COMMENTATIONES MATHEMATICAE Vol. 54, No. 1 (2014), 51-64

Monika Wrzosek*

Newton's method for first-order stochastic functional partial differential equations

Abstract. We apply Newton's method to hyperbolic stochastic functional partial differential equations of the first order driven by a multidimensional Brownian motion. We prove a first-order convergence and a second-order convergence in a probabilistic sense.

2010 Mathematics Subject Classification: 60H15, 35R60, 35R10.

Key words and phrases: Stochastic functional partial differential equations.

1. Introduction. First-order functional partial differential equations are widely studied in numerous papers. Problems being discussed include existence theory of solutions (e.g. classical solutions [4], semiclassical solutions [11], generalized solutions in the Carathéodory sense [19] and Cinquini Cibrario solutions [13]), functional differential inequalities and their applications [20], numerical methods for initial and mixed problems. Applications of Chaplygin's and Newton's methods or in general quasilinearization methods can be found in [3, 5].

Random transport equations in nonfunctional setting are considered in [7, 8, 17]. The article [18] focuses on the use of difference methods in order to approximate the solutions of SPDE of Itô-type, in particular hyperbolic equations. The Cauchy problem for SPDE is considered in [15]. In [10] the authors present a Wiener chaos approach to solve hyperbolic stochastic partial differential equations.

Newton's methods for stochastic differential equations are studied by Kawabata and Yamada in [12] and Amano in [1]. In [2] Amano develops techniques to prove a probabilistic second-order convergence of Newton's methods. In [21] we derive further nontrivial generalizations to the case of stochastic functional differential equations with Hale functionals. However, our approach is different from Amano's

*Supported by grant BW-UG 538-5100-B151-13 from the University of Gdańsk

as it is not possible to construct explicit solutions of linear stochastic functional differential equations using his methods. Instead, we define suitable chains of sets, formulate Itô isometry-type lemma (see Lemma 4.1 [21]), utilize the Gronwall-type inequality and the Chebyshev inequality [9]. It is worth mentioning that existence and uniqueness results in the stochastic functional case are known in the literature [16]. Appropriate assumptions on the given functions imply the existence and uniqueness of solutions and convergence of Newton's sequence.

Our goal is to extend existing results to the case of partial functional differential equations of the first order. Consider any transport equation [6]

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = f, \quad u(0,x) = v(x).$$

If c, f depends on t, x and unknown function u, the quasilinearization method is easy to apply. In the same way one can approximate solutions of initial-boundary value problems for generalized McKendrick's equation [14]

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \lambda u, \qquad \begin{array}{l} u(0,x) = v(x) \text{ for } x \in \mathbb{R}_+ \\ u(t,0) = \tilde{v}(t) \text{ for } t \in [0,T]. \end{array}$$

By adding white noise to this problem with intensity dependent on some functionals operating on the unknown function u (e.g. weighted average with respect to x) we arrive at the stochastic functional differential equation

$$\frac{\partial}{\partial t}u(t,x) + \frac{\partial}{\partial x}u(t,x) = \lambda u(t,x) + \sigma(t)\hat{u}(t)\dot{B}_t,$$

where

$$\hat{u}(t) = \int_{\mathbb{R}} G(x)u(t,x)dx, \quad \int_{\mathbb{R}} G(x)dx = 1.$$

The Cauchy problem for this equation is easy to solve. However, with λ additionaly dependent on t, x or $u(t,x)$), it turns out to be nontrivial. Moreover, $\sigma(t)\hat{u}(t)$ can be replaced by a function dependent on t and $\hat{u}$ up to time t, e.g.

$$S\left(t, \int_0^t \hat{u}(s)ds\right).$$

Observe that white noise coefficient $\sigma\hat{u}$ (or g in model ()) is independent on x. Additional spatial dependence ($g = g(t, x, u_{(t,x)})$) may lead to nontrivial problems with estimating the Itô integral with respect to the norm:

$$||v||_{D_t}^2 = \mathbb{E}\left[\sup_{\tilde{t}\leqslant t, x\in\mathbb{R}^m} |v(\tilde{t},x)|^2\right]. \tag{1}$$

For instance, we cannot use the Doob inequality in the same way as in the proof of Gronwall-type inequality (see Lemma 3.1). The Doob inequality does not relate to the spatial parameter x, but bounds with respect to the time variable t of the stochastic process:

$$\mathbb{E}\left[\sup_{\tilde{t}\leqslant t}|X_{\tilde{t}}|^2\right]\leqslant 4\,\mathbb{E}\left[|X_t|^2\right]$$

for a martingale or a nonnegative submartingale X. One may ask if the following modification of norm (1):

$$||v||^2=\sup_{x\in\mathbb{R}^m}\mathbb{E}\left[\sup_{\tilde{t}\leqslant t}|v(\tilde{t},x)|^2\right]$$

could solve the problem. However, in this case it is not obvious (or nontrivial to show) if

$$Y_t=\int_0^t g\left(s,y^{t,x}(s),u_{(s,y^{t,x}(s))}\right)dB_s$$

is a martingale or a nonnegative submartingale.

The paper is organized as follows. In Section 2 we introduce basic notations and formulate the hyperbolic problem. We prove the existence of solutions by means of successive approximations (Section 4). Next we establish a first-order convergence (Section 5) and a probabilistic second-order convergence (Section 6) of Newton's method. The results in Section 4 and 5 base on the Gronwall-type inequality presented in Section 3.

