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ON A CONJECTURE OF MĄKOWSKI AND SCHINZEL CONCERNING THE COMPOSITION OF THE ARITHMETIC FUNCTIONS σ AND ϕ

BY

A. GRYTCZUK (ZIELONA GÓRA), F. LUCA (BIELEFELD AND PRAHA), AND M. WÓJTOWICZ (ZIELONA GÓRA)

Abstract. For any positive integer n let $\phi(n)$ and $\sigma(n)$ be the Euler function of n and the sum of divisors of n, respectively. In [5], Mąkowski and Schinzel conjectured that the inequality $\sigma(\phi(n)) \ge n/2$ holds for all positive integers n. We show that the lower density of the set of positive integers satisfying the above inequality is at least 0.74.

1. Introduction. For any positive integer k let $\phi(k)$ and $\sigma(k)$ be the Euler totient function and the divisor sum of k, respectively.

In 1964, A. Mąkowski and A. Schinzel [5] proved the following relations concerning ϕ and σ:

$$\begin{aligned} &\liminf \frac{\sigma(\sigma(n))}{n} = 1, \quad \limsup \frac{\phi(\sigma(n))}{n} = \infty, \\ &\limsup \frac{\phi(\phi(n))}{n} = \frac{1}{2}, \quad \liminf \frac{\sigma(\phi(n))}{n} \le \inf_{4\mid n} \frac{\sigma(\phi(n))}{n} \le \frac{1}{2} + \frac{1}{2^{34}-1}. \end{aligned} \tag{1}$$

They noted that K. Kuhn checked that

$$\frac{\sigma(\phi(n))}{n} \ge \frac{1}{2} \tag{2}$$

holds for all positive integers n having at most six prime factors, and that in this case equality in (2) occurs only for $n = 2(2^{2^k} - 1)$ and $0 \le k \le 5$. Accordingly, they asked if inequality (2) holds for all positive integers n. In the same paper, they pointed out that it was not even known whether

$$\liminf \frac{\sigma(\phi(n))}{n} > 0 \tag{3}$$

but C. Pomerance has since then proved inequality (3) by using Brun's method [7].

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In 1992, M. Filaseta, S. W. Graham and C. Nicol [3] verified (2) for the positive integers n which are the product of the first k primes for sufficiently large values of k. In 1994, U. Balakrishnan [1] proved that (2) holds for all squarefull positive integers n. Recently, G. L. Cohen [2] checked that (2) holds for various classes of positive integers n including:

1°. Any positive integer n of the form $2^a m$ where:

(i) the distinct prime factors of m are either Fermat primes or primes $p \equiv 1 \pmod 3$, with at most eight of the latter;

(ii) m is a product of primes of the form $2^b r + 1$ with $b \geq 1$ and r prime.

2°. Any positive integer n which is a product of primes less than 1780.

Moreover, G. L. Cohen and R. Gupta proved independently that (2) holds for all positive integers n provided that it holds for all squarefree integers n. More precisely, they proved the following

Cohen–Gupta Theorem [2]. *We have* $\sigma(\phi(n))/n \geq \sigma(\phi(n'))/n'$, *where* n' *is the squarefree part of* n.

For various other results concerning this inequality as well as related inequalities the reader should also consult [4] and [6].

In this paper, we give some evidence that (2) may hold for all positive integers n, by showing in Theorem 2 that the lower density of the set of all integers satisfying (2) is greater than 0.74. Theorem 2 claims, intuitively, that at least 74% of all positive integers n satisfy inequality (2). Of course, this is a very weak result when compared with the Mąkowski–Schinzel Conjecture, but at least it offers some evidence that the conjecture might be true.

2. The results. Let

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}, \tag{4}$$

where $2 \leq p_1 < \ldots < p_r$ are distinct primes and $\alpha_i \geq 1$ for $i = 1, \ldots, r$. Let $\Omega(n)$ denote the set $\{p_1, \ldots, p_r\}$ and put

$$\omega(n) = r \ (= \operatorname{card} \Omega(n)), \tag{5}$$

$$f(n) = \operatorname{ord}_2(p_1 - 1) + \ldots + \operatorname{ord}_2(p_r - 1) + 1, \tag{6}$$

$$P(n) = 4\left(1 - \frac{1}{2^{f(n)}}\right)\left(1 + \frac{1}{p_r}\right)\left(1 - \frac{1}{p_1}\right)\ldots\left(1 - \frac{1}{p_{r-1}}\right). \tag{7}$$

Notice that $f(n) \geq \omega(n)+1$ for n odd and $f(n) \geq \omega(n)$ for n even. Our first result is:

Theorem 1. *Let* n *be a positive integer. If*

$$P(n) \geq 1, \tag{8}$$

then n *satisfies inequality* (2). *In particular*, *inequality* (2) *holds for all* n *odd with* $\omega(n) \leq 21$ *and for all even* n *with* $f(n) \leq 5$.

By using the above theorem, we prove the following:

Theorem 2. *Let* ϱ *be the lower density of the set of positive integers* n *satisfying* (2). *Then* $\varrho > .74$.

3. The proof of Theorem 1. Let $n' = \prod_{i=1}^{r} p_i$ be the squarefree part of n. Notice that $P(n) = P(n')$. By the Cohen–Gupta Theorem, it suffices to prove Theorem 1 for n'. Thus, we may assume that n is squarefree. By Cohen's result included in 1°(i), we may also assume that $\phi(n)$ is not a power of 2. Write

$$\phi(n) = \prod_{i=1}^{r}(p_i - 1) = 2^{f(n)-1} \prod_{k=1}^{t} q_k^{\beta_k}, \tag{9}$$

where $t \geq 1$, $q_1 < \ldots < q_t$ are odd primes and $\beta_k \geq 1$ for $k = 1, \ldots, t$. We now get

$$\begin{aligned} \frac{\sigma(\phi(n))}{n} &= \frac{\sigma(\phi(n))}{\phi(n)} \cdot \frac{\phi(n)}{n} = \frac{2^{f(n)} - 1}{2^{f(n)-1}} \prod_{k=1}^{t} \frac{q_k^{\beta_k+1} - 1}{q_k^{\beta_k}(q_k - 1)} \cdot \prod_{i=1}^{r}\Big(1 - \frac{1}{p_i}\Big) \\ &= 2\Big(1 - \frac{1}{2^{f(n)}}\Big)\Big(1 - \frac{1}{p_r}\Big) \prod_{k=1}^{t} \frac{q_k^{\beta_k+1} - 1}{q_k^{\beta_k}(q_k - 1)} \cdot \prod_{i=1}^{r-1}\Big(1 - \frac{1}{p_i}\Big). \end{aligned} \tag{10}$$

Since $q_1 \leq (p_r - 1)/2$, we get

$$\begin{aligned} \Big(1 - \frac{1}{p_r}\Big) \prod_{k=1}^{t} \frac{q_k^{\beta_k+1} - 1}{q_k^{\beta_k}(q_k - 1)} &\geq \Big(1 - \frac{1}{p_r}\Big)\Big(1 + \frac{1}{q_1}\Big) \\ &\geq \Big(1 - \frac{1}{p_r}\Big)\Big(1 + \frac{2}{p_r - 1}\Big) = 1 + \frac{1}{p_r}. \end{aligned} \tag{11}$$

From inequalities (10) and (11), we have

$$\frac{\sigma(\phi(n))}{n} \geq 2\Big(1 - \frac{1}{2^{f(n}}\Big)\Big(1 + \frac{1}{p_r}\Big) \prod_{i=1}^{r-1}\Big(1 - \frac{1}{p_i}\Big) = \frac{P(n)}{2}.$$

The first assertion of Theorem 1 is now obvious.

Assume now that n is odd and $\omega(n) \leq 21$. Notice that $f(n) \geq r + 1 = \omega(n) + 1$ in this case. By analyzing all cases, it follows easily that the infimum over all $P(n)$ for n odd and $\omega(n) \leq 21$ is at least 1.008; hence, $P(n) > 1$ for such n's.

Assume now that n is even and $f(n) \leq 5$. It follows that n is divisible by at most 4 odd primes. By analyzing all cases, it follows that the infimum of all $P(n)$ for n even and $f(n) \leq 5$ is at least 1.006; hence, $P(n) > 1$ for all such n's. ■

4. The proof of Theorem 2. Let $n = 2^s p_1^{\alpha_1} \ldots p_r^{\alpha_r}$, where $s \geq 0$, $2 < p_1 < \ldots < p_r$ are distinct odd primes and $\alpha_i \geq 1$ for $i = 1, \ldots, r$. Define

$$E(n) = \prod_{i=1}^{r-1} \left(1 - \frac{1}{p_i}\right)$$

and

$$F(n) = \begin{cases} \left(4\left(1 - \frac{1}{2^{f(n)}}\right)\left(1 + \frac{1}{p_r}\right)\right)^{-1} & \text{if } n \text{ is odd,} \\ \left(2\left(1 - \frac{1}{2^{f(n)}}\right)\left(1 + \frac{1}{p_r}\right)\right)^{-1} & \text{if } n \text{ is even.} \end{cases} \tag{12}$$

Notice that

$$P(n) = \frac{E(n)}{F(n)}. \tag{13}$$

Let

$$c_0 = 2\left(1 - \frac{1}{2^6}\right), \tag{14}$$

$$c_1 = 4\left(1 - \frac{1}{2^{23}}\right). \tag{15}$$

Notice that by Theorem 1, if a positive integer n does not satisfy inequality (2), then $E(n) < F(n)$. Moreover, by Theorem 1 again, if n does not satisfy inequality (2), then either n is odd and $\omega(n) \geq 22$, or n is even and $f(n) \geq 6$. These arguments, combined with the fact that $f(n) \geq \omega(n) + 1$ when n is odd, show that if n does not satisfy inequality (2) and $n \equiv i \pmod 2$, then

$$E(n) < c_i^{-1} \qquad \text{for } i \in \{0, 1\}. \tag{16}$$

Let x be an arbitrary positive number. For any $s \geq 0$, let

$$A_s(x) = \{n < x \mid 2^s \,\|\, n\}, \tag{17}$$

$$B_s(x) = \{n \in A_s(x) \mid n \text{ does not satisfy (2)}\}. \tag{18}$$

Notice that $A_s(x)$ (hence, $B_s(x)$ too) is empty when $s > \log_2(x)$. We use inequality (16) to bound the cardinality $b(x, s)$ of $B_s(x)$ in terms of s and x.

Set

$$T(x, s) = \prod_{n \in A_s(x)} E(n).$$

Since

$$E(n) > \prod_{i=1}^{r}\left(1 - \frac{1}{p_i}\right),$$

and since from $n \in A_s(x)$ we get $n/2^s < x/2^s$, it follows that

$$T(x,s) \ge \prod_{3\le p<x}\left(1 - \frac{1}{p}\right)^{\frac{1}{2}(\frac{x}{2^s p}+1)}.$$

On the other hand, $T(x,s) < c_i^{-b_2(x,s)}$, where $i = 1$ if $s = 0$ (i.e., for the odd values of n) and $i = 0$ if $s > 0$ (i.e., for the even values of n). Hence,

$$b(x,s)\log c_i < \frac{1}{2}\cdot\frac{x}{2^s}\left(S_0 - \frac{\log 2}{2}\right) + \frac{1}{2}S_1 \tag{19}$$

where

$$S_0 = \sum_{p\ge 2}\frac{1}{p}\log\left(1 + \frac{1}{p-1}\right) < .58006, \qquad S_1 = \sum_{3\le p\le x}\log\left(1 + \frac{1}{p-1}\right).$$

Since

$$\log\left(1 + \frac{1}{p-1}\right) < \frac{1}{p-1}, \qquad \sum_{3\le p\le x}\frac{1}{p-1} = O(\log\log x),$$

it follows that

$$b(x,s)\log c_i < \frac{x}{2}\cdot\frac{1}{2^s}\left(S_0 - \frac{\log 2}{2}\right) + C\log\log x \tag{20}$$

where C is a constant.

When $s = 0$, we get

$$\frac{b(x,0)}{x} < \frac{1}{2\log c_1}\left(S_0 - \frac{\log 2}{2}\right) + o(x). \tag{21}$$

For $s \ge 1$, we sum up inequalities (20) for all $s \le \log_2(x)$ and use the fact that $\sum_{s\ge 1} 1/2^s = 1$ to get

$$\frac{1}{x}\sum_{s\ge 1} b(x,s) < \frac{1}{2\log c_0}\left(S_0 - \frac{\log 2}{2}\right) + o(x). \tag{22}$$

Now let $b(x) = \sum_{s\ge 0} b(x,s)$. From formulae (21) and (22), we have

$$\limsup_{x\to\infty}\frac{b(x)}{x} \le \frac{1}{2}\left(\frac{1}{\log c_0} + \frac{1}{\log c_1}\right)\left(S_0 - \frac{\log 2}{2}\right) < .25655.$$

This implies that $\varrho > 1 - .25655 > .74$. ∎

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Institute of Mathematics
T. Kotarbiński Pedagogical University
65-069 Zielona Góra, Poland
E-mail: agryt@lord.wsp.zgora.pl
mwojt@lord.wsp.zgora.pl

Fachbereich Mathematik
Universität Bielefeld
33501 Bielefeld, Germany
E-mail: fluca@mathematik.uni-bielefeld.de

Mathematical Institute
Czech Academy of Sciences
Žitná 25
115 67 Praha 1, Czech Republic
E-mail: luca@matsrv.math.cas.cz

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