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MULTIPLIER THEOREM ON GENERALIZED HEISENBERG GROUPS II

BY

WALDEMAR HEBISCH AND JACEK ZIENKIEWICZ (WROCŁAW)

We prove that on a product of generalized Heisenberg groups, a Hörmander type multiplier theorem for Rockland operators is true with the critical index $n/2+\varepsilon$, $\varepsilon>0$, where n is the euclidean (topological) dimension of the group.

1. Introduction. Let L be a positive Rockland operator on a homogeneous group G (cf. [4]) and let d be the homogeneous degree of L (cf. Section 2). Let

$$Lf=\int_0^\infty \lambda\, dE(\lambda)f$$

be its spectral resolution (on $L^2(G)$), and for $m\in L^\infty(\mathbb{R}_+)$ let

$$m(L)f=\int_0^\infty m(\lambda)\, dE(\lambda)f.$$

Conditions on the function m which guarantee boundedness of $m(L)$ on $L^p(G)$, $1<p<\infty$, have a long history. In 1960 L. Hörmander proved that if G is abelian and if for a nonzero $\phi\in C_c^\infty(\mathbb{R}_+)$,

$$\sup_{t>0}\|\phi m(t\cdot)\|_{H(s)}<\infty$$

for an s greater than half the (topological) dimension of G, then $m(L)$ is of weak type 1-1 and bounded on L^p, $1<p<\infty$.

For sublaplaceans on general stratified groups M. Christ [1] and G. Mauceri and S. Meda [15] showed that the Hörmander theorem holds if the topological dimension is replaced by the homogeneous dimension. Recently D. Müller and E. M. Stein [16] showed that if L is the canonical sublaplacean and G is a cartesian product of copies of Heisenberg groups and abelian groups then, in fact, in the Hörmander theorem s greater than half the

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topological dimension suffices. A bit earlier J. Randall [17] obtained estimates for the heat kernel on generalized Heisenberg groups which imply a multiplier theorem with s greater than half the euclidean dimension plus a constant, so if the dimension of the center is large this is less than half the homogeneous dimension.

The present paper should be considered a companion paper to [9]. We extend the result of [9] to Rockland operators and the proof is somewhat simpler.

2. Preliminaries. Let G be a graded nilpotent Lie algebra, that is,

$$G = \bigoplus_{\alpha \geq 1} V_\alpha,$$

and $[V_\alpha, V_\beta] \subset V_{\alpha+\beta}$ for all $\alpha, \beta \geq 1$. We assume that $V_1 \neq \{0\}$.

A *dilation structure* on a graded Lie algebra G is a one-parameter group $\{\delta_t\}_{t>0}$ of automorphisms of G determined by

$$\delta_t X = t^\alpha X \quad \text{ for } X \in V_\alpha.$$

If we consider G as a nilpotent Lie group with multiplication given by the Campbell–Hausdorff formula

$$xy = x + y + \frac{1}{2}[x, y] + \ldots,$$

then $\{\delta_t\}_{t>0}$ forms a group of automorphisms of the group G, and the nilpotent Lie group G equipped with the dilations $\{\delta_t\}_{t>0}$ is said to be a *graded homogeneous group.*

The *homogeneous dimension* of G is the number Q determined by

$$\int_G f(\delta_t x)\, dx = t^{-Q} \int_G f(x)\, dx,$$

where dx is a Haar measure on G. It is evident that

$$Q = \sum_\alpha \alpha \dim V_\alpha.$$

We fix a basis $\mathbf{e}_k$ in G consisting of homogeneous vectors, that is,

$$\delta_t \mathbf{e}_k = t^{\alpha_k} \mathbf{e}_k.$$

Then we define right-invariant vector fields X_k by

$$X_k f(x) = \frac{d}{dt}\bigg|_{t=0} f(\exp(t\mathbf{e}_k)x).$$

If $I = (i_1, \ldots, i_{\dim(G)})$ is a multi-index, then we put

$$X^I = X_1^{i_1} \ldots X_{\dim(G)}^{i_{\dim(G)}}.$$

The number $|I| = \sum_{k=1}^{\dim(G)} \alpha_k i_k$ is called the *homogeneous length* of I and determines the homogeneous degree of the operator X^I.

