove that $T_1 : C[I,E] \to C[I,E]$.

Let $t_1,\ t_2 \in I$ and (without loss of generality assume that $T_1y(t_2) - T_1y(t_1) \neq 0$)

$$\begin{aligned}
T_1y(t_2) - T_1y(t_1) &= g_1(t_2) - g_1(t_1) \\
&\quad + \int_0^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, y(s))\, ds - \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, y(s))\, ds \\
&= g_1(t_2) - g_1(t_1) + \int_0^{t_1} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, y(s))\, ds \\
&\quad + \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, y(s))\, ds - \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, y(s))\, ds \\
&\leqslant g_1(t_2) - g_1(t_1) + \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, y(s))\, ds \\
&\quad + \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, y(s))\, ds - \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, y(s))\, ds \\
&\leqslant g_1(t_2) - g_1(t_1) + \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, y(s))\, ds.
\end{aligned}$$

Therefore, as a consequence of Proposition 1.3, we obtain

$$\begin{aligned}
||T_1y(t_2) - T_1y(t_1)|| &= \phi(T_1y(t_2) - T_1y(t_1)) \\
&= \phi(g_1(t_2) - g_1(t_1)) + \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} \phi(f_1(s, y(s)))\, ds \\
&= ||g_1(t_2) - g_1(t_1)|| + \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} ||f_1(s, y(s))||\, ds \\
&\leqslant ||g_1(t_2) - g_1(t_1)|| + M_1 \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)}\, ds \\
&\leqslant ||g_1(t_2) - g_1(t_1)|| + M_1 \frac{(t_2 - t_1)^{\alpha}}{\Gamma(\alpha + 1)}.
\end{aligned}$$

As done above, we can show that

$$||T_2x(t_2) - T_2x(t_1)|| \leqslant ||g_2(t_2) - g_2(t_1)|| + M_2 \frac{(t_2 - t_1)^{\beta}}{\Gamma(\beta + 1)}.$$

Now, we shall prove that $T:\ C[I, E] \times C[I, E] \to C[I, E] \times C[I, E]$

$$\begin{aligned}
Tu(t_2) - Tu(t_1) &= T(x, y)(t_2) - T(x, y)(t_1) \\
&= (T_1y(t_2),\ T_2x(t_2)) - (T_1y(t_1),\ T_2x(t_1)) \\
&= (T_1y(t_2) - T_1y(t_1),\ T_2x(t_2) - T_2x(t_1)),
\end{aligned}$$

then we have

$$\begin{aligned}
||\ Tu(t_2) - Tu(t_1)\ || &\leqslant ||T_1y(t_2) - T_1y(t_1)|| + ||T_2x(t_2) - T_2x(t_1)|| \\
&\leqslant ||g_1(t_2) - g_1(t_1)|| + M_1 \frac{(t_2 - t_1)^{\alpha}}{\Gamma(\alpha + 1)}
\end{aligned}$$

$$+ ||g_2(t_2) \; - \; g_2(t_1)|| \; + \; M_2 \; \frac{(t_2 - t_1)^{\beta}}{\Gamma(\beta + 1)},$$

and

$$\begin{aligned} ||T_1 \; y(t)|| \; &= \; \phi(T_1 y(t)) \; = \; \phi(g_1(t)) \; + \; \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \; \phi(f_1(s, y(s))) \; ds \\ &= \; ||g_1|| \; + \; \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \; ||f_1(s, y(s))|| \; ds \\ &\leqslant \; ||g_1|| \; + \; M_1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \; ds \\ &\leqslant \; ||g_1|| \; + \; \frac{M_1}{\Gamma(\alpha + 1)}. \end{aligned}$$

By a similar way as done above we can prove that

$$||T_2 x(t)|| \; \leqslant \; ||g_2|| \; + \; \frac{M_2}{\Gamma(\beta + 1)}.$$

Then, $T_1, \; T_2$ are well defined on the sets

$$Q_1 = \; \{ \; y \in C[I, E] : \; ||y|| \leqslant \; ||g_1|| \; + \; \frac{M_1}{\Gamma(\alpha + 1)} \; \},$$

and

$$Q_2 = \; \{ \; x \in C[I, E] : \; ||x|| \leqslant \; ||g_2|| \; + \; \frac{M_2}{\Gamma(\beta + 1)} \},$$

respectively.
Now, define the set Q by

$$Q \; = \; \{ \; u = (x, y) \in C[I, E] \times C[I, E] : \; ||u|| \leqslant \; ||g_1|| \; + \; \frac{M_1}{\Gamma(\alpha + 1)} \; + \; ||g_2|| \; + \; \frac{M_2}{\Gamma(\beta + 1)} \; \}.$$

Then, for any $u \in Q$ we have

$$\begin{aligned} ||Tu(t)|| \; &= \; ||T(x, y)(t)|| \; = \; ||(T_1 y(t), T_2 x(t))|| \; \leqslant \; ||T_1 y(t)|| \; + \; ||T_2 x(t)|| \\ &\leqslant \; ||g_1|| \; + \; \frac{M_1}{\Gamma(\alpha + 1)} \; + \; ||g_2|| \; + \; \frac{M_2}{\Gamma(\beta + 1)}. \end{aligned}$$

i.e. $\forall u \in Q \Rightarrow Tu \in Q \Rightarrow TQ \subset Q$. Thus $T : Q \rightarrow Q$.
Then Q is nonempty, uniformly bounded and strongly equi-continuous subset of $C[I, E] \times C[I, E]$. Also, it can be shown that Q is convex and closed. As a consequence of Proposition 1.2, then TQ is relatively weakly compact. It remains to prove that T is weakly sequentially continuous.
Let $\{y_n(t)\}$ and $\{x_n(t)\}$ be two sequences in $Q_1, \; Q_2$ converge weakly to $x(t), \; y(t)$ respectively $\forall t \in I$. Since $f_1(t, y(t))$ and $f_2(t, x(t))$ are weakly sequentially continuous in the second argument, then $f_1(t, y_n(t))$ and $f_2(t, x_n(t))$ converge weakly to $f_1(t, y(t))$ and $f_2(t, x(t))$ respectively and hence $\phi(f_1(t, y_n(t)))$ and $\phi(f_2(t, x_n(t)))$ converge strongly to $\phi(f_1(t, y(t)))$ and $\phi(f_2(t, x(t)))$ respectively.

Using assumption (iii)) and applying Lebesgue Dominated Convergence Theorem for Pettis integral, then we get

$$\phi(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, y_n(s))\, ds = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(f_1(s, y_n(s)))\, ds$$

$$\to \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(f_1(s, y(s)))\, ds \qquad \forall \phi \in E^*, \quad t \in I,$$

and

$$\phi(\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2(s, x_n(s))\, ds = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \phi(f_2(s, x_n(s)))\, ds$$

$$\to \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\alpha)} \phi(f_2(s, x(s)))\, ds \qquad \forall \phi \in E^*, \quad t \in I,$$

Then T is weakly sequentially continuous. Since all conditions of Theorem 1.1 are satisfied, then the operator T has at least one fixed point $u \in Q$ which competes the proof. ■

As a particular case of Theorem 2.2 we can deduce the existence of weak solutions belonging to the space $C(I, E) \times C(I, E)$ of the following coupled system:

$$\begin{aligned} x(t) &= g_1(t) + \int_0^t f_1(s, y(s))\, ds, \qquad t \in I, \\ y(t) &= g_2(t) + \int_0^t f_2(s, x(s))\, ds, \qquad t \in I. \end{aligned}$$

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