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Finite difference inequalities

Abstract. Discrete inequalities of Gronwall type are considered in vector lattices.

In this note we wish to establish the discrete analogues of the Gronwall's differential inequality [2]. Discrete inequalities of this type but in other form were considered by many authors, see for example Pachpatte [3]–[6], Popenda and Werbowski [8], Willett and Wong [9]. Gronwall's type inequalities play an important role in the qualitative theory of differential, difference, integral and summary equations.

Before giving the main results, we first recollect a few of the basic definitions and notions (see [1]). Throughout the paper E denotes any vector lattice, E_0 – a subset of nonnegative elements of E, N – the set of positive integers. Denote by $\bigwedge_{j=1}^{n} x_j$ the greatest lower bound and by $\bigvee_{j=1}^{n} x_j$ the least upper bound for the system $\{x_j\}_{j=1,\ldots,n}$ of elements of the lattice E. For any two elements we shall always denote these two bounds by $x \wedge y$ and $x \vee y$; they exist by the definition of a lattice. The symbol "$\leqslant$" is used for order relation. For $x \in E$, $(x)_+ := x \vee 0$. Furthermore we define

$$x \vee \bigvee_{j=1}^{0} x_j = x \wedge \bigwedge_{j=1}^{0} x_j = x, \qquad \sum_{j=1}^{0} x_j = 0, \qquad \prod_{j=1}^{0} a_j = 1.$$

The order relation in a vector lattice has the following properties:

(i) $$x_k \leqslant \bigvee_{j=1}^{n} x_j, \quad \bigwedge_{j=1}^{n} x_j \leqslant x_k \quad \text{for any } x_k,\ k = 1, \ldots, n,\ x_k \in E;$$

(ii) $$\bigwedge_{j=1}^{n} x_j \leqslant \bigwedge_{j=1}^{n} y_j, \quad \bigvee_{j=1}^{n} x_j \leqslant \bigvee_{j=1}^{n} y_j, \quad \sum_{j=1}^{n} x_j \leqslant \sum_{j=1}^{n} y_j$$

for any $x_j, y_j \in E$ such that $x_j \leqslant y_j$, $j = 1, \ldots, n$;

(iii) $$\bigvee_{j=1}^{n} (x_j+y_j) \leqslant \bigvee_{j=1}^{n} x_j + \bigvee_{j=1}^{n} y_j, \quad \bigvee_{\substack{j=1\\k=1}}^{n} (x_j+y_k) = \bigvee_{j=1}^{n} x_j + \bigvee_{j=1}^{n} y_j$$

for any $x_j, y_k \in E$, $j, k = 1, \ldots, n$;

(iv) if $x \leqslant y$ then $ax \leqslant ay$ for any $a \in \langle 0, \infty)$, $x, y \in E$;
if $x \in E_0$ and $a \leqslant b$, $a, b \in (-\infty, \infty)$ then $ax \leqslant bx$;

(v) $$x \vee (y+z) \leqslant x \vee y + (z)_+ \quad \text{for any } x, y, z \in E.$$

LEMMA. *Let* $y, z\colon N \to E$, $a\colon N \to (0, \infty)$. *If*

$$y_{n+1} \leqslant a_n y_n + z_n, \quad n \in N, \tag{1}$$

then

$$y_n \leqslant \prod_{j=1}^{n-1} a_j \Big[y_1 + \sum_{k=1}^{n-1} \Big(\prod_{j=1}^{k} a_j^{-1}\Big) z_k\Big], \quad n \in N. \tag{2}$$

Proof. Taking into account property (iv), we have from (1)

$$\Big(\prod_{j=1}^{k} a_j^{-1}\Big) y_{k+1} - \Big(\prod_{j=1}^{k-1} a_j^{-1}\Big) y_k \leqslant \Big(\prod_{j=1}^{k} a_j^{-1}\Big) z_k.$$

Hence, by (ii),

$$\Big(\prod_{j=1}^{n-1} a_j^{-1}\Big) y_n - y_1 \leqslant \sum_{k=1}^{n-1} \Big(\prod_{j=1}^{k} a_j^{-1}\Big) z_k.$$

Estimation (2) follows now from the above inequality and (iv).

