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A NOTE ON A CONJECTURE OF JEŚMANOWICZ

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Abstract. Let a, b, c be relatively prime positive integers such that $a^2+b^2=c^2$. Jeśmanowicz conjectured in 1956 that for any given positive integer n the only solution of $(an)^x+(bn)^y=(cn)^z$ in positive integers is $x=y=z=2$. If $n=1$, then, equivalently, the equation $(u^2-v^2)^x+(2uv)^y=(u^2+v^2)^z$, for integers $u>v>0$, has only the solution $x=y=z=2$. We prove that this is the case when one of u, v has no prime factor of the form $4l+1$ and certain congruence and inequality conditions on u, v are satisfied.

1. Introduction. Let a, b, c be relatively prime positive integers such that $a^2+b^2=c^2$, and let n be a positive integer. Then the Diophantine equation

$$(na)^x+(nb)^y=(nc)^z \tag{1}$$

has solution $x=y=z=2$. Jeśmanowicz [4] conjectured in 1956 that there are no other solutions of (1). Building on the work of Dem'yanenko [2], we proved in [3] that the conjecture is true when $n>1$, $c=b+1$ and certain further divisibility conditions are satisfied.

If $n=1$, (1) is equivalent to

$$(u^2-v^2)^x+(2uv)^y=(u^2+v^2)^z, \tag{2}$$

where u, v are integers such that $u>v>0$, $\gcd(u,v)=1$, and one of u, v is even, the other odd. A number of special cases of Jeśmanowicz's conjecture have been settled. Sierpiński [8] and Jeśmanowicz [4] proved it for $(u,v)=(2,1)$ and $(u,v)=(3,2)$, $(4,3)$, $(5,4)$ and $(6,5)$, respectively. Lu [7] proved it when $v=1$, and Dem'yanenko [2] when $v=u-1$. Takakuwa [9] proved the conjecture in a number of special cases in which, in particular, $v\equiv 1 \pmod 4$, and, in [10], when u is exactly divisible by 2 and $v=3$, 7, 11 or 15. Le [6] proved it when uv is exactly divisible by 2, $v\equiv 3 \pmod 4$ and $u\ge 81v$. Chao Ko [5] and Jingrun Chen [1] proved the conjecture when uv has no prime factor of the form $4l+1$ and certain congruence and inequality conditions on u, v are satisfied.

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In this note, we shall prove that the conjecture is true if one of u, v has no prime factor of the form $4l+1$, and certain congruence and inequality conditions on u, v are satisfied.

2. Main results

THEOREM 1. *Suppose u is even with no prime factor of the form $4l+1$, $u > v > 0$ and $\gcd(u,v) = 1$. Write $u = 2m$ and suppose also that one of the following is true*:

(i) *$m \equiv 1 \pmod 2$, $v \equiv 1 \pmod 4$, $u^2 - v^2$ has a prime factor of the form $8l+5$ or $u - v$ has a prime factor of the form $8l+3$;*

(ii) *$m \equiv 1 \pmod 2$, $v \equiv 3 \pmod 4$, $u + v$ has a prime factor of the form $4l+3$;*

(iii) *$m \equiv 2 \pmod 4$, $v \equiv 3, 7 \pmod 8$;*

(iv) *$m \equiv 2 \pmod 4$, $v \equiv 5 \pmod 8$, $u + v$ has a prime factor of the form $8l+7$;*

(v) *$m \equiv 2 \pmod 4$, $v \equiv 1 \pmod 8$, $u + v$ has a prime factor of the form $4l+3$;*

(vi) *$m \equiv 0 \pmod 4$, $v \equiv 1 \pmod 8$, $u + v$ has a prime factor of the form $4l+3$, $u^2 - v^2$ has a prime factor of the form $8l+5$ or $u - v$ has a prime factor of the form $8l+3$;*

(vii) *$m \equiv 0 \pmod 4$, $v \equiv 3, 5 \pmod 8$;*

(viii) *$m \equiv 0 \pmod 4$, $v \equiv 7 \pmod 8$, $u^2 - v^2$ has a prime factor of the form $8l+3$ or $8l+5$.*

Then the Diophantine equation (2) *has no positive integer solution other than $x = y = z = 2$.*

Proof. Modulo 4, (2) becomes $(-1)^x \equiv 1$, so x is even. We now show that z is also even, and that, except perhaps in case (ii), y is even.