Given a unitary representation ϱ of G and a right-invariant differential operator L on G we define the image $\varrho(L)$ of L under ϱ by the formula

$$(\varrho(L)f, g) = L(\phi_{f,g})(e), \quad \text{where} \quad \phi_{f,g}(x) = (\varrho(x)f, g).$$

Then $\varrho(L)$ is well defined for $f \in C^\infty(\varrho)$.

A right-invariant differential operator L on G is called a *Rockland operator* if L is homogeneous of some degree $d > 0$, that is,

$$L(f \circ \delta_t) = t^d (Lf) \circ \delta_t \quad \text{for } f \in C^\infty(G),$$

and for every nontrivial irreducible unitary representation π of G the operator $\pi(L)$ is injective on C^∞ vectors.

The operator L satisfies the following subelliptic estimates proved by B. Helffer and J. Nourrigat [11]: for every multi-index I there is a constant C such that

$$\|X^I f\|_{L^2(G)} \le C \|L^{|I|/d} f\|_{L^2(G)}, \quad f \in C_c^\infty(G).$$

The estimate above remains true in any (unitary) representation of G.

For a positive-definite Rockland operator L, Theorem (4.25) of [4] asserts that the closure $-\overline{L}$ of the essentially selfadjoint operator $-L$ is the infinitesimal generator of a semigroup of linear operators on $L^2(G)$ which has the form

$$T_t f = p_t * f, \quad t > 0,$$

where the p_t belong to the Schwartz space $S(G)$.

We fix a *homogeneous norm* on G, that is, a continuous, nonnegative, symmetric function $x \mapsto |x|$ smooth away from 0 which vanishes only for $x = 0$, and satisfies $|\delta_t x| = t|x|$. Henceforth we will assume that our homogeneous norm is subadditive, that is, $|xy| \le |x| + |y|$ (cf. e.g. [10]).

We have

$$Lf = D * f$$

where D is a distribution on G. We write

$$Rf = f * D \quad \text{and} \quad A = \frac{1}{2}(R + L).$$

The one-parameter semigroups generated by $-L$ and $-R$ are given by

$$e^{-tL} f = p_t * f \quad \text{and} \quad e^{-tR} f = f * p_t.$$

In other words, $e^{-tL}\delta_0 = e^{-tR}\delta_0$ and, since L and R commute,

$$e^{-tA}\delta_0 = e^{-tL/2} e^{-tR/2} \delta_0 = e^{-tL}\delta_0.$$

Let $\chi(x) = x^{-1}$ and $L^{\mathrm{T}} f = (L(f \circ \chi)) \circ \chi$. We put

$$\widetilde{G} = G \times G, \quad \widetilde{A} f(x_1, x_2) = \frac{1}{2}((Lf(\cdot, x_2))(x_1) + (L^{\mathrm{T}} f(x_1, \cdot))(x_2)).$$

It is easy to see that if L is a Rockland operator on G, then $\widetilde{A}$ is a Rockland operator on $\widetilde{G}$. We define the action of $\widetilde{G}$ on G by

$$(x_1, x_2)g = x_1 g x_2^{-1}.$$

Then A is the image of $\widetilde{A}$ under this action. Let $\check{X}_j$ be the left invariant vector field such that $\check{X}_j(e) = X_j(e)$. By the Helffer–Nourrigat theorem, we get the following subelliptic inequalities for A:

$$\|X^{I_1}\check{X}^{I_2} f\|_{L^2(G)} \le C_{I_1,I_2}\|A^{(|I_1|+|I_2|)/d} f\|_{L^2(G)} \quad \text{for } f \in C_c^\infty(G).$$

We say that a step two nilpotent Lie algebra G is a *generalized Heisenberg Lie algebra* if there is a scalar product $\langle\cdot,\cdot\rangle$ on G and an orthogonal decomposition

$$G = W \oplus [G, G]$$

such that for each $x \in W$ of length 1 the mapping ad_x^* is an isometry from $[G, G]^*$ into W^*. We call W the *generating subspace* of G. We identify Lie algebras with Lie groups (using the exponential map), and we say that G is a *generalized Heisenberg group* if, as a Lie algebra, it is a generalized Heisenberg Lie algebra. With this identification 0 is the neutral element in our groups.