THEOREM 1. *Let* $x, w\colon N \to E_0$, $v\colon N \to E$, $b\colon N \to (0, \infty)$. *If*

$$x_{n+1} \leqslant v_n + \sum_{j=1}^{n} w_j \vee b_j x_j, \quad n \in N, \tag{3}$$

then

$$x_{n+1} \leqslant v_n + \prod_{j=2}^{n} (1+b_j)\Big[w_1 \vee b_1 x_1 + \sum_{k=1}^{n-1} \Big(\prod_{j=2}^{k+1} (1+b_j)^{-1}\Big)(w_{k+1} \vee b_{k+1} v_k)\Big], \quad n \in N. \tag{4}$$

Proof. By putting

$$p_n = \sum_{j=1}^{n} w_j \vee b_j x_j, \quad n \in N, \tag{5}$$

we have from (3) obviously

$$x_{n+1} \leqslant v_n + p_n. \tag{6}$$

Applying (iv), (ii) together with (6) we get from (5)

$$p_{n+1} \leqslant p_n + w_{n+1} \vee b_{n+1}(v_n + p_n).$$

Observe that $p_n \geqslant 0$; therefore by (v) we obtain

$$p_{n+1} \leqslant (1 + b_{n+1}) p_n + w_{n+1} \vee b_{n+1} v_n.$$

Hence, by lemma, we obtain the following estimation:

$$p_n \leqslant \prod_{j=2}^{n} (1+b_j)\left[w_1 \vee b_1 x_1 + \sum_{k=1}^{n-1} \left(\prod_{j=2}^{k+1} (1+b_j)^{-1}\right)(w_{k+1} \vee b_{k+1} v_k)\right]. \tag{7}$$

Estimation (7) together with (6) and (ii) imply (4).

THEOREM 2. *Let* $x, v, w\colon N \to E$, $b\colon N \to (0, \infty)$. *If*

$$x_{n+1} \leqslant v_n + \sum_{j=1}^{n} w_j \wedge b_j x_j, \qquad n \in N, \tag{8}$$

then

$$x_{n+1} \leqslant \left(w_1 \wedge b_1 x_1 + \sum_{k=2}^{n} w_k\right) \wedge \prod_{j=2}^{n} (1+b_j)\left[w_1 \wedge b_1 x_1 + \sum_{k=1}^{n-1} \left(\prod_{j=2}^{k+1} (1+b_j)^{-1}\right) b_{k+1} v_k\right] + v_n, \qquad n \in N. \tag{9}$$

Proof. Writing

$$p_n = \sum_{j=1}^{n} w_j \wedge b_j x_j, \tag{10}$$

we obtain inequalities

$$p_{n+1} \leqslant p_n + w_{n+1} \wedge b_{n+1} x_{n+1} \leqslant p_n + w_{n+1},$$

from which we get, by the lemma,

$$p_n \leqslant w_1 \wedge b_1 x_1 + \sum_{k=2}^{n} w_k := c_n. \tag{11}$$

On the other hand, applying (i) and (iv) we get from (10)

$$p_{n+1} \leqslant p_n + b_{n+1} x_{n+1} \leqslant p_n + b_{n+1}(v_n + p_n) = (1+b_{n+1}) p_n + b_{n+1} v_n.$$

Hence, by lemma, we obtain the following estimation of p_n:

$$p_n \leqslant \prod_{j=2}^{n} (1+b_j)\left[w_1 \wedge b_1 x_1 + \sum_{k=1}^{n-1} \left(\prod_{j=2}^{k+1} (1+b_j)^{-1}\right) b_{k+1} v_k\right] := d_n. \tag{12}$$

Comparing (11), (12) we have, by (ii),

$$p_n \leqslant c_n \wedge d_n.$$

Hence follows the desired estimation (9).