The following simple congruences are required:

$$(3)\qquad \begin{aligned} 2uv &\equiv 2v^2 \pmod{u-v}, & u^2+v^2 &\equiv 2v^2 \pmod{u-v},\\ 2uv &\equiv -2v^2 \pmod{u+v}, & u^2+v^2 &\equiv 2v^2 \pmod{u+v}. \end{aligned}$$

In case (i), we have $2m + v \equiv 3 \pmod 4$, so $u + v$ has either a prime factor, p say, of the form $8l+3$, or a prime factor, q say, of the form $8l+7$ (or both). In the former case, from (2) and (3),

$$(-2v^2)^y \equiv (2v^2)^z \pmod p,$$

and it follows that

$$\begin{aligned} 1 &= \left(\frac{-2}{p}\right)^y = \left(\frac{-2v^2}{p}\right)^y = \left(\frac{(-2v^2)^y}{p}\right)\\ &= \left(\frac{(2v^2)^z}{p}\right) = \left(\frac{2v^2}{p}\right)^z = \left(\frac{2}{p}\right)^z = (-1)^z, \end{aligned}$$

where $\left(\frac{\cdot}{\cdot}\right)$ is Legendre's symbol. So z is even. In the latter case, we find in the same way that y is even.

If $u^2 - v^2$ has a prime factor of the form $8l + 5$, or $u - v$ has a prime factor of the form $8l + 3$, then, again in the same way, we find that $y \equiv z \pmod 2$. Then y and z are even in case (i), and we may similarly obtain the same conclusion in cases (vi) and (viii).

In case (ii), since $u + v$ has a prime factor of the form $8l + 3$ or $8l + 7$, we find as above that z is even or y is even. If y is even, then $y > 1$, and, recalling that x is even, from (2) we have $5^z \equiv 1 \pmod 8$. It follows that, in case (ii), z must be even.

Consider case (iii). If $v \equiv 3 \pmod 8$, then $u+v \equiv 7 \pmod 8$. From (2) and (3), we have $(-2v^2)^y \equiv (2v^2)^z \pmod{u+v}$, so that

$$(-1)^y = \left(\frac{-2v^2}{u+v}\right)^y = \left(\frac{2v^2}{u+v}\right)^z = 1,$$

where $\left(\frac{\cdot}{\cdot}\right)$ is Jacobi's symbol. Then y is even. From (2), $1 \equiv 9^z \pmod{16}$, which implies z is even. If $v \equiv 7 \pmod 8$, then, considering (2) modulo $u+v$ and $u - v$, respectively, we may similarly show that y and z are even. This also follows in a similar fashion in cases (iv), (v) and (vii).

In all cases except one, we have now shown that y and z are both even. The exception is case (ii), in which we know only that z is even. We show now that y must be even in this case as well.

Write $x = 2x_1$ and $z = 2z_1$. Then, from (2),

$$(4mv)^y = ((4m^2 + v^2)^{z_1} + (4m^2 - v^2)^{x_1})((4m^2 + v^2)^{z_1} - (4m^2 - v^2)^{x_1}).$$

If x_1 is even, then $(4m^2 + v^2)^{z_1} + (4m^2 - v^2)^{x_1} \equiv 2 \pmod 4$. Let p be an odd prime factor of m, so that, by hypothesis, $p \equiv 3 \pmod 4$. Since $\gcd(m, v) = 1$, and since -1 is a quadratic nonresidue of p, we have

$$(4m^2 + v^2)^{z_1} + (4m^2 - v^2)^{x_1} \equiv v^{2z_1} + v^{2x_1} \not\equiv 0 \pmod p, \tag{4}$$