2. Formulation of the problem. Let $(\Omega,\mathcal{F},P)$ be a complete probability space, $(B_t)_{t\in[0,T]}$ the standard Brownian motion and $(\mathcal{F}_t)_{t\in[0,T]}$ its natural filtration. We recall that $L^2(\Omega)$ is the space of all random variables $Y:\Omega\to\mathbb{R}$ such that $\|Y\|^2=\mathbb{E}[Y^2]<\infty$. By $C([0,T],L^2(\Omega))$ we denote the space of all continuous and $\mathcal{F}_t$-adapted processes $y:[0,T]\to L^2(\Omega)$ with the norm

$$||y||_t^2=\mathbb{E}\left[\sup_{\tilde{t}\leqslant t}|y(\tilde{t})|^2\right].$$

Let $0\leqslant r<\infty$ and $T>0$. For $D_T=[-r,T]\times\mathbb{R}^m$ let $\mathcal{C}_{D_T}$ denote the space of these continuous and $\mathcal{F}_t$-adapted processes $v:D_T\to L^2(\Omega)$ which have the bounded norm

$$||v||_{D_t}^2=\mathbb{E}\left[\sup_{(\tilde{t},x)\in D_t}|v(\tilde{t},x)|^2\right]<\infty.$$

Set $\mathcal{F}_t=\mathcal{F}_0$ for $-r\leqslant t\leqslant 0$. For any process $u\in\mathcal{C}_{D_T}$ and any point $(t,x)\in[0,T]\times\mathbb{R}^m$, the Hale-type operator $u_{(t,x)}$ is defined by

$$u_{(t,x)}(\tau,\theta)=u(t+\tau,x+\theta)\quad\text{for}\quad(\tau,\theta)\in D_0:=[-r,0]\times\mathbb{R}^m\,.$$

We consider the following initial value problem for a first-order stochastic functional partial differential equation

(2)
$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + a(t,x)\cdot\nabla u(t,x) = f\left(t,x,u_{(t,x)}\right) + g\left(t,u_{(t,0)}\right)\dot{B}_t \text{ for } (t,x)\in[0,T]\times\mathbb{R}^m, \\ u(t,x) = \varphi(t,x) \text{ for } (t,x)\in D_0, \end{cases}$$

where

$$\begin{aligned} &a : [0,T]\times\mathbb{R}^m \to \mathbb{R}^m, \\ &\varphi : [-r,0]\times\mathbb{R}^m \to \mathbb{R}, \\ &f : [0,T]\times\mathbb{R}^m \times C(D_0,\mathbb{R}) \to \mathbb{R}, \\ &g : [0,T]\times C(D_0,\mathbb{R}) \to \mathbb{R}^p, \end{aligned}$$

$\dot{B}_t$ is the formal derivative of a p-dimensional Brownian motion B_t and

$$\nabla u = \left(\frac{\partial u}{\partial x_1},\ldots,\frac{\partial u}{\partial x_m}\right).$$

The characteristic equation takes the form

$$y(s) = x - \int_s^t a(\tau,y(\tau))d\tau.$$

Let $y^{t,x}(s)$ denote its solution. By the differentation chain rule

$$d\left(u\left(s,y(s)\right)\right) = \left(\frac{\partial u}{\partial s}\left(s,y(s)\right) + a(s,y(s))\cdot\nabla u\left(s,y(s)\right)\right)ds.$$

Hence we get

$$u(t,x) = \varphi(0,y^{t,x}(0)) + \int_0^t f\left(s,y^{t,x}(s),u_{(s,y^{t,x}(s))}\right)ds + \int_0^t g\left(s,u_{(s,0)}\right)dB_s, \tag{3}$$

for $0\leqslant s\leqslant t\leqslant T$.

3. Gronwall-type inequality. By $(C(D_0,\mathbb{R}))^*$ we denote the space of all linear and bounded functionals on $C(D_0,\mathbb{R})$. Let $\mathcal{L}\left(C(D_0,\mathbb{R}),\mathbb{R}^p\right)$ be the space of all linear and bounded maps from $C(D_0,\mathbb{R})$ to $\mathbb{R}^p$ with the norm

$$||A||_{\mathcal{L}(C(D_0,\mathbb{R}),\mathbb{R}^p)} := \sup_{\sup_{(s,x)\in D_0}|v(s,x)|\leqslant 1} |Av|$$

for $A\in\mathcal{L}\left(C(D_0,\mathbb{R}),\mathbb{R}^p\right)$. The main supremum is taken over all $v\in C(D_0,\mathbb{R})$, whose uniform norms do not exceed 1. Denote $D_t = [-r,t]\times\mathbb{R}^m$ for $t\in[0,T]$.

Lemma 3.1 *Suppose that $\alpha^{(1)} : [0,T]\times\mathbb{R}^m \to \mathcal{C}_{D_T}$, $\alpha^{(2)} : [0,T]\to\mathcal{C}_{D_T}$ are continuous, $A^{(1)}\in(C(D_0,\mathbb{R}))^*$, $A^{(2)}\in\mathcal{L}\left(C(D_0,\mathbb{R}),\mathbb{R}^p\right)$ and there exists a nonnegative constant M such that*

$$\begin{aligned} &||A^{(1)}(t,x)||_{(C(D_0,\mathbb{R}))^*} \leqslant M \quad \textit{for } (t,x)\in[0,T]\times\mathbb{R}^m, \\ &||A^{(2)}(t)||_{\mathcal{L}(C(D_0,\mathbb{R}),\mathbb{R}^p)} \leqslant M \quad \textit{for } t\in[0,T]. \end{aligned} \tag{4}$$

If $u \in \mathcal{C}_{D_T}$ satisfies the following stochastic functional integral equation

$$
\begin{aligned}
u(t,x) &= \int_0^t \left\{ \alpha^{(1)}(s,x) + A^{(1)}(s,x) u_{(s,x)} \right\} ds \\
&+ \int_0^t \left\{ \alpha^{(2)}(s) + A^{(2)}(s) u_{(s,0)} \right\} dB_s \quad \textit{for } (t,x) \in [0,T] \times \mathbb{R}^m \\
u(t,x) &= 0 \quad \textit{for } (t,x) \in D_0,
\end{aligned}
$$

then

$$
||u||^2_{D_t} \leqslant 4e^{4M^2(t+4)t} \int_0^t \left(t||\alpha^{(1)}||^2_{D_s} + 4||\alpha^{(2)}||^2_s \right) ds \quad \textit{for } t \in [0,T].
$$