Remark. Let us consider inequality (8), where $w_n \geqslant b_n x_n$ for all $n \in N$; then of course $w_n \wedge b_n x_n = b_n x_n$ and (8) reduces to

$$x_{n+1} \leqslant v_n + \sum_{j=1}^{n} b_j x_j, \qquad n \in N. \tag{13}$$

It is easily verified by using the method of Theorem 2 that then

$$x_{n+1} \leqslant v_n + \prod_{j=2}^{n}(1+b_j)\Big[b_1 x_1 + \sum_{k=1}^{n-1}\Big(\prod_{j=2}^{k+1}(1+b_j)\Big)^{-1} b_{k+1} v_k\Big], \qquad n \in N. \tag{14}$$

THEOREM 3. *Let* $x\colon N \to E_0$, v, $w\colon N \to E$, $b\colon N \to (0, \infty)$. *If*

$$x_{n+1} \leqslant v_n \vee \sum_{j=1}^{n} w_j + b_j x_j, \qquad n \in N, \tag{15}$$

then

(16)

$$x_{n+1} \leqslant v_n \vee \sum_{j=1}^{n} w_j + \prod_{j=2}^{n}(1+b_j)\Big[b_1 x_1 + \sum_{k=1}^{n-1}\Big(\prod_{j=2}^{k+1}(1+b_j)\Big)^{-1} b_{k+1}\Big(v_k \vee \sum_{t=1}^{k} w_t\Big)\Big],$$
$$n \in N.$$

Proof. Let us observe that by (v)

$$x_{n+1} \leqslant \Big(v_n \vee \sum_{j=1}^{n} w_j\Big) + \sum_{j=1}^{n} b_j x_j.$$

The above inequality is of the form (13). Hence, by remark, we obtain (16).

THEOREM 4. *Let* $x\colon N \to E_0$, v, $w\colon N \to E$, $b\colon N \to (0, \infty)$. *If*

$$x_{n+1} \leqslant v_n + \bigvee_{\substack{j=1\\k=1}}^{n} w_j + b_k x_k, \qquad n \in N, \tag{17}$$

then

$$\begin{aligned} x_{n+1} \leqslant v_n + \bigvee_{j=1}^{n} w_j + \prod_{j=2}^{n} \max[1, b_j]\Big[b_1 x_1 + \\ + \sum_{k=1}^{n-1}\Big(\prod_{j=2}^{k+1} \max[1, b_j]\Big)^{-1}\Big(b_{k+1}\Big(v_k + \bigvee_{t=1}^{k} w_t\Big)\Big)_+\Big], \qquad n \in N. \end{aligned} \tag{18}$$

Proof. In virtue of property (iii), inequality (17) is equivalent to

$$x_{n+1} \leqslant v_n + \bigvee_{j=1}^{n} w_j + \bigvee_{j=1}^{n} b_j x_j. \tag{19}$$

By putting

$$p_n = \bigvee_{j=1}^{n} b_j x_j, \qquad z_n = v_n + \bigvee_{j=1}^{n} w_j$$

and applying (iv), (ii), (v) we have

$$\begin{aligned} p_{n+1} &= p_n \vee b_{n+1} x_{n+1} \leqslant p_n \vee (b_{n+1} z_n + b_{n+1} p_n) \\ &\leqslant p_n \vee b_{n+1} p_n + (b_{n+1} z_n)_+ \leqslant \max[1, b_{n+1}]\, p_n + (b_{n+1} z_n)_+ . \end{aligned}$$

Hence, by lemma we obtain an estimation of p_n, which together with (19) give us (18).

Remark. Let us observe that estimation (18) holds if $x\colon N \to E_0$, $v, w\colon N \to E$, $b\colon N \to (0, \infty)$ but instead of (17) the following assumption is satisfied:

$$x_{n+1} \leqslant v_n + \bigvee_{j=1}^{n} w_j + b_j x_j, \qquad n \in N.$$

It is evident since by (iii), inequality (19) remains true.