It follows that

$$(4m^2 + v^2)^{z_1} + (4m^2 - v^2)^{x_1} = 2v_1^y, \tag{5}$$

$$(4m^2 + v^2)^{z_1} - (4m^2 - v^2)^{x_1} = 2^{2y-1}m^y v_2^y, \tag{6}$$

where $v = v_1 v_2$. We will show that $v_2 > 1$. In case (ii), $v \equiv 3 \pmod 4$, so v has a prime factor $q \equiv 3 \pmod 4$ and, as in (4),

$$(4m^2 + v^2)^{z_1} + (4m^2 - v^2)^{x_1} \equiv (2m)^{2z_1} + (2m)^{2x_1} \not\equiv 0 \pmod q.$$

In fact, this implies that $v_1 \equiv 1 \pmod 4$ and $v_2 \equiv 3 \pmod 4$. Since $v_2 > 1$, we now have $2^{2y-1}m^y v_2^y > (2m)^y > v^y > 2v_1^y$, whence (5) and (6) cannot both hold. Hence x_1 is odd.

We then have, as above,

$$(4m^2 + v^2)^{z_1} + (4m^2 - v^2)^{x_1} = 2^{2y-1}m^y v_3^y,$$

$$(4m^2+v^2)^{z_1}-(4m^2-v^2)^{x_1}=2v_4^y,$$

where $v=v_3v_4$, so that

$$(4m^2+v^2)^{z_1}=2^{2y-2}m^yv_3^y+v_4^y, \tag{7}$$

$$(4m^2-v^2)^{x_1}=2^{2y-2}m^yv_3^y-v_4^y. \tag{8}$$

From (7), $\gcd(v_3,v_4)=1$, $y>1$ and $v^{2z_1}\equiv v_4^y \pmod 4$. But, in case (ii), as shown above for v_2, we have $v_4\equiv 3 \pmod 4$, so $1\equiv 3^y \pmod 4$, and it follows that y is even, as required.

We now complete the proof of Theorem 1.

Notice first that x_1 must be odd. To confirm this, consider again the passage above in which it was assumed that x_1 is even. Then, since $y\geq 2$, it follows that $2^{2y-1}m^yv_2^y\geq 2^{y-1}(2m)^y>2^{y-1}v^y\geq 2v_1^y$, so, again, (5) and (6) cannot both hold. With x_1 odd, we may refer again to (7) and (8).

Write $y=2y_1$. From (8),

$$(4m^2-v^2)^{x_1}=(2^{2y_1-1}m^{y_1}v_3^{y_1}+v_4^{y_1})(2^{2y_1-1}m^{y_1}v_3^{y_1}-v_4^{y_1}).$$

Since $\gcd(v_3,v_4)=1$, the factors on the right are relatively prime. Let $2^{2y_1-1}m^{y_1}v_3^{y_1}+v_4^{y_1}=s^{x_1}$ and $2^{2y_1-1}m^{y_1}v_3^{y_1}-v_4^{y_1}=t^{x_1}$. Then

$$st=4m^2-v^2,\qquad \gcd(s,t)=1,\ s\geq t+2. \tag{9}$$

We have

$$s^{x_1}+t^{x_1}=2^{y_1}(2m)^{y_1}v_3^{y_1}>2^{y_1}v^{y_1}v_3^{y_1}=2^{y_1}v_3^{2y_1}v_4^{y_1}=2^{y_1-1}v_3^{2y_1}(s^{x_1}-t^{x_1}),$$

from which

$$\begin{aligned}(2^{y_1-1}v_3^{2y_1}+1)t^{x_1}&>(2^{y_1-1}v_3^{2y_1}-1)s^{x_1}\geq(2^{y_1-1}v_3^{2y_1}-1)(t+2)^{x_1}\\&\geq(2^{y_1-1}v_3^{2y_1}-1)t^{x_1}+2(2^{y_1-1}v_3^{2y_1}-1)x_1t^{x_1-1}.\end{aligned}$$