Proof By the fact that $(x+y)^2 \leqslant 2(x^2+y^2)$ we have

$$
\begin{aligned}
||u||^2_{D_t} &\leqslant 2\,\mathbb{E}\left[\sup_{(\tilde{t},x) \in D_t} \left| \int_0^{\tilde{t}} \left\{ \alpha^{(1)}(s,x) + A^{(1)}(s,x) u_{(s,x)} \right\} ds \right|^2 \right] \\
&+ 2\,\mathbb{E}\left[\sup_{\tilde{t} \leqslant t} \left| \int_0^{\tilde{t}} \left\{ \alpha^{(2)}(s) + A^{(2)}(s) u_{(s,0)} \right\} dB_s \right|^2 \right] \\
&= 2I_1 + 2I_2.
\end{aligned}
$$

Using the Schwarz inequality and (4) we obtain

$$
\begin{aligned}
I_1 &\leqslant t\,\mathbb{E}\left[\sup_{(\tilde{t},x) \in D_t} \int_0^{\tilde{t}} \left| \alpha^{(1)}(s,x) + A^{(1)}(s,x) u_{(s,x)} \right|^2 ds \right] \\
&\leqslant 2t\,\mathbb{E}\left[\int_0^t \sup_{x \in \mathbb{R}^m} \left| \alpha^{(1)}(s,x) \right|^2 ds \right] + 2t\,\mathbb{E}\left[\int_0^t \sup_{x \in \mathbb{R}^m} \left| A^{(1)}(s,x) u_{(s,x)} \right|^2 ds \right] \\
&\leqslant 2t \int_0^t \mathbb{E}\left[\sup_{(\tilde{s},x) \in D_s} \left| \alpha^{(1)}(\tilde{s},x) \right|^2 \right] ds + 2M^2 t \int_0^t \mathbb{E}\left[\sup_{(\tilde{s},x) \in D_s} |u(\tilde{s},x)|^2 \right] ds \\
&\leqslant 2t \int_0^t ||\alpha^{(1)}||^2_{D_s} ds + 2M^2 t \int_0^t ||u||^2_{D_s} ds
\end{aligned}
$$

By the Doob martingale inequality and the Itô isometry:

$$
\begin{aligned}
I_2 &\leqslant 4\,\mathbb{E}\left[\left| \int_0^t \left\{ \alpha^{(2)}(s) + A^{(2)}(s) u_{(s,0)} \right\} dB_s \right|^2 \right] \\
&= 4\,\mathbb{E}\left[\int_0^t \left| \alpha^{(2)}(s) + A^{(2)}(s) u_{(s,0)} \right|^2 ds \right] \\
&\leqslant 8 \int_0^t \mathbb{E}\left[\sup_{\tilde{s} \leqslant s} \left| \alpha^{(2)}(\tilde{s}) \right|^2 \right] ds + 8M^2 \int_0^t \mathbb{E}\left[\sup_{(\tilde{s},x) \in D_s} |u(\tilde{s},x)|^2 \right] ds \\
&= 8 \int_0^t ||\alpha^{(2)}||^2_s ds + 8M^2 \int_0^t ||u||^2_{D_s} ds
\end{aligned}
$$

Hence

$$||u||_{D_t}^2 \leqslant 4\int_0^t \left(t||\alpha^{(1)}||_{D_s}^2 + 4||\alpha^{(2)}||_s^2\right) ds + 4M^2(t+4)\int_0^t ||u||_{D_s}^2 ds.$$

For a fixed t_0 such that $0 \leqslant t_0 \leqslant T$ we have

$$||u||_{D_t}^2 \leqslant 4\int_0^{t_0} \left(t_0||\alpha^{(1)}||_{D_s}^2 + 4||\alpha^{(2)}||_s^2\right) ds + 4M^2(t_0+4)\int_0^t ||u||_{D_s}^2 ds.$$

for $0 \leqslant t \leqslant t_0$. We apply the Gronwall inequality and obtain

$$||u||_{D_t}^2 \leqslant 4e^{4M^2(t_0+4)t}\int_0^{t_0} \left(t_0||\alpha^{(1)}||_{D_s}^2 + 4||\alpha^{(2)}||_s^2\right) ds, \quad 0 \leqslant t \leqslant t_0.$$

Since t_0 is fixed arbitrarily, we get

$$||u||_{D_t}^2 \leqslant 4e^{4M^2(t+4)t}\int_0^{t} \left(t||\alpha^{(1)}||_{D_s}^2 + 4||\alpha^{(2)}||_s^2\right) ds, \quad t \in [0,T].$$

This completes the proof. ∎

4. Existence of solutions. We formulate an iterative scheme for problem (3). Let

$$u^{(0)} \in \mathcal{C}_{D_T},\ u^{(0)}(t,x) = \varphi(t,x) \text{ for } t \in [-r,0),\ x \in \mathbb{R}^m$$

and

$$\begin{aligned} u^{(k+1)}(t,x) &= \varphi(0,y^{t,x}(0)) + \int_0^t f\left(s, y^{t,x}(s), u^{(k)}_{(s,y^{t,x}(s))}\right) ds \\ &\quad + \int_0^t g\left(s, u^{(k)}_{(s,0)}\right) dB_s, \quad (t,x) \in [0,T]\times\mathbb{R}^m \\ u^{(k+1)}(t,x) &= \varphi(t,x), \quad (t,x) \in [-r,0]\times\mathbb{R}^m\,. \end{aligned} \tag{5}$$