As a matter of fact, we use only two properties of a generalized Heisenberg group, one that the dimension of its center is at most half the topological dimension of G, second that

$$\sum \langle s, [x, \mathbf{e}_i]\rangle^2 \ge c|s|^2|\pi_W(x)|^2,$$

where π_W is the projection on W, and $s \in [G, G]^*$. In fact, the inequality above becomes an equality if the norms are chosen properly.

In the sequel we assume that $G = \prod G_i$, each G_i being a generalized Heisenberg group with the generating subspace W_i. Let $|x|_i$ be the length of x in W_i (we fix a scalar product). We write $W = \bigoplus W_i$. We may consider G as the direct sum of W_i and $[G_i, G_i]$, so the projection $\pi_i : G \to W_i$ is well defined. Put

$$w_i(x) = |\pi_i(x)|_i.$$

G also has a natural structure of a homogeneous group: elements in W are of degree 1 and elements in $[G, G]$ are of degree 2.

3. Results

(3.1) THEOREM. *If G is a product of generalized Heisenberg groups, L a positive-definite Rockland operator on G, $n = \dim(G)$, $s > n/2$, $\phi \in C_c^\infty(\mathbb{R}_+)$, $\phi \ne 0$, and*

$$\sup_{t>0} \|\phi m(t)\|_{H(s)} < \infty$$

then $m(L)$ is of weak type 1-1 and bounded on L^p, $1 < p < \infty$.

(3.2) THEOREM. *For every* $0 < \alpha_i < \dim([G_i, G_i])$ *there exists* C *such that if* $f \in C_c^\infty(\mathbb{R}_+)$ *and* $\operatorname{supp} f \subset [1/2, 2]$, *then*

$$\int \prod w_i^{\alpha_i} |f(L)|^2(x)\,dx \le C\|f\|_{L^2}^2.$$

Remark. From [12] and [14] we know that $f(L)$ is a well-defined rapidly decaying (Schwartz class) function, so all we have to do is to get the estimate.

First note that since $e^{-tA}\delta_0 = e^{-tL}\delta_0$, also $f(L) = f(L)\delta_0 = f(A)\delta_0$, so we may replace L by A.

Let $\widetilde{\pi}$ be the representation of $\widetilde{G}$ on $L^2(G)$ corresponding to the action of $\widetilde{G}$ on G. Of course, for any $x \in \widetilde{G}$ central translations on G commute with $\widetilde{\pi}(x)$. Hence spectral decomposition of translations from $[G,G]$ (given by the Fourier transform on $[G,G]$) also decomposes $\widetilde{\pi}$. By the Plancherel formula on $[G,G]$, we have

$$\int_G w^2 |f(L)|^2(x)\,dx = C \int_{[G,G]^*} \|wf(A_s)\delta_0\|_{L^2(W)}^2\,ds,$$

where A_s is the Fourier transform of A in $[G,G]$ directions (note that coefficients of A are independent of the central coordinates).

(3.3) LEMMA. *There exists* C *such that for all* $s \in [G,G]^*$ *and all* $f \in C_c^\infty(\mathbb{R}_+)$ *with* $\operatorname{supp} f \subset [1/2, 2]$ *we have*

$$\int_1^2 \|f(A_{ts})\delta_0\|_{L^2(W)}^2\,dt \le C\|f\|_{L^2(W)}^2.$$

We define D_t, for $t > 0$, by

$$(D_t\phi)(x) = t^{-\dim(W)}\phi(t^{-1}x) \quad \text{for } \phi \in L^1(W)$$

and extend it by continuity to measures. One easily checks that

$$D_{t^{-1/2}}(f(A_{ts})\delta_0) = f(D_{t^{-1/2}} A_s D_{t^{1/2}}) D_{t^{-1/2}}\delta_0 = f(tA_s)\delta_0.$$

We have $\|D_t\phi\|_{L^2(W)} = t^{-\dim(W)/2}\|\phi\|_{L^2(W)}$, so

$$\int_1^2 \|f(A_{ts})\delta_0\|_{L^2(W)}^2\,dt \le \int_1^2 \|D_{t^{-1/2}}(f(A_{ts})\delta_0)\|_{L^2(W)}^2\,dt$$
$$= \int_1^2 \|f(tA_s)\delta_0\|_{L^2(W)}^2\,dt.$$