It follows that

$$t>(2^{y_1-1}v_3^{2y_1}-1)x_1\geq 2^{y_1-1}v_3^{2y_1}-1. \tag{10}$$

But, from (8), we have

$$\begin{aligned}0&\equiv(4m^2-v^2)^{x_1}=2^{y-2}(2m)^yv_3^y-v_4^y=2^{2(y_1-1)}(4m^2)^{y_1}v_3^{2y_1}-v_4^{2y_1}\\&\equiv 2^{2(y_1-1)}v^{2y_1}v_3^{2y_1}-v_4^{2y_1}\pmod{4m^2-v^2},\end{aligned}$$

so that $v_4^y(2^{2(y_1-1)}v_3^{2y}-1)\equiv 0 \pmod{st}$, by (9). Since $\gcd(v_4,st)=1$, we have $2^{2(y_1-1)}v_3^{2y}-1\equiv 0 \pmod{st}$. If $v_3>1$ or $y_1>1$, then the left-hand side is positive, and we must have $t^2<st\leq 2^{2(y_1-1)}v_3^{2y}-1$, so that $t\leq 2^{y_1-1}v_3^{2y_1}-1$, contradicting (10).

Hence $v_3=y_1=1$, and, from (7), $x_1=z_1=1$. Thus $x=y=z=2$, completing the proof of Theorem 1.

Theorem 2. *Suppose u is even, $25v>2u>2v>0$, $\gcd(u,v)=1$ and v has no prime factor of the form $4l+1$. Write $u=2m$ and suppose also* *that one of conditions* (i)–(viii) *in Theorem* 1 *is true. Then the Diophantine equation* (2) *has no positive integer solution other than* $x = y = z = 2$.

Proof. When one of conditions (i)–(viii) in the statement of Theorem 1 is satisfied, we may show, as in the proof of Theorem 1, that x and z are even, and, except in case (ii), y is even. We show first that y is even in this case as well. Let $x = 2x_1$ and $z = 2z_1$. In much the same way as before, we may show that x_1 is odd and

$$(4m^2 + v^2)^{z_1} + (4m^2 - v^2)^{x_1} = 2^{2y-1}m_1^y,$$
$$(4m^2 + v^2)^{z_1} - (4m^2 - v^2)^{x_1} = 2m_2^y v^y,$$

where $m = m_1 m_2$ and $m_2 \equiv 1 \pmod 4$. We have

$$(4m^2 + v^2)^{z_1} = 2^{2y-2}m_1^y + m_2^y v^y, \tag{11}$$
$$(4m^2 - v^2)^{x_1} = 2^{2y-2}m_1^y - m_2^y v^y. \tag{12}$$

From (11), $y > 1$ so that, in case (ii), $1 \equiv 3^y \pmod 4$. Hence y is even.

Let $y = 2y_1$. From (12),

$$(4m^2 - v^2)^{x_1} = (2^{2y_1-1}m_1^{y_1} + m_2^{y_1}v^{y_1})(2^{2y_1-1}m_1^{y_1} - m_2^{y_1}v^{y_1}).$$

As in the corresponding part of the proof of Theorem 1, we may put

$$2^{2y_1-1}m_1^{y_1} + m_2^{y_1}v^{y_1} = s^{x_1} \quad \text{and} \quad 2^{2y_1-1}m_1^{y_1} - m_2^{y_1}v^{y_1} = t^{x_1},$$

so that

$$st = 4m^2 - v^2, \quad \gcd(s,t) = 1, \; s \ge t + 2 \tag{13}$$

and

$$s^{x_1} + t^{x_1} = 2^{2y_1}m_1^{y_1}, \quad s^{x_1} - t^{x_1} = 2m_2^{y_1}v^{y_1}. \tag{14}$$

If $m_2 \ne 1$, then $m_2 \ge 5$. From (14), $(4m_1)^{y_1} > 2(m_2 v)^{y_1}$, so $4m_1 > m_2 v$. Then $4m > m_2^2 v \ge 25v$, contradicting the hypothesis that $2u < 25v$. Thus, $m_2 = 1$, and if $y_1 > 1$ then we may use (13) and (14) to obtain a contradiction, much as in the closing part of the proof of Theorem 1, by showing both $t \ge 2^{y_1-1}$ and $t < 2^{y_1-1}$.

Hence $m_2 = y_1 = 1$, and it follows from (11) and (12) that $x_1 = z_1 = 1$. Therefore, $x = y = z = 2$, completing the proof of Theorem 2.

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