Since $\mathcal{F}_t := \mathcal{F}_0$ for $t \in [-r,0)$, the process φ is deterministic, thus independent of the Brownian motion on $[0,T]$. If we denote

$$\Delta u^{(k)}(t,x) = u^{(k+1)}(t,x) - u^{(k)}(t,x)$$

then we have

$$\begin{aligned} \Delta u^{(k+1)}(t,x) &= \int_0^t \left\{ f\left(s, y^{t,x}(s), u^{(k+1)}_{(s,y^{t,x}(s))}\right) - f\left(s, y^{t,x}(s), u^{(k)}_{(s,y^{t,x}(s))}\right)\right\} ds \\ &+ \int_0^t \left\{ g\left(s, u^{(k+1)}_{(s,0)}\right) - g\left(s, u^{(k)}_{(s,0)}\right)\right\} dB_s. \end{aligned}$$

THEOREM 4.1 *Suppose that the functions*

$$f : [0,T] \times \mathbb{R}^m \times C_b(D_0,\mathbb{R}) \to \mathbb{R},$$
$$g : [0,T] \times C_b(D_0,\mathbb{R}) \to \mathbb{R}^p,$$

are continuous and satisfy the Lipschitz condition with respect to the functional variable

$$|f(t,x,v) - f(t,x,\bar{v})| \leqslant L\,|v-\bar{v}|\,, \quad |g(t,v) - g(t,\bar{v})| \leqslant L\,|v-\bar{v}|$$

for $v,\bar{v} \in C(D_0,\mathbb{R})$. *Then the sequence* $u^{(k)} = (u^{(k)})_{k\in\mathbb{N}}$ *defined by (5) converges to the unique solution* u *of equation (2) in the following sense*

$$\lim_{k\to\infty} \left\| u^{(k)} - u \right\|_{D_T} = 0.$$

PROOF We show that $(u^{(k)})_{k\in\mathbb{N}}$ satisfies the Cauchy condition with respect to the norm $||\cdot||_{D_T}$. By the Lipschitz condition we have

$$\begin{aligned} \left| f\left(s, y^{t,x}(s), u^{(k+1)}_{(s,y^{t,x}(s))}\right) - f\left(s, y^{t,x}(s), u^{(k)}_{(s,y^{t,x}(s))}\right) \right| &\leqslant L \left| u^{(k+1)}_{(s,y^{t,x}(s))} - u^{(k)}_{(s,y^{t,x}(s))} \right| \\ &\leqslant L \sup_{(\tilde{s},\tilde{x})\in D_s} \left| \Delta u^{(k)}(\tilde{s},\tilde{x}) \right|. \end{aligned}$$

Similarly

$$\left| g\left(s, u^{(k+1)}_{(s,0)}\right) - g\left(s, u^{(k)}_{(s,0)}\right) \right| \leqslant L \left| u^{(k+1)}_{(s,0)} - u^{(k)}_{(s,0)} \right| \leqslant L \sup_{(\tilde{s},\tilde{x})\in D_s} \left| \Delta u^{(k)}(\tilde{s},\tilde{x}) \right|.$$

Applying Lemma 3.1 with

$$\begin{aligned} &\alpha^{(1)}(s,x) = f\left(s, y^{t,x}(s), u^{(k+1)}_{(s,y^{t,x}(s))}\right) - f\left(s, y^{t,x}(s), u^{(k)}_{(s,y^{t,x}(s))}\right), \\ &\alpha^{(2)}(s) = g\left(s, u^{(k+1)}_{(s,0)}\right) - g\left(s, u^{(k)}_{(s,0)}\right), \\ &A^{(1)}(s,x) \equiv 0, \quad A^{(2)}(s) \equiv 0 \end{aligned}$$

we obtain

$$\left\| \Delta u^{(k+1)} \right\|^2_{D_t} \leqslant 2L^2(T+4) \int_0^t \left\| \Delta u^{(k)} \right\|^2_{D_s} ds.$$

We have the recurrence inequality

$$\left\| \Delta u^{(k+1)} \right\|^2_{D_t} \leqslant C_T \int_0^t \left\| \Delta u^{(k)} \right\|^2_{D_s} ds, \tag{6}$$

where $C_T = 2L^2(T+4)$. The recursive use of (6) leads to

$$\left\| \Delta u^{(k+1)} \right\|^2_{D_t} \leqslant \frac{C_T^{k+1} t^{k+1}}{(k+1)!} \left\| \Delta u^{(0)} \right\|^2_{D_t}, \quad k = 0,1,\ldots.$$

Hence

$$\left\| u^{(k+p)} - u^{(k)} \right\|^2_{D_t} \leqslant \left(\frac{C_T^{k+p-1} t^{k+p-1}}{(k+p-1)!} + \ldots + \frac{C_T^k t^k}{k!} \right) \left\| \Delta u^{(0)} \right\|^2_{D_t}$$

for $k = 0, 1, \ldots$. We conclude that $(u^{(k)})_{k\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $\mathcal{C}_{D_T}$. Therefore it is convergent to the solution u. ■