THEOREM 5. *Let* $x, v, w\colon N \to E$, $b\colon N \to (-\infty, \infty)$. *If for* $n \in N$

$$\text{(20)}\quad \text{(a)}\qquad x_{n+1} \leqslant v_n \vee \bigwedge_{j=1}^{n} w_j + b_j x_j,$$

$$\text{(b)}\qquad x_{n+1} \leqslant v_n \wedge \bigwedge_{j=1}^{n} w_j + b_j x_j,$$

$$\text{(c)}\qquad x_{n+1} \leqslant v_n + \bigwedge_{j=1}^{n} w_j + b_j x_j,$$

$$\text{(d)}\qquad x_{n+1} \leqslant v_n + \bigwedge_{j=1}^{n} w_j \wedge b_j x_j,$$

$$\text{(e)}\qquad x_{n+1} \leqslant v_n + \bigwedge_{j=1}^{n} w_j \vee b_j x_j.$$

Then

$$\text{(21)}\quad \text{(a)}\qquad x_{n+1} \leqslant v_n \vee (w_1 + b_1 x_1),$$

$$\text{(b)}\qquad x_{n+1} \leqslant v_n \wedge (w_1 + b_1 x_1),$$

$$\text{(c)}\qquad x_{n+1} \leqslant v_n + (w_1 + b_1 x_1),$$

$$\text{(d)}\qquad x_{n+1} \leqslant v_n + (w_1 \wedge b_1 x_1),$$

$$\text{(e)}\qquad x_{n+1} \leqslant v_n + (w_1 \vee b_1 x_1),$$

respectively, for all $n \in N$.

Proof. Estimations (21) can easily be checked from (20) using properties (i) and (ii).

Remark. Estimations (21) are not the best possible. If for instance $b_n x_n \leqslant 0$ for all $n \in N$, then $w_n + b_n x_n \leqslant w_n$, $n \in N$. From this we infer that $\bigwedge_{j=1}^{n} w_j + b_j x_j \leqslant \bigwedge_{j=1}^{n} w_j$; hence for functions satisfying (20a) we obtain the estimation $x_{n+1} \leqslant v_n \vee \bigwedge_{j=1}^{n} w_j$, $n \in N$, may be, better than (21a).

We shall prove the theorem given by Pachpatte [4] in the real domain. The presented proof shows us that many theorems considered for real sequences hold in vector lattices. For the function $y\colon N\to E$ we denote $\Delta y\colon N\to E$ by $\Delta y_n = y_{n+1}-y_n$ for $n\in N$.

THEOREM 6. *Let* x, $\Delta x\colon N\to E_0$, a, $b\colon N\to\langle 0,\infty)$. *If*

$$\Delta x_n \leqslant x_1+\sum_{j=1}^{n-1} a_j(x_j+\Delta x_j)+\sum_{j=1}^{n-1} a_j\sum_{i=1}^{j-1} b_i\,\Delta x_i,\qquad n\in N, \tag{22}$$

then

$$\Delta x_n \leqslant \Big[1+2\sum_{j=1}^{n-1} a_j\prod_{i=1}^{j-1}(2+a_i+b_i)\Big]x_1,\qquad n\in N. \tag{23}$$

Proof. Putting

$$p_n = x_1+\sum_{j=1}^{n-1} a_j(x_j+\Delta x_j)+\sum_{j=1}^{n-1} a_j\sum_{i=1}^{j-1} b_i\,\Delta x_i, \tag{24}$$

we obviously have

$$\Delta x_n \leqslant p_n. \tag{25}$$

From (24), (25) conditions (ii) and (iv) we have

$$\Delta p_n = a_n(x_n+\Delta x_n)+a_n\sum_{j=1}^{n-1} b_j\,\Delta x_j \leqslant a_n\Big(x_n+p_n+\sum_{j=1}^{n-1} b_j p_j\Big). \tag{26}$$

Again from (25) and (ii) we obtain

$$x_n \leqslant x_1+\sum_{j=1}^{n-1} p_j. \tag{27}$$

Using (27) in (26) in virtue of (ii) and (iv) we get

$$\Delta p_n \leqslant a_n\Big(x_1+\sum_{j=1}^{n-1} p_j+p_n+\sum_{j=1}^{n-1} b_j p_j\Big). \tag{28}$$

