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ON EULER–VON MANGOLDT'S EQUATION

BY

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The title equation

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0,$$

elementarily equivalent to the prime number theorem, was conjectured by Euler and proved by von Mangoldt. We shall prove here a much more general theorem:

THEOREM. *Let F be an algebraic number field and A an arbitrary non-empty set of prime ideals of the ring of integers $\mathcal{O}_F$. Then*

$$\sum_{\mathbf{a}} \frac{\mu_A(\mathbf{a})}{N(\mathbf{a})} = \prod_{\mathbf{p}\in A} \left(1 - \frac{1}{N(\mathbf{p})}\right), \tag{1}$$

where the sum is extended over all integral ideals of $\mathcal{O}_F$ and μ_A is defined by $\mu_A(\mathbf{a}) = \mu(\mathbf{a})$ if $\mathbf{a}$ is not divisible by any prime ideal outside A and $\mu_A(\mathbf{a}) = 0$ otherwise.

LEMMA 1 (Axer's theorem, [1]). *Let $f(\mathbf{a})$ be an ideal-theoretic function satisfying*

$$\sum_{N(\mathbf{a})\le x} |f(\mathbf{a})| = O(x),$$
$$\sum_{N(\mathbf{a})\le x} f(\mathbf{a}) = o(x).$$

Then

$$\sum_{N(\mathbf{a})\le x} \left(\alpha \frac{x}{N(\mathbf{a})} - \left[\frac{x}{N(\mathbf{a})}\right]_F\right) f(\mathbf{a}) = o(x),$$

where

$$[x]_F = |\{\mathbf{a} \lhd \mathcal{O}_F : N(\mathbf{a}) \le x\}| = \alpha x + O(x^{1-1/k}),$$

$k = (F : \mathbb{Q})$, *and* $\alpha = \alpha_F$.

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Lemma 2 (the simplest sieve). *Let B be a set of prime ideals of $\mathcal{O}_F$ such that*

$$\sum_{\mathbf{p}\in B}\frac{1}{N(\mathbf{p})}=\infty.$$

Then

$$\mathcal{N}_{P\setminus B}(x)=|\{\mathbf{a}\lhd\mathcal{O}_F : N(\mathbf{a})\le x \text{ and } \mathbf{p}|\mathbf{a}\Rightarrow \mathbf{p}\in P\setminus B\}|=o(x).$$

Proof of the Theorem. We shall distinguish three cases.

Case 1: $\sum_{\mathbf{p}\in A}1/N(\mathbf{p})<\infty$. This implies that the product on the right-hand side of (1) is absolutely convergent. Hence (1) holds.

Case 2: $\sum_{\mathbf{p}\in P\setminus A}1/N(\mathbf{p})<\infty$. This case is a classical one; we multiply two convergent Dirichlet series, the second of them is absolutely convergent:

$$\sum_{\mathbf{a}}\frac{\mu(\mathbf{a})}{N(\mathbf{a})}\cdot\sum_{\mathbf{a}\in\mathcal{N}_{P\setminus A}}\frac{1}{N(\mathbf{a})}=\sum_{\mathbf{a}}\frac{\mu_A(\mathbf{a})}{N(\mathbf{a})}.$$

The first one is convergent to 0 by Landau [3] (the special case $F=\mathbb{Q}$ by von Mangoldt [4]) and because the above multiplication is allowed we obtain (1) also in this case.

Case 3: $\sum_{\mathbf{p}\in A}1/N(\mathbf{p})=\infty$ and $\sum_{\mathbf{p}\in P\setminus A}1/N(\mathbf{p})=\infty$. This is the most interesting case. We estimate as follows:

$$\Big|\alpha x\sum_{N(\mathbf{a})\le x}\frac{\mu_A(\mathbf{a})}{N(\mathbf{a})}\Big|\le\Big|\sum_{N(\mathbf{a})\le x}\Big(\alpha\frac{x}{N(\mathbf{a})}-\Big[\frac{x}{N(\mathbf{a})}\Big]_F\Big)\mu_A(\mathbf{a})\Big| + \Big|\sum_{N(\mathbf{a})\le x}\Big[\frac{x}{N(\mathbf{a})}\Big]_F\mu_A(\mathbf{a})\Big|. \tag{2}$$

The first term on the right-hand side is $o(x)$ by Lemma 1, applied to $f(\mathbf{a})=\mu_A(\mathbf{a})$; the assumptions of Lemma 1 are satisfied in view of Lemma 2 and the inequality

$$\sum_{N(\mathbf{a})\le x}|\mu_A(\mathbf{a})|\le\mathcal{N}_A(x).$$

The second term on the right-hand side of (2) is $o(x)$ by the identity

$$\sum_{N(\mathbf{a})\le x}\Big[\frac{x}{N(\mathbf{a})}\Big]_F\mu_A(\mathbf{a})=\sum_{N(\mathbf{c})\le x}\sum_{\mathbf{a}|\mathbf{c}}\mu_A(\mathbf{a})=\mathcal{N}_{P\setminus A}(x)$$

and Lemma 2. Hence

$$\alpha x\sum_{N(\mathbf{a})\le x}\frac{\mu_A(\mathbf{a})}{N(\mathbf{a})}=o(x)+o(x)=o(x)$$

and the proof of the Theorem has been finished. ∎

Remark 1. In the case $F = \mathbb{Q}$ one can dispense with Axer's theorem because of the straightforward estimation

$$\Big|\sum_{n\le x}\Big(\frac{x}{n}-\Big[\frac{x}{n}\Big]\Big)\mu_A(n)\Big| \le \sum_{n\le x}|\mu_A(n)| = o(x)$$

and the proof becomes very easy!

Remark 2. Using a standard sieve result ([2], p. 72, Corollary 2.3.1) and the above method of estimation one can obtain quantitative versions of the Theorem.

PROPOSITION. *Assume that* $A_1 \cup A_2 = P$, $A_1 \cap A_2 = \emptyset$, *and for each* $i = 1, 2$ *there exist* $\delta_i > 0$ *and* $M_i \in \mathbb{R}$ *such that*

$$\sum_{p\le x,\, p\in A_i}\frac{1}{p} \ge \delta_i \log\log x - M_i.$$

Then

$$\sum_{n\le x}\frac{\mu_{A_i}(n)}{n} = O((\log x)^{-\min(\delta_1,\delta_2)}).$$

As a very special corollary we obtain

COROLLARY. *Let* $A_i = \{p \in P : p \equiv i \bmod 4\}$ *for* $i = 1, 3$. *Then*

$$\sum_{n\le x}\frac{\mu_{A_i}(n)}{n} = O\Big(\frac{1}{\sqrt{\log x}}\Big) \quad \text{for } i = 1, 3.$$

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