COLLOQUIUM MATHEMATICUM

VOL. 83 2000 NO. 1

QUASITILTED ALGEBRAS HAVE PREPROJECTIVE COMPONENTS

BY

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Abstract. We show that a quasitilted algebra has a preprojective component. This is proved by giving an algorithmic criterion for the existence of preprojective components.

1. Introduction. This paper provides an extension of work by Coelho–Happel [3]. They showed that if Λ is a quasitilted k-algebra with k an algebraically closed field, then the Auslander–Reiten quiver of Λ contains a preprojective component. As the main result we show here that this is true in general. That is, let k be any field, and assume that Λ is a quasitilted k-algebra. Then the Auslander–Reiten quiver of Λ contains a preprojective component. Unlike Coelho–Happel we make no assumption in our proof that our algebras are quasitilted but not tilted algebras. Hence we obtain an independent proof of the fact that the Auslander–Reiten quiver of a tilted algebra contains a preprojective component, which was proved by Strauss [11].

Let R be a commutative Artin ring. All our algebras are R-algebras, and finitely generated as R-modules. We assume that R acts centrally on any bimodule. For an algebra Λ we denote by $\operatorname{mod}\Lambda$ the category of finitely generated left Λ-modules, and by $\operatorname{ind}\Lambda$ the full subcategory of $\operatorname{mod}\Lambda$ consisting of indecomposable modules. Let M be a Λ-module. We denote by $\operatorname{pd}_\Lambda M$ the projective dimension of M, by $\operatorname{id}_\Lambda M$ the injective dimension of M, and by $\operatorname{gl.dim}\Lambda$ the global dimension of Λ. The Auslander–Reiten quiver of Λ is denoted by Γ_Λ. The vertices of Γ_Λ are in one-to-one correspondence with the isomorphism classes of indecomposable finitely generated Λ-modules. There is an arrow from an indecomposable module X to an indecomposable module Y if and only if there is an irreducible morphism from X to Y. The arrow has valuation (a,b) if there is a minimal right almost split morphism $aX \oplus V \to Y$, where X is not a direct summand of V, and a minimal left almost split morphism $X \to bY \oplus W$, where Y is not a direct summand of W. A connected component $\mathcal{P}$ of Γ_Λ is called a *preprojective component* if $\mathcal{P}$

2000 *Mathematics Subject Classification*: 16E10, 16E30, 16G10, 16G20, 16G70.

does not contain an oriented cycle, and each $X \in \mathcal{P}$ is of the form $(\mathrm{TrD})^i P$ for some $i \in \mathbb{N}$ and an indecomposable projective module P.

The next section provides the necessary background for quasitilted algebras. In Section 3 we generalize a result of Dräxler–de la Peña [5], giving an algorithmic criterion for the existence of preprojective components. In Section 4 we prove that each quasitilted algebra has a preprojective component. The main idea of the proof is to investigate the conditions on a Λ-module M, where Λ is quasitilted, such that the triangular matrix algebra $\left(\begin{smallmatrix} F & 0 \\ M & \Lambda \end{smallmatrix}\right)$ is quasitilted, where $F \subseteq \mathrm{End}_\Lambda(M)^{\mathrm{op}}$ is a division algebra. For general background on Artin algebras we refer to Auslander–Reiten–Smalø [1].

I thank Professor Sverre O. Smalø for his advice and helpful suggestions during the preparation of this paper.

2. Preliminaries. In this section we recall some basic facts on quasitilted algebras, and give some results which we need later. For basic reference on quasitilted algebras we refer to Happel–Reiten–Smalø [7].

A *path* from an indecomposable module X_0 to an indecomposable module X_t in $\mathrm{mod}\,\Lambda$ is a sequence of morphisms $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \ldots \xrightarrow{f_{t-2}} X_{t-1} \xrightarrow{f_{t-1}} X_t$ in $\mathrm{ind}\,\Lambda$, where $t \geq 1$ and each f_i is nonzero and not an isomorphism. We say that such a path has *length* t. If there is a path from an indecomposable module M to an indecomposable module N, or $N \simeq M$, we denote this by $M \rightsquigarrow N$ and say that M is a *predecessor* of N, and that N is a *successor* of M. We say that M *lies on a cycle* if there is a path from M to M, and the number of morphisms in the path is called the *length* of the cycle. If the length of the cycle is 1 or 2, we say the path is a *short cycle*. We say that a path $Z_0 \xrightarrow{f_0} Z_1 \xrightarrow{f_1} \ldots \xrightarrow{f_{t-1}} Z_t \xrightarrow{f_t} Z_{t+1}$ of irreducible morphisms is *sectional* if $Z_i \not\simeq \mathrm{DTr}\, Z_{i+2}$ for $1 \leq i \leq t-1$. Let