5. First-order convergence of Newton's method. We formulate Newton's scheme for problem (3). Let

$$u^{(0)} \in \mathcal{C}_{D_T},\ u^{(0)}(t,x) = \varphi(t,x) \text{ for } (t,x) \in D_0$$

and consider the following sequence of functional integral problems

$$(7)\quad \begin{aligned} u^{(k+1)}(t,x) &= \varphi(0, y^{t,x}(0)) \\ &+ \int_0^t \left\{ f\left(s, y^{t,x}(s), u^{(k)}_{(s,y^{t,x}(s))}\right) + f_v\left(s, y^{t,x}(s), u^{(k)}_{(s,y^{t,x}(s))}\right) \Delta u^{(k)}_{(s,y^{t,x}(s))} \right\} ds \\ &+ \int_0^t \left\{ g\left(s, u^{(k)}_{(s,0)}\right) + g_v\left(s, u^{(k)}_{(s,0)}\right) \Delta u^{(k)}_{(s,0)} \right\} dB_s, \quad (t,x) \in [0,T] \times \mathbb{R}^m \\ u^{(k+1)}(t,x) &= \varphi(t,x), \quad (t,x) \in [-r, 0] \times \mathbb{R}^m, \end{aligned}$$

where $\Delta u^{(k)}(t,x) = u^{(k+1)}(t,x) - u^{(k)}(t,x)$ and

$$f : [0,T] \times \mathbb{R}^m \times C_b(D_0, \mathbb{R}) \to \mathbb{R}, \quad g : [0,T] \times C_b(D_0, \mathbb{R}) \to \mathbb{R}^p$$

are continuous functions,

$$f_v \in \left(C(D_0, \mathbb{R})\right)^*, \quad g_v \in \mathcal{L}\left(C(D_0, \mathbb{R}), \mathbb{R}^p\right)$$

are Fréchet derivatives of f and g with respect to the functional variable $v \in C(D_0, \mathbb{R})$. We have the following integral equation for increments $\Delta u^{(k+1)}$:

$$(8)\quad \begin{aligned} &\Delta u^{(k+1)}(t,x) \\ &= \int_0^t \left\{ \Delta f^{(k)}(s,x) - f_v^{(k)}(s,x) \Delta u^{(k)}_{(s,y^{t,x}(s))} + f_v^{(k+1)}(s,x) \Delta u^{(k+1)}_{(s,y^{t,x}(s))} \right\} ds \\ &+ \int_0^t \left\{ \Delta g^{(k)}(s) - g_v^{(k)}(s) \Delta u^{(k)}_{(s,0)} + g_v^{(k+1)}(s) \Delta u^{(k+1)}_{(s,0)} \right\} dB_s \end{aligned}$$

for $(t,x) \in [0,T] \times \mathbb{R}^m$ and

$$\begin{aligned} \Delta f^{(k)}(s,x) &:= f\left(s, y^{t,x}(s), u^{(k+1)}_{(s,y^{t,x}(s))}\right) - f\left(s, y^{t,x}(s), u^{(k)}_{(s,y^{t,x}(s))}\right) \\ \Delta g^{(k)}(s) &:= g\left(s, u^{(k+1)}_{(s,0)}\right) - g\left(s, u^{(k)}_{(s,0)}\right) \\ f_v^{(k)}(s,x) &:= f_v\left(s, y^{t,x}(s), u^{(k)}_{(s,y^{t,x}(s))}\right) \\ g_v^{(k)}(s) &:= g_v\left(s, u^{(k)}_{(s,0)}\right). \end{aligned}$$

Theorem 5.1 *Suppose that there exists a nonnegative constant L such that*

$$\|f_v(t,x)\|_{(C(D_0,\mathbb{R}))^*} \leqslant L, \quad \|g_v(t)\|_{\mathcal{L}(C(D_0,\mathbb{R}),\mathbb{R}^p)} \leqslant L. \tag{9}$$

Then the Newton sequence $(u^{(k)})_{k\in\mathbb{N}}$ defined by (7) converges to the unique solution u of equation (2) in the following sense:

$$\lim_{k\to\infty} \left\|u^{(k)} - u\right\|_{D_T} = 0.$$

Proof We show that $(u^{(k)})_{k\in\mathbb{N}}$ satisfies the Cauchy condition with respect to the norm $||\cdot||_{D_T}$. Notice that (9) implies the Lipschitz condition for $f(t,x,v)$ and $g(t,v)$:

$$|f(t,x,v) - f(t,x,\bar{v})| \leqslant L\,|v-\bar{v}|\,, \quad |g(t,v) - g(t,\bar{v})| \leqslant L\,|v-\bar{v}| \tag{10}$$

for $v,\bar{v} \in C(D_0,\mathbb{R})$. We apply Lemma 3.1 with

$$\alpha^{(1)}(s,x) = \Delta f^{(k)}(s,x) - f_v^{(k)}(s,x)\Delta u^{(k)}_{(s,y^{t,x}(s))}, \qquad A^{(1)}(s,x) = f_v^{(k+1)}(s,x)$$
$$\alpha^{(2)}(s) = \Delta g^{(k)}(s) - g_v^{(k)}(s)\Delta u^{(k)}_{(s,0)}, \qquad A^{(2)}(s) = g_v^{(k+1)}(s)$$

and obtain

$$\begin{aligned}&||\Delta u^{(k+1)}||^2_{D_t}\\ &\quad\leqslant\ 4e^{4L^2(t+4)t}\int_0^t \left[t||\Delta f^{(k)} - f_v^{(k)}\Delta u^{(k)}||^2_{D_s} + 4||\Delta g^{(k)} - g_v^{(k)}\Delta u^{(k)}||^2_s\right] ds.\end{aligned}$$

By the Lipschitz condition and (9) we have the estimate

$$||\Delta u^{(k+1)}||^2_{D_t} \quad\leqslant\quad 16L^2(t+4)e^{4L^2(t+4)t}\int_0^t \|\Delta u^{(k)}\|^2_{D_s} ds.$$

Since $t \leqslant T$, we have

$$||\Delta u^{(k+1)}||^2_{D_t} \quad\leqslant\quad C_T\int_0^t \|\Delta u^{(k)}\|^2_{D_s} ds, \tag{11}$$

where

$$C_T = 16L^2(T+4)e^{4L^2(T+4)T}.$$

The recursive use of (11) leads to

$$\|\Delta u^{(k+1)}\|^2_{D_t} \leqslant \frac{C_T^{k+1}t^{k+1}}{(k+1)!}\|\Delta u^{(0)}\|^2_{D_t}.$$

Thus

$$\|u^{(k+p)} - u^{(k)}\|^2_{[0,t]} \quad\leqslant\quad \left(\frac{C_T^{k+p-1}t^{k+p-1}}{(k+p-1)!} + \ldots + \frac{C_T^k t^k}{k!}\right)\|\Delta u^{(0)}\|^2_{D_T}.$$

We conclude that $(u^{(k)})_{k\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $\mathcal{C}_{D_T}$. Therefore it is convergent to u, which is the solution to problem (2). ∎

6. Probabilistic second-order convergence of Newton's method. The following theorem establishes a second-order convergence of Newton's method in a probabilistic sense.

THEOREM 6.1 *Suppose that the general assumptions of Section 5 are satisfied and there exists a nonnegative constant M such that*

$$\|f_v(t,x,v) - f_v(t,x,\bar v)\|_{(C(D_0,\mathbb{R}))^*} \leqslant M\,|v-\bar v|\,, \tag{12}$$

$$\|g_v(t,v) - g_v(t,\bar v)\|_{\mathcal{L}(C(D_0,\mathbb{R}),\mathbb{R}^p)} \leqslant M\,|v-\bar v| \tag{13}$$

for all $v,\bar v \in C(D_0,\mathbb{R}^n)$. *Then there exists a nonnegative constant C (independent of T) such that for any $T>0$*

$$P\left(\sup_{(t,x)\in D_T} |\Delta u^{(k)}(t,x)| \leqslant \rho \quad\Rightarrow\quad \sup_{(t,x)\in D_T} |\Delta u^{(k+1)}(t,x)| \leqslant R\rho^2\right) \geqslant 1 - e^{C(T^2+1)}TR^{-2}$$

for all $R>0$, $0<\rho\leqslant 1$, $k=0,1,2,\ldots$.

PROOF Define the sets

$$A^{(k)}_{\rho,t} = \left\{\omega : \sup_{(s,x)\in D_t} |\Delta u^{(k)}(s,x)| \leqslant \rho\right\} \quad \text{for } 0<\rho\leqslant 1,\ 0\leqslant t\leqslant T,\ k=0,1,2,\ldots.$$

We consider the sequence $(\Delta u^{(k)})_{k\in\mathbb{N}}$ restricted to the sets $A^{(k)}_{\rho,t}$. For this reason we multiply equation (8) by $\mathbf{1}_{A^{(k)}_{\rho,t}}$, the characteristic function of the set $A^{(k)}_{\rho,t}$, and have

$$\begin{aligned}
&\mathbf{1}_{A^{(k)}_{\rho,t}}\,\Delta u^{(k+1)}(t,x)\\
&= \mathbf{1}_{A^{(k)}_{\rho,t}} \int_0^t \left\{\Delta f^{(k)}(s,x) - f_v^{(k)}(s,x)\Delta u^{(k)}_{(s,y^{t,x}(s))} + f_v^{(k+1)}(s,x)\Delta u^{(k+1)}_{(s,y^{t,x}(s))}\right\} ds\\
&+ \mathbf{1}_{A^{(k)}_{\rho,t}} \int_0^t \left\{\Delta g^{(k)}(s) - g_v^{(k)}(s)\Delta u^{(k)}_{(s,0)} + g_v^{(k+1)}(s)\Delta u^{(k+1)}_{(s,0)}\right\} dB_s
\end{aligned}$$

for $(t,x)\in[0,T]\times\mathbb{R}^m$. If we denote

$$\begin{aligned}
F(s,x) &= \Delta f^{(k)}(s,x) - f_v^{(k)}(s,x)\Delta u^{(k)}_{(s,y^{t,x}(s))} + f_v^{(k+1)}(s,x)\Delta u^{(k+1)}_{(s,y^{t,x}(s))},\\
G(s) &= \Delta g^{(k)}(s) - g_v^{(k)}(s)\Delta u^{(k)}_{(s,0)} + g_v^{(k+1)}(s)\Delta u^{(k+1)}_{(s,0)},
\end{aligned}$$

then we have

$$\begin{aligned}
&||\,\mathbf{1}_{A^{(k)}_{\rho,t}}\,\Delta u^{(k+1)}||^2_{D_t}\\
&\leqslant 2\,\mathbb{E}\left[\mathbf{1}_{A^{(k)}_{\rho,t}} \sup_{(\tilde t,x)\in D_t}\left|\int_0^{\tilde t} F(s,x)ds\right|^2\right] + 2\,\mathbb{E}\left[\mathbf{1}_{A^{(k)}_{\rho,t}} \sup_{\tilde t\leqslant t}\left|\int_0^{\tilde t} G(s)dB_s\right|^2\right]\\
&:= 2I_1 + 2I_2.
\end{aligned}$$

By the fact that for $s \leqslant t$:

$$A^{(k)}_{\rho,t} \subset A^{(k)}_{\rho,s} \;\Rightarrow\; \mathbf{1}_{A^{(k)}_{\rho,t}} = \mathbf{1}_{A^{(k)}_{\rho,t}}\, \mathbf{1}_{A^{(k)}_{\rho,s}}$$

we have

$$I_1 \;\leqslant\; \mathbb{E}\left[\mathbf{1}_{A^{(k)}_{\rho,t}} \sup_{(\tilde t,x)\in D_t} \left| \int_0^{\tilde t} \mathbf{1}_{A^{(k)}_{\rho,s}} F(s,x) ds \right|^2 \right] \leqslant \mathbb{E}\left[\sup_{(\tilde t,x)\in D_t} \left| \int_0^{\tilde t} \mathbf{1}_{A^{(k)}_{\rho,s}} F(s,x) ds \right|^2 \right]$$

Hence by the Schwarz inequality:

$$\begin{aligned}
I_1 \;&\leqslant\; t\,\mathbb{E}\left[\sup_{(\tilde t,x)\in D_t} \int_0^{\tilde t} \mathbf{1}_{A^{(k)}_{\rho,s}} |F(s,x)|^2\, ds \right] \leqslant t\,\mathbb{E}\left[\int_0^{t} \mathbf{1}_{A^{(k)}_{\rho,s}} \sup_{(\tilde s,x)\in D_s} |F(\tilde s,x)|^2\, ds \right] \\
&\leqslant\; 2t\,\mathbb{E}\left[\int_0^{t} \mathbf{1}_{A^{(k)}_{\rho,s}} \sup_{(\tilde s,x)\in D_s} \left| \Delta f^{(k)}(\tilde s,x) - f^{(k)}_v(\tilde s,x)\Delta u^{(k)}_{(\tilde s, y^{t,x}(\tilde s))} \right|^2 ds \right] \\
&+\; 2t\,\mathbb{E}\left[\int_0^{t} \mathbf{1}_{A^{(k)}_{\rho,s}} \sup_{(\tilde s,x)\in D_s} \left| f^{(k+1)}_v(\tilde s,x)\Delta u^{(k+1)}_{(\tilde s, y^{t,x}(\tilde s))} \right|^2 ds \right]
\end{aligned}$$

From the fundamental theorem of calculus and (12) it follows that

$$\begin{aligned}
&\left| \Delta f^{(k)}(s,x) - f^{(k)}_v(s,x)\Delta u^{(k)}_{(s, y^{t,x}(s))} \right| \\
&\quad\leqslant\; \sup_{(\tilde s,x)\in D_s} \left| \Delta u^{(k)}_{(\tilde s, y^{t,x}(\tilde s))} \right| \\
&\qquad\times \int_0^1 \left\| f_v\left(\tilde s, x, u^{(k)}_{(\tilde s, y^{t,x}(\tilde s))} + \theta \Delta u^{(k)}_{(\tilde s, y^{t,x}(\tilde s))}\right) - f_v\left(\tilde s, x, u^{(k)}_{(\tilde s, y^{t,x}(\tilde s))}\right) \right\|_{\mathcal{L}(C(D_0,\mathbb{R}),\mathbb{R})} d\theta \\
&\quad\leqslant\; \frac{1}{2} M \sup_{(\tilde s,x)\in D_s} \left| \Delta u^{(k)}_{(\tilde s, y^{t,x}(\tilde s))} \right|^2 .
\end{aligned}$$

Hence by (9)

$$\begin{aligned}
I_1 \;&\leqslant\; \frac{1}{2} t M^2\, \mathbb{E}\left[\int_0^{t} \mathbf{1}_{A^{(k)}_{\rho,s}} \sup_{(\tilde s,x)\in D_s} \left| \Delta u^{(k)}_{(\tilde s, y^{t,x}(\tilde s))} \right|^4 ds \right] \\
&+\; 2tL^2\, \mathbb{E}\left[\int_0^{t} \mathbf{1}_{A^{(k)}_{\rho,s}} \sup_{(\tilde s,x)\in D_s} \left| \Delta u^{(k)}_{(\tilde s, y^{t,x}(\tilde s))} \right|^2 ds \right].
\end{aligned}$$

Recall that $|\Delta u^{(k)}(\tilde s, x)| \leqslant \rho$ on $A^{(k)}_{\rho,s}$ for $0 \leqslant \tilde s \leqslant s, x \in \mathbb{R}^m$. Thus

$$I_1 \leqslant \frac{1}{2} t^2 M^2 \rho^4 + 2tL^2 \int_0^t \left\| \mathbf{1}_{A^{(k)}_{\rho,s}} \Delta u^{(k+1)} \right\|^2_{D_s} ds.$$

We have

$$\begin{aligned} I_2 &= \mathbb{E}\left[\mathbf{1}_{A^{(k)}_{\rho,t}} \sup_{\tilde t\leqslant t}\left|\int_0^{\tilde t} G(s)dB_s\right|^2\right] \\ &\leqslant \mathbb{E}\left[\mathbf{1}_{A^{(k)}_{\rho,t}} \sup_{\tilde t\leqslant t}\left|\int_0^{\tilde t} \mathbf{1}_{A^{(k)}_{\rho,s}} G(s)dB_s\right|^2\right] \\ &\leqslant \mathbb{E}\left[\sup_{\tilde t\leqslant t}\left|\int_0^{\tilde t} \mathbf{1}_{A^{(k)}_{\rho,s}} G(s)dB_s\right|^2\right] \end{aligned}$$

Applying the Doob martingale inequality and the Itô isometry we obtain

$$I_2 \leqslant 4\,\mathbb{E}\left[\left|\int_0^t \mathbf{1}_{A^{(k)}_{\rho,s}} G(s)dB_s\right|^2\right] = 4\,\mathbb{E}\left[\int_0^t \mathbf{1}_{A^{(k)}_{\rho,s}} G^2(s)ds\right]$$

Hence

$$I_2 \leqslant 2tM^2\rho^4 + 8L^2\int_0^t \left\|\mathbf{1}_{A^{(k)}_{\rho,s}} \Delta u^{(k+1)}\right\|^2_{D_s} ds.$$

We have the estimate

$$\begin{aligned} ||\,\mathbf{1}_{A^{(k)}_{\rho,t}} \Delta u^{(k+1)}||^2_{D_t} &\leqslant 2I_1 + 2I_2 \\ &\leqslant t(T+4)M^2\rho^4 + 4(T+4)L^2\int_0^t \left\|\mathbf{1}_{A^{(k)}_{\rho,s}} \Delta u^{(k+1)}\right\|^2_{D_s} ds. \end{aligned}$$

Applying the Gronwall inequality we obtain

$$||\,\mathbf{1}_{A^{(k)}_{\rho,t}} \Delta u^{(k+1)}||^2_{D_t} \leqslant t(T+4)M^2\rho^4 e^{4T(T+4)L^2}.$$

The Chebyshev inequality yields

$$\begin{aligned} &P\left(\sup_{(s,x)\in D_t}|\Delta u^{(k)}(s,x)|\leqslant\rho \quad\wedge\quad \sup_{(s,x)\in D_t}|\Delta u^{(k+1)}(s,x)|>R\rho^2\right) \\ &= P\left(\mathbf{1}_{A^{(k)}_{\rho,t}}\sup_{(s,x)\in D_t}|\Delta u^{(k+1)}(s,x)|>R\rho^2\right) \leqslant \frac{1}{R^2\rho^4}||\,\mathbf{1}_{A^{(k)}_{\rho,t}} \Delta u^{(k+1)}||^2_{D_t} \\ &\leqslant (T+4)M^2 e^{4T(T+4)L^2} tR^{-2}. \end{aligned}$$

Hence for any $R>0$, $0<\rho\leqslant 1$ we have

$$P\left(\sup_{(s,x)\in D_t}|\Delta u^{(k)}(s,x)|\leqslant\rho \quad\Rightarrow\quad \sup_{(s,x)\in D_t}|\Delta u^{(k+1)}(s,x)|\leqslant R\rho^2\right) \geqslant 1-e^{C(T^2+1)}tR^{-2},$$

which completes the proof. ■

References

[1] K. Amano, *A note on Newton's method for stochastic differential equations and its error estimate*, Proc. Japan Acad., **84** (2008), Ser. A, 1-3.

[2] K. Amano, *Newton's method for stochastic differential equations and its probabilistic second-order error estimate, Electron. J.Differential Equations*, **2012** (2012), No. 03, 1-8.

[3] Y.-Z. Chen, *The generalized quasilinearization method for partial differential equations of first order*, J. Math. Anal. Appl. 197 (1996) 726-734.

[4] T. Człapiński, *On the mixed problem for quasilinear partial differential-functional equations of the first order*, Z. Anal. Anwendungen **16**, No. 2, (1997), 463478.

[5] S. G. Deo, S. Sivasundaram *Extension of the method of quasilinearization to hyperbolic partial differential equations of first order*, Appl. Anal. **59** (1995), no. 1-4, 153162.

[6] L.C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, American Mathematical Society, 2 edition, 2010.

[7] T. Funaki *Construction of a solution of random transport equation with boundary condition*, J. Math. Soc. Japan. **31** (1979), 719-744.

[8] H. Holden, B. Øksendal, J. Ubøe, T. Zhang, *Stochastic partial differential equations. A modeling, white noise functional approach*, Probability and its Applications, Birkhuser Boston.

[9] O. Kallenberg, *Foundations of Modern Probability,* Springer, 1997.

[10] E.A. Kalpinelli, N.E. Frangos, A. N. Yannacopoulos, *A Wiener Chaos Approach to Hyperbolic SPDEs*, Stochastic Analysis and Application, vol.**29**, no.2, (2011), 237-258.

[11] Z. Kamont, *First order partial functional differential equations with state dependent delays*, Nonlinear Studies **12**, 2005, 135-157.

[12] S. Kawabata, and T. Yamada, *On Newton's method for stochastic differential equations*, Seminaire de Probabilites, XXV, Lecture Notes in Math., **1485**, Springer, Berlin, (1991), pp.121-137.

[13] H. Leszczyński, *On CC-solutions to the initial-boundary-value problem for systems of nonlinear first-order partial differential-functional equations*, Rendiconti Mat. **15**, Roma (1995), 173-209.

[14] A.G. McKendrick, *Applications of mathematics to medical problems*, Pro. Edinburgh Math. Soc., 44:98130, 1926

[15] T. M. Mestechkina, *On the solution of the Cauchy problem for a class of first-order stochastic partial differential equations*, Izv. Vyssh. Uchebn. Zaved. Mat., 1991, no. 8, 42-44; translation in SovietMath. (Iz. VUZ) 35 (1991), no. 8, 39-41.

[16] S.E.A. Mohammed, *Stochastic Functional Differential Equations,* Pitman Publishing Program, Boston, 1984.

[17] A.E. Rodkina, *On existence and uniqueness of solution of stochastic differential equation with hereditary*, Stochastics **12** (1984), 187-200.

[18] C. Roth, *Weak approximations of solutions of a first order hyperbolic stochastic partial differential equation*, Monte Carlo Methods Appl. **13**, no. 2, (2007), 117-133.

[19] J. Turo, *Mixed problems for quasilinear hyperbolic systems*, Proceedings of the Second World Congress of Nonlinear Analysis, Part 4 (Athens, 1996). Nonlinear Anal. **30** No. 4, (1997), 23292340.

[20] T.D. Van, M. Tsuji, N.D. Thai Son, *The Characteristic Method and its Generalizations for First-Order Nonlinear Partial Differential Equations*, Boca Raton, London, Chapman & Hall/CRC, 2000.

[21] M. Wrzosek, *Newton's method for stochastic functional differential equations*, Electron. J.Differential Equations, **2012**, No. 130, (2012), 1-10.

Monika Wrzosek
Institute of Mathematics, University of Gdańsk
Wita Stwosza 57, 80-952 Gdańsk
E-mail: mwrzosek@mat.ug.edu.pl

(Received: 16.10.2013)