For $E_s(\lambda)$ being the spectral measure of A_s we write $d\mu(\lambda) = d(E_s(\lambda)e^{-A_s}\delta_0, e^{-A_s}\delta_0)$. Note that

$$\|e^{-A_s}\delta_0\|_{L^2(W)} \le \|e^{-A_s}\|_{L^1(W),L^2(W)} = \|e^{-A_s}\|_{L^2(W),L^\infty(W)} \le C.$$

The last inequality follows from a subelliptic estimate (uniform in s) and the Sobolev embedding. We have

$$\int_1^2 \|f(tA_s)\delta_0\|^2_{L^2(W)}\,dt \le \int_1^2 \int |f(t\lambda)|^2 e^{2\lambda}\,d\mu(\lambda)\,dt$$
$$\le C\|f\|^2_{L^2(W)} \int d\mu \le C\|f\|^2_{L^2(W)},$$

which gives (3.3).

For step two nilpotent G, the Campbell–Hausdorff formula takes the form

$$xy = x + y + \frac{1}{2}[x,y],$$

hence

$$X_i = \partial_{\mathbf{e}_i} + \frac{1}{2}\partial_{[x,\mathbf{e}_i]}, \qquad \check{X}_i = \partial_{\mathbf{e}_i} - \frac{1}{2}\partial_{[x,\mathbf{e}_i]}$$

and

$$\sum_i (X_i - \check{X}_i)^2 = \frac{1}{4}\sum_i \partial^2_{[x,\mathbf{e}_i]}.$$

Since $\widetilde{A}$ is a Rockland operator and the Fourier transform on $[G,G]$ decomposes the natural representation of $\widetilde{G}$ on $L^2(G)$ we obtain

$$\Big\| \Big(\sum \langle s, [\cdot,\mathbf{e}_i]\rangle^2\Big)^{\beta/2} f\Big\|_{L^2} \le \|A_s^{1/d} f\|_{L^2}.$$

Put

$$\|s\| = \max|s_i|, \qquad s^{(\alpha)} = \prod |s_i|^{\alpha_i}.$$

Consequently,

$$s^{(\alpha)}\|wf(A_s)\delta_0\|^2_{L^2} \le C\|A_s^{|\alpha|/d} f(A_s)\delta_0\|^2_{L^2} \le C'\|f(A_s)\delta_0\|^2_{L^2}.$$

Also, if s is large enough, then $A_s \ge 2$. Therefore if $\|s\| \ge C$, then $f(A_s) = 0$. We need a version of polar coordinates: there exist measures η_k such that for all positive Borel measurable ϕ we have

$$\int_{[G,G]^*} \phi = C\sum_k 2^k \int_{\|s\|=2^k} \int_1^2 t^{\dim([G,G])-1}\phi(ts)\,dt\,d\eta_k(s).$$

Using these observations and (3.3) we have

$$C\int_{[G,G]^*} \|wf(A_s)\delta_0\|^2\,ds \le C\int_{\|s\|\le C} s^{(-\alpha)}\|f(A_s)\delta_0\|^2\,ds$$
$$\le C\sum_{k=k_0}^{\infty} 2^{-k} \int_{\|s\|=2^{-k}} \int_1^2 s^{(-\alpha)}\|f(A_{ts})\delta_0\|^2\,dt\,d\eta_k(s)$$

$$\leq C\|f\|_{L^2}^2 \sum_{k=k_0}^{\infty} 2^{-k} \int\limits_{\|s\|=2^{-k}} \int\limits_1^2 s^{(-\alpha)}\, dt\, d\eta_k(s)$$
$$\leq C\|f\|_{L^2}^2 \int\limits_{\|s\|\leq C} s^{(-\alpha)}\, ds \leq C\|f\|_{L^2}^2,$$

which ends the proof of (3.2).

From (3.2) we get (3.1) by a (by now) standard argument (see for example [9]).

Remark. The method presented here allows us to improve the multiplier theorems of [1] and [15]. Namely, for a large class of homogeneous G (for example all G with one-dimensional center) the multiplier theorem holds if $s > (Q-1)/2$.

Remark. Using the methods of [3] together with our argument we can prove an analog of (3.2) for regular nondifferential Rockland operators (cf. [6]).

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INSTITUTE OF MATHEMATICS
UNIVERSITY OF WROCŁAW
PL. GRUNWALDZKI 2/4
50-384 WROCŁAW, POLAND

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