We denote

$$q_n = x_1+\sum_{j=1}^{n-1} p_j+p_n+\sum_{j=1}^{n-1} b_j p_j. \tag{29}$$

Observe that p_n is nonnegative for all $n\in N$; hence $p_n\leqslant q_n$ for $n\in N$. Therefore, it follows from (28), (29), (ii) and (iv) that

$$\Delta q_n = \Delta p_n+p_n+b_n p_n \leqslant a_n q_n+q_n+b_n q_n,$$

whence

$$q_{n+1}\leqslant (2+a_n+b_n)q_n.$$

Since $q_1 = 2x_1$, applying lemma to the above inequality, we have the following estimation

(30) $$q_n \leqslant 2\prod_{j=1}^{n-1}(2+a_j+b_j)x_1.$$

Estimation (30) together with (28) and (29) give us by (ii), (iv)

(31) $$\Delta p_n \leqslant a_n\Big[2\prod_{j=1}^{n-1}(2+a_j+b_j)\Big]x_1.$$

Applying (ii) to (31) and the fact that $p_1 = x_1$ we obtain

(32) $$p_n \leqslant \Big[1+2\sum_{j=1}^{n-1} a_j\prod_{i=1}^{j-1}(2+a_i+b_i)\Big]x_1.$$

Estimation (23) follows now from (32) and (25).

In concluding this paper we note that theorems presented here can be used to study boundedness, stability, asymptotic equivalence and other properties of summary difference equations for instance in real domain.

Consider the sequence given by recurrence formula

(33) $$x_{n+1} = v_n + \sum_{j=1}^{n} w_j \vee b_j x_j, \qquad n \in N,$$

where $w\colon N \to E_0$, $v\colon N \to E_0$, $b\colon N \to (0, \infty)$, $x_1 \in E_0$. Let furthermore

(34) (a) $\{v_n\}_{n\in N}$ is bounded,

(b) $\sum_{j=1}^{\infty} b_j$ converges,

(c) $\{\sum_{j=1}^{n} w_{j+1} \vee b_{j+1} v_j\}_{n\in N}$ is bounded;

then the set $\{x_n\}_{n\in N}$ is bounded.

Since $x_1 \in E_0$ so by assumptions on w, v, b we have $x_n \in E_0$ for all $n \in N$. Therefore applying Theorem 1 we get

(35) $$x_{n+1} \leqslant v_n + \prod_{j=2}^{n}(1+b_j)(w_1 \vee b_1 x_1) + \sum_{k=1}^{n-1}\Big(\prod_{j=k+2}^{n}(1+b_j)\Big)(w_{k+1} \vee b_{k+1} v_k).$$

It follows from (34b) that there exists a constant $\bar{b}$ such that $\prod_{j=k}^{n}(1+b_j) < \bar{b}$ for all $k, n \in N$. Boundedness of $\{x_n\}_{n\in N}$ follows now by (ii) and (iv) from (34a), (34c) and (35).

The definition of an (o)-convergent series of elements in an ordered vector space is the usual

$$(o)-\sum_{n=1}^{\infty} x_n = (o)-\lim_n \sum_{j=1}^{n} x_j$$

if the right-hand side limit does exist.

The lattice E is said to be a *relatively σ-complete lattice* if any countable, bounded subset of E admits the greatest lower bound and the least upper bound.

Consider recurrence equation (33) in the relatively σ-complete lattice under the same conditions as previously. Since the set $\{x_n\}_{n\in N}$ is bounded, there exists $\bigvee_{n\in N} x_n$. Therefore, by (ii) and (iv),

$$\sum_{k=1}^{n} b_k x_k \leqslant \sum_{k=1}^{n} b_k \bigvee_{n\in N} x_n = \bigvee_{n\in N} x_n \sum_{k=1}^{n} b_k, \qquad n\in N.$$

Hence, from the above inequality and (34b) follows boundedness of the set $\{\sum_{k=1}^{n} b_k x_k\}_{n\in N}$. Taking into account the definition of a relatively σ-complete lattice we have that this set admits its least upper bound. Since the sequence $\{\sum_{k=1}^{n} b_k x_k\}_{n\in N}$ is increasing, it has the limit. Therefore the series $\sum_{k=1}^{\infty} b_k x_k$ is (o)-convergent.

We obtain (o)-summability with the wedge b of the solution of equation (33) in a relatively σ-complete vector lattice.

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