$$(*) \qquad M \xrightarrow{f_0} M_1 \xrightarrow{f_1} \ldots \xrightarrow{f_{t-1}} M_t \xrightarrow{f_t} N$$

be a path in $\mathrm{ind}\,\Lambda$. A path $M \to M_{0,1} \to \ldots \to M_{0,n_0} \to M_1 \to M_{1,1} \to \ldots \to M_{1,n_1} \to M_2 \to \ldots \to M_t \to M_{t,1} \to \ldots \to N$ is called a *refinement* of $(*)$, and it is called a *refinement of irreducible morphisms* if all the morphisms in the refinement are irreducible. Further, a *walk* is a sequence of indecomposable modules X_0—X_1—X_2—…—X_{t-1}—X_t, where X_i—X_{i+1} means that there is either a nonzero morphism $X_i \to X_{i+1}$ or a nonzero morphism $X_{i+1} \to X_i$ for all $1 \leq i \leq t-1$. The number of morphisms in a walk is called the *length* of the walk.

Let R be a commutative Artin ring. An algebra Λ is called a *quasitilted algebra* if there exists a locally finite hereditary abelian R-category $\mathcal{H}$ and a tilting object $T \in \mathcal{H}$ such that $\Lambda = \mathrm{End}_{\mathcal{H}}(T)^{\mathrm{op}}$. According to Happel–Reiten–Smalø [7] the ordinary valued quiver of a quasitilted algebra Λ contains no oriented cycles and therefore the center of Λ is a field. Hence there is no harm to consider just finite-dimensional algebras over a field k when dealing with quasitilted algebras. In this paper we use the following homological characterization of quasitilted algebras given in [7].

THEOREM 1. *The following are equivalent for an algebra* Λ:

(1) Λ *is quasitilted.*
(2) Λ *satisfies the following two conditions*:

(a) $\mathrm{gl.dim}\,\Lambda \le 2$.
(b) *If* X *is a finitely generated indecomposable* Λ*-module*, *then either* $\mathrm{pd}_\Lambda X \le 1$ *or* $\mathrm{id}_\Lambda X \le 1$.

Let Λ be an Artin algebra. The following two subclasses of $\mathrm{ind}\,\Lambda$ are of interest to us. Let $\mathcal{L}_\Lambda$ denote the subclass of $\mathrm{ind}\,\Lambda$ given by $\mathcal{L}_\Lambda = \{X \in \mathrm{ind}\,\Gamma \mid \mathrm{pd}_\Gamma Y \le 1$ for all Y with $Y \rightsquigarrow X\}$ and let $\mathcal{R}_\Lambda$ denote the subclass of $\mathrm{ind}\,\Lambda$ given by $\mathcal{R}_\Lambda = \{X \in \mathrm{ind}\,\Gamma \mid \mathrm{id}_\Gamma Y \le 1$ for all Y with $X \rightsquigarrow Y\}$. Using this we have the following characterization of quasitilted algebras [7, Theorem II.1.14].

THEOREM 2. *The following are equivalent for an Artin algebra* Λ:

(1) Λ *is quasitilted.*
(2) $\mathcal{R}_\Lambda$ *contains all injective modules in* $\mathrm{ind}\,\Lambda$.
(3) $\mathcal{L}_\Lambda$ *contains all projective modules in* $\mathrm{ind}\,\Lambda$.
(4) *Any path in* $\mathrm{mod}\,\Lambda$ *starting in an injective module and ending in a projective module has a refinement of irreducible morphisms and any such refinement is sectional.*

The proof of the following result is essentially due to Happel–Reiten–Smalø [8, Lemma 1.2].

LEMMA 3. *Let* Λ *be a quasitilted algebra and* $M \xrightarrow{f_0} X_1 \xrightarrow{f_1} \ldots \xrightarrow{f_{t-1}} X_t \xrightarrow{f_t} N$ *a path. If* M *belongs to* $\mathcal{R}_\Lambda$ *or if* N *belongs to* $\mathcal{L}_\Lambda$, *then there exist an indecomposable module* Z *and nonzero morphisms* $M \to Z$ *and* $Z \to N$. *In particular*, *an indecomposable* Λ*-module* M *belongs to a cycle if and only if it belongs to a short cycle.*

P r o o f. We only give the proof when M is in $\mathcal{R}_\Lambda$. The proof for the case of N in $\mathcal{L}_\Lambda$ is dual.

Assume that M belongs to $\mathcal{R}_\Lambda$. The proof is by induction on the length of the path. If the length is 1 or 2, then there is nothing to show.

So assume that we have shown the assertion for all paths of length less than $t+1$, and let the path be $M \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \ldots \xrightarrow{f_{t-1}} X_t \xrightarrow{f_t} N$, with $t \ge 2$. We can choose our path $M \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3$ so that $l(X_1)+l(X_2)$, the sum of the lengths of X_1 and X_2, is minimal among the paths with three morphisms connecting M to X_3. We can also assume that all compositions $f_i f_{i-1}$ are 0, since otherwise we would have a shorter path. In particular, $f_1 f_0 = 0$ and $f_2 f_1 = 0$. Thus $\operatorname{Im} f_0 \subseteq \operatorname{Ker} f_1 = K$. We show that K is indecomposable.

Assume that K is decomposable, say $K = K_1 \oplus K_2$, with K_1 indecomposable and K_2 nonzero. We may also assume that $p_1 f_0 \neq 0$, where $p_1 : K \to K_1$ is the projection according to the given decomposition. We have the exact sequence $0 \to K \to X_1 \xrightarrow{\widehat{f_1}} \operatorname{Im} f_1 \to 0$, where $\widehat{f_1}$ is the induced morphism. Consider the pushout diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, "p_1"] & X_1 \arrow[r, "\widehat{f_1}"] \arrow[d] & \operatorname{Im} f_1 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K_1 \arrow[r, "f"] & Y \arrow[r, "g"] & \operatorname{Im} f_1 \arrow[r] & 0 \end{tikzcd}$$

Since X_1 is indecomposable and $\operatorname{Im} f_1 \neq 0$, it follows that K_1 cannot be a summand of X_1. Hence the sequence $0 \to K_1 \xrightarrow{f} Y \xrightarrow{g} \operatorname{Im} f_1 \to 0$ does not split. Since $f : K_1 \to Y$ is a monomorphism, there is a decomposition $Y = Y_1 \oplus Y_2$, with Y_1 indecomposable and such that $q_1 f p_1 f_0 : M \to Y_1$ is nonzero, where $q_1 : Y \to Y_1$ is the projection onto Y_1 according to the given decomposition of Y, and where we have also denoted the induced morphism $M \to K$ by f_0. Now the sequence $0 \to K_1 \xrightarrow{f} Y \xrightarrow{g} \operatorname{Im} f_1 \to 0$ does not split, so $g(Y_1) \neq 0$. Hence we have a path $M \to Y_1 \to X_2 \to X_3$ with $l(Y_1) < l(X_1)$. This contradicts the choice of the path $M \to X_1 \to X_2 \to X_3$. We conclude that K is indecomposable, and hence $\operatorname{id}_\Lambda K \leq 1$, since K is a successor of $M \in \mathcal{R}_\Lambda$.

Since $\operatorname{id}_\Lambda K \leq 1$ we have an exact sequence

$$\operatorname{Ext}^1_\Lambda(C, K) \to \operatorname{Ext}^1_\Lambda(C, X_1) \to \operatorname{Ext}^1_\Lambda(C, \operatorname{Im} f_1) \to 0$$

for any C in $\operatorname{mod} \Lambda$. Consider the exact sequence $0 \to \operatorname{Im} f_1 \xrightarrow{h} X_2 \xrightarrow{t} C \to 0$, with $C = \operatorname{Coker} f_1$. The exact sequence of Ext-groups above gives rise to a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & X_1 \arrow[r, "f'"] \arrow[d, "u"] & W \arrow[r, "g'"] \arrow[d, "v"] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \operatorname{Im} f_1 \arrow[r, "h"] & X_2 \arrow[r, "t"] & C \arrow[r] & 0 \end{tikzcd}$$

with exact rows. Let $W = \bigoplus_{i=1}^s W_i$ be a decomposition of W into a direct sum of indecomposable modules. Let $q_i : W_i \to W$ and $p_i : W \to W_i$ denote the corresponding inclusions and projections for $i = 1, \ldots, s$. The sequence $0 \to \operatorname{Im} f_1 \xrightarrow{h} X_2 \xrightarrow{t} C \to 0$ does not split, since X_2 is indecomposable and

$\operatorname{Im} f_1 \neq 0$ and $C = \operatorname{Coker} f_1 \neq 0$. Hence the sequence $0 \to X_1 \xrightarrow{f'} W \xrightarrow{g'} C \to 0$ does not split. Since X_1 is indecomposable, $p_i f' : X_1 \to W_i$ is not an isomorphism for any i. The diagram above gives rise to an exact sequence

$$(*) \qquad 0 \to X_1 \xrightarrow{\binom{f'}{-\widehat{f}_1}} W \oplus \operatorname{Im} f_1 \xrightarrow{(v,h)} X_2 \to 0.$$

Since $f_1 f_0 = 0$, the morphism $f_1 : X_1 \to X_2$ is not a monomorphism. Hence $\widehat{f_1} : X_1 \to \operatorname{Im} f_1$ is a proper epimorphism, and thus not a split monomorphism. Since $f : X_1 \to W$ is also not a split monomorphism and X_1 is indecomposable, it follows that $(*)$ does not split. Since in addition X_2 is indecomposable, the morphisms $vq_i : W_i \to X_2$ are nonzero nonisomorphisms for any i. Since $f' : X_1 \to W$ is a monomorphism and $f_0 : M \to X_1$ is nonzero, there is some i with $p_i f' f_0 : M \to W_i$ nonzero. Further, since $v : W \to X_2$ is an epimorphism and $f_2 : X_2 \to X_3$ is nonzero, there is some j with $f_2 v q_j : W_j \to X_3$ nonzero. If $i = j$, then we have a path $M \to W_i \to X_3$. If $i \neq j$, then consider the paths $M \to X_1 \to W_j \to X_3$ and $M \to W_i \to X_2 \to X_3$. We have $l(X_1) + l(W_i) + l(W_j) + l(X_2) < 2(l(X_1) + l(X_2))$ by using the exact sequence $0 \to X_1 \to W \oplus \operatorname{Im} f_1 \to X_2 \to 0$. Hence we have $l(X_1) + l(W_j) < l(X_1) + l(X_2)$ or $l(W_i) + l(X_2) < l(X_1) + l(X_2)$, which contradicts our choice of the path $M \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3$.

Therefore we have a path $M \to W_i \to X_3 \to \ldots \to X_t \to N$ of length less than $t + 1$, and we are done by the induction hypothesis. ■

It was shown by Happel–Reiten–Smalø [7] that a nonsemisimple quasitilted algebra Λ is always of the form $\Lambda = \left(\begin{smallmatrix} F & 0 \\ M & A \end{smallmatrix}\right)$ where A is a quasitilted algebra, M an A-module and $F \subseteq \operatorname{End}_A(M)^{\mathrm{op}}$ a division algebra. We now recall some results which will be needed later.

LEMMA 4. *Let A be an Artin algebra, let M be a finitely generated A-module with $F \subseteq \operatorname{End}_A(M)^{\mathrm{op}}$ a division algebra and let $\Lambda = \left(\begin{smallmatrix} F & 0 \\ M & A \end{smallmatrix}\right)$. Then* $\operatorname{gl.dim} \Lambda \le 2$ *if and only if* $\operatorname{gl.dim} A \le 2$ *and* $\operatorname{pd}_A M \le 1$.

Proof. See [1, Proposition III.2.7]. ■

LEMMA 5. *Let A be an Artin algebra with* $\operatorname{gl.dim} A \le 2$, *and let $\Lambda = \left(\begin{smallmatrix} F & 0 \\ M & A \end{smallmatrix}\right)$ for an A-module M and $F \subseteq \operatorname{End}_A(M)^{\mathrm{op}}$ a division algebra. Let (V, X, f) be in* $\operatorname{mod} \Lambda$. *Then*:

(i) *If* $\operatorname{Ker} f$ *is not projective, then* $\operatorname{pd}_\Lambda(V, X, f) \ge 2$.

(ii) *Assume that* $\operatorname{pd}_A \operatorname{Coker} f \le 1$. *Then* $\operatorname{pd}_\Lambda(V, X, f) \le 1$ *if and only if* $\operatorname{Ker} f$ *is projective.*

(iii) $\operatorname{id}_\Lambda(V, X, f) \le 1$ *if and only if* $\operatorname{id}_A X \le 1$ *and* $\operatorname{Ext}^1_A(M, X) = 0$.

Proof. See [7, Lemma III.2.1, 2.2]. ■

PROPOSITION 6. *Let A be an Artin algebra, and let $\Lambda = \left(\begin{smallmatrix} F & 0 \\ M & A \end{smallmatrix}\right)$ for an A-module M and $F \subseteq \operatorname{End}_A(M)^{\mathrm{op}}$ a division algebra. If Λ is quasitilted, then so is A.*

Proof. See [7, Proposition III.2.3]. ∎

The proof of the next result is a slight modification of the proof given in [7, Proposition III.2.4].

PROPOSITION 7. *Let A be an Artin algebra, and let $\Lambda = \left(\begin{smallmatrix} F & 0 \\ M & A \end{smallmatrix}\right)$ for an A-module M and $F \subseteq \operatorname{End}_A(M)^{\mathrm{op}}$ a division algebra. If Λ is quasitilted, then M is in $\operatorname{add} \mathcal{L}_A$.*

The next result is a generalization of a result by Coelho–Happel [3, Lemma 1.4].

LEMMA 8. *Let A be an Artin algebra. Let $M = M_1 \oplus M_2$ be an A-module with $M_1 \neq 0 \neq M_2$, and let $F \subseteq \operatorname{End}_A(M)^{\mathrm{op}}$ be a division algebra. Let Λ be the triangular matrix algebra $\Lambda = \left(\begin{smallmatrix} F & 0 \\ M & A \end{smallmatrix}\right)$. Let X_1 and X_2 be two indecomposable nonisomorphic A-modules and let $f_i : M_i \to X_i$ be nonzero morphisms for $i = 1, 2$. Then the Λ-module $\left(F, X_1 \oplus X_2, \left(\begin{smallmatrix} f_1 & 0 \\ 0 & f_2 \end{smallmatrix}\right)\right)$ is indecomposable.*

Proof. Let $f = \left(\begin{smallmatrix} f_1 & 0 \\ 0 & f_2 \end{smallmatrix}\right)$. If $(F, X_1 \oplus X_2, f)$ is decomposable, then there exists an i such that $(0, X_i, 0)$ is isomorphic to a direct summand of $(F, X_1 \oplus X_2, f)$. We may assume that $i = 1$. Then there exists a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & M_1 \oplus M_2 & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ X_1 & \xrightarrow{g} & X_1 \oplus X_2 & \xrightarrow{h} & X_1 \end{array}$$

with $hg = \mathrm{id}_{X_1}$. Writing $g = \left(\begin{smallmatrix} g_1 \\ g_2 \end{smallmatrix}\right)$ and $h = (h_1, h_2)$, we obtain $h_1 f_1 = 0 = h_2 f_2$ and $h_1 g_1 + h_2 g_2 = \mathrm{id}_{X_1}$. Since X_1 is indecomposable and $X_1 \not\simeq X_2$, we see that $h_2 g_2$ is nilpotent. Thus $h_1 g_1 = \mathrm{id}_{X_1} - h_2 g_2$ is invertible. In particular, h_1 is invertible, and therefore $f_1 = 0$, a contradiction. We conclude that $(F, X_1 \oplus X_2, f)$ is indecomposable. ∎

We have the following direct observation [3, Lemma 1.5].

LEMMA 9. *Let A be an Artin algebra. Let M be an A-module, and let $F \subseteq \operatorname{End}_A(M)^{\mathrm{op}}$ be a division algebra. Let $\Lambda = \left(\begin{smallmatrix} F & 0 \\ M & A \end{smallmatrix}\right)$. Let X be an indecomposable A-module and let $f : M \to X$ be a nonzero morphism. Then the Λ-module (F, X, f) is indecomposable.*

Let Λ be an algebra. Let M be a Λ-module, not necessarily indecomposable. Following Happel–Ringel [9] we say that M is *nondirecting* if there exist indecomposable direct summands M_1 and M_2 of M and an indecomposable nonprojective module W such that $M_1 \rightsquigarrow \operatorname{DTr} W$ and $W \rightsquigarrow M_2$.

Otherwise we say that M is *directing*. A path $M \rightsquigarrow \mathrm{DTr}\, W \rightsquigarrow W \rightsquigarrow N$ is called a *hook path*.

The next result is due to Enge–Slungård–Smalø [6, Theorem 9].

THEOREM 10. *Let Λ be an algebra and let M be a decomposable Λ-module. Let $G \subseteq \mathrm{End}_\Lambda(M)^{\mathrm{op}}$ be a local subalgebra of $\mathrm{End}_\Lambda(M)^{\mathrm{op}}$. If the triangular matrix algebra $\Gamma = \left(\begin{smallmatrix} G & 0 \\ M & \Lambda \end{smallmatrix}\right)$ is quasitilted, then G is a division algebra, and M is either directing or of the form $M = M_1 \amalg P$ where M_1 is indecomposable nondirecting, P is hereditary projective and the only hook paths from M to M are the ones both starting and ending in M_1.*

Next we consider subsets of preprojective components which we will need later. Let Λ be an algebra, and let $\mathcal{P}$ be a preprojective component in the Auslander–Reiten quiver of Λ. Let J be the direct sum of one copy of each indecomposable injective module lying in $\mathcal{P}$. Let $\mathcal{P}(J\rightsquigarrow) = \{X \in \mathcal{P} \mid I \rightsquigarrow X$ for some indecomposable direct summand I of $J\}$. Moreover, for $N \in \mathcal{P}$, let $\mathcal{P}(N\rightsquigarrow) = \{X \in \mathcal{P} \mid N \rightsquigarrow X\}$.

A *walk* between different DTr-orbits $\{(\mathrm{TrD})^i X\}_{i\in\mathbb{Z}}$ and $\{(\mathrm{TrD})^i Y\}_{i\in\mathbb{Z}}$ in $\mathcal{P}$ is a walk $M—Z_1—\ldots—Z_t = N$ of irreducible morphisms with $M \in \{(\mathrm{TrD})^i X\}_{i\in\mathbb{Z}}$ and $N \in \{(\mathrm{TrD})^i Y\}_{i\in\mathbb{Z}}$. The *distance* between two DTr-orbits $\{(\mathrm{TrD})^i X\}_{i\in\mathbb{Z}}$ and $\{(\mathrm{TrD})^i Y\}_{i\in\mathbb{Z}}$ in $\mathcal{P}$ is the minimal length of walks $M—Z_1—\ldots—Z_t = N$ with $M \in \{(\mathrm{TrD})^i X\}_{i\in\mathbb{Z}}$ and $N \in \{(\mathrm{TrD})^i Y\}_{i\in\mathbb{Z}}$.

Using this we obtain the following result.

LEMMA 11. *Let Λ be an algebra, and let $\mathcal{P}$ be a preprojective component in the Auslander–Reiten quiver of Λ. Then*:

(a) *If $\mathcal{P}$ contains some injective module, then $\mathcal{P} \setminus \mathcal{P}(J\rightsquigarrow)$ is finite.*

(b) *If $\mathcal{P}$ contains no injective module, then $\mathcal{P} \setminus \mathcal{P}(N\rightsquigarrow)$ is finite for any $N \in \mathcal{P}$.*

Proof. We prove (a). The proof of (b) is similar.

It suffices to show that every DTr-orbit in $\mathcal{P}$ has an element in $\mathcal{P}(J\rightsquigarrow)$. Let $\mathcal{D}$ be the set of DTr-orbits in $\mathcal{P}$ with no element in $\mathcal{P}(J\rightsquigarrow)$. Note that any DTr-orbit in $\mathcal{D}$ is infinite. Assume that $\mathcal{D}$ is nonempty. Since $\mathcal{P}$ is connected there is a walk between any DTr-orbit in $\mathcal{D}$ and any DTr-orbit not in $\mathcal{D}$. Take the minimal distance between DTr-orbits in $\mathcal{D}$ and DTr-orbits not in $\mathcal{D}$. Again, since $\mathcal{P}$ is connected, this minimal distance has to be one. Thus there is a DTr-orbit $\{(\mathrm{TrD})^i X\}_{i\in\mathbb{Z}} \in \mathcal{D}$ and a DTr-orbit $\{(\mathrm{TrD})^i Y\}_{i\in\mathbb{Z}} \notin \mathcal{D}$ with $M \in \{(\mathrm{TrD})^i X\}_{i\in\mathbb{Z}}$ and $N \in \{(\mathrm{TrD})^i Y\}_{i\in\mathbb{Z}}$ such that there is a walk $M—N$. We may assume $N \in \mathcal{P}(J\rightsquigarrow)$, otherwise we consider $(\mathrm{TrD})^i N$ for some $i \geq 1$. Then we have an irreducible morphism $M \to N$, and $\mathrm{TrD}\, M = 0$, since $\{(\mathrm{TrD})^i M\}_{i\in\mathbb{Z}} \in \mathcal{D}$. Hence M is injective, which gives the desired contradiction. ■

This gives the immediate result.

LEMMA 12. *Let A be an indecomposable quasitilted algebra, and let P be an indecomposable projective A-module which is not contained in a preprojective component of Γ_A. Then no preprojective component of Γ_A contains an injective A-module.*

Proof. Since A is indecomposable there exists an indecomposable projective A-module P' contained in a preprojective component $\mathcal{P}$ of Γ_A and such that there exists a nonzero morphism $f : P' \to P$ with P not in a preprojective component. By the choice of P and P', we have $f \in \mathrm{rad}_A^{\infty}(P', P)$. For all nonzero $f : P' \to P$ and $m \in \mathbb{N}$, there is a direct sum of modules of the form $(\mathrm{TrD})^j X_i \in \mathcal{P}$ with $j \geq m$ such that f factors through $\bigoplus_i (\mathrm{TrD})^j X_i$. Thus for each m there exist $j \geq m$ and i such that $\mathrm{Hom}_A((\mathrm{TrD})^j X_i, P) \neq 0$.

Assume $\mathcal{P}$ contains some injective modules. Let J be the direct sum of one copy of each indecomposable injective module in $\mathcal{P}$. By Lemma 11 the subset $\mathcal{P} \setminus \mathcal{P}(J\rightsquigarrow)$ is finite, where $\mathcal{P}(J\rightsquigarrow) = \{X \in \mathcal{P} \mid I \rightsquigarrow X$ for some indecomposable direct summand I of $J\}$. Hence there exists $m \in \mathbb{N}$ with $I \rightsquigarrow (\mathrm{TrD})^j X_i$ for $j \geq m$. So we obtain a path $I \rightsquigarrow (\mathrm{TrD})^{j-1} X_i \to E \to (\mathrm{TrD})^j X_i \to P$ for some j, where I is an indecomposable direct summand of J. By Theorem 2, A is not quasitilted, which is a contradiction. ∎

3. Existence of preprojective components. In this section we give necessary and sufficient conditions for the existence of preprojective components in the Auslander–Reiten quiver of an Artin algebra. This result is essentially proved by Dräxler–de la Peña [5]. Here we repeat the arguments with the necessary modifications for the general case.

Recall that with any Artin algebra A we may associate a valued quiver Q, that is, a quiver with at most one arrow from a vertex i to a vertex j, and an ordered pair of positive integers assigned to each arrow. The vertices of Q are the isomorphism classes $[S]$ of simple A-modules. There is an arrow from $[S_i]$ to $[S_j]$ if $\mathrm{Ext}_A^1(S_i, S_j) \neq 0$, and we assign to this arrow the pair of integers $(\dim_{\mathrm{End}_A(S_j)} \mathrm{Ext}_A^1(S_i, S_j), \dim_{\mathrm{End}_A(S_i)^{\mathrm{op}}} \mathrm{Ext}_A^1(S_i, S_j))$. Let A be an Artin algebra such that Q has no oriented cycles. For a vertex $c \in Q$ we denote by S_c the corresponding simple A-module, and by P_c the projective cover of S_c. We consider a partial order on the vertices of Q by defining $a \preccurlyeq b$ if there is a path from a to b in Q. Note that this implies that there is a path from P_b to P_a in $\mathrm{mod}\, A$. Given any A-module N, we define the *support algebra* of N as the factor algebra of A modulo the ideal generated by all idempotents that annihilate N. Let x be a vertex in Q. We denote by A^x the support algebra of $\bigoplus_{a \not\preccurlyeq x} S_a$. The indecomposable projective A-module P_x has radical $\mathrm{rad}\, P_x$ which is an A^x-module. Let $\mathrm{rad}\, P_x = \bigoplus_{i=1}^{n_x} R_x(i)$ be its decomposition into indecomposable summands.

The next result is due to Happel–Ringel [9] and Skowroński–Wenderlich [10].

THEOREM 13. *Let x be a vertex in Q. Then P_x is directing in* $\operatorname{mod} A$ *if and only if* $\operatorname{rad} P_x$ *is directing in* $\operatorname{mod} A$. *Moreover*, *if x is a source*, *then P_x is directing in* $\operatorname{mod} A$ *if and only if* $\operatorname{rad} P_x$ *is directing in* $\operatorname{mod} A^x$.

The next result gives an algorithmic criterion for the existence of preprojective components.

THEOREM 14. *Let A be an Artin algebra such that the valued quiver Q of A has no oriented cycles. Then the Auslander–Reiten quiver Γ_A of A has a preprojective component if and only if for each vertex $x \in Q$ one of the following conditions is satisfied*:

(1) *There is a preprojective component $\mathcal{P}$ of Γ_{A^x} such that $R_x(i) \notin \mathcal{P}$ for each $i \in \{1, \ldots, n_x\}$.*

(2) *For each $i \in \{1, \ldots, n_x\}$ the set of predecessors $\{Y \in \Gamma_{A^x} \mid Y \rightsquigarrow R_x(i)\}$ of $R_x(i)$ in* $\operatorname{mod} A^x$ *is finite and formed by directing modules. Moreover*, *if x is a source*, *then* $\operatorname{rad} P_x$ *is directing in* $\operatorname{mod} A^x$.

Proof. Assume first that $\mathcal{P}$ is a preprojective component of Γ_A. Let x be a vertex in Q. If the projective module P_x belongs to $\mathcal{P}$, condition (2) holds for x. So assume $P_x \notin \mathcal{P}$. We show that $\mathcal{P}$ is formed by A^x-modules. Let $X \in \mathcal{P}$, and assume that $\operatorname{Hom}_A(P_y, X) \neq 0$ for a vertex $y \preccurlyeq x$. Then $P_x \rightsquigarrow P_y \rightsquigarrow X$ in $\operatorname{mod} A$, thus $P_x \in \mathcal{P}$, which contradicts our assumption. We conclude that $\mathcal{P}$ is a preprojective component of Γ_{A^x} and $R_x(i) \notin \mathcal{P}$ for every $1 \leq i \leq n_x$. Thus condition (1) is satisfied for the vertex x.

In order to prove the converse we first assume that for all vertices $x \in Q$ condition (2) is satisfied. We then claim that for every $x \in Q$ the following holds:

(3) For each $i \in \{1, \ldots, n_x\}$ the set of predecessors $\{X \in \Gamma_A \mid X \rightsquigarrow R_x(i)\}$ of $R_x(i)$ in $\operatorname{mod} A$ is finite and formed by directing modules.

Indeed, let X be a predecessor of $R_x(i)$ in Γ_A and assume that X is not an A^x-module. Now there is a vertex y with $y \preccurlyeq x$ such that $\operatorname{Hom}_A(P_y, X) \neq 0$. In $\operatorname{mod} A$ we then get $P_y \rightsquigarrow X \rightsquigarrow R_x(i) \rightsquigarrow P_x \rightsquigarrow P_y$. By assumption $\operatorname{rad} P_y$ is directing in $\operatorname{mod} A^y$. Thus by Theorem 13 we see that y is not a source in Q since P_y is not directing in $\operatorname{mod} A$. Let z be a source which is a proper predecessor of y in Q. We see that P_y is a nondirecting predecessor of some indecomposable direct summand of $\operatorname{rad} P_z$. By assumption, condition (2) is satisfied for a vertex z, so some of the modules M in the path $P_y \rightsquigarrow P_y$ are not A^z-modules. Hence $\operatorname{Hom}_A(P_z, M) \neq 0$, and P_z is not directing in $\operatorname{mod} A$, a contradiction to Theorem 13.

Following Bongartz [2] we can then construct inductively full subquivers C_n of Γ_A satisfying

(i) C_n is finite, connected, contains no oriented cycle and is closed under predecessors,

(ii) $\mathrm{TrD}\, C_n \cup C_n \subseteq C_{n+1}$.

Then $\bigcup_{n\in\mathbb{N}} C_n$ forms the desired preprojective component. Let $C_1 = \{S\}$, where S is a simple projective A-module. To get C_{n+1} from C_n number the modules $M_1, \ldots, M_t$ of C_n with $\mathrm{TrD}\, M_i \notin C_n$ in such a way that $i < j$ provided that $M_i \rightsquigarrow M_j$. If $t = 0$, we let $C_{n+1} = C_n$, and we have obtained a finite preprojective component.

Otherwise, let $D_0 = C_n$, and for each $0 \le i \le t-1$ let D_{i+1} be the full subquiver of Γ_A with vertices those in D_i and all predecessors of $\mathrm{TrD}\, M_{i+1}$. Consider the almost split sequence $0 \to M_{i+1} \to X \to \mathrm{TrD}\, M_{i+1} \to 0$, $0 \le i \le t-1$. We show that each indecomposable summand Y of X has only finitely many predecessors and does not lie on a cycle. If Y is nonprojective then $\mathrm{DTr}\, Y$ belongs to C_n, hence Y belongs to D_i and we are done. If Y is projective, say $Y = P_y$ for a vertex $y \in Q$, then condition (3) states that for each $i \in \{1, \ldots, n_y\}$ the set of predecessors of $R_y(i)$ in $\operatorname{mod} A$ is finite and formed by directing modules. By Theorem 13, P_y is directing in $\operatorname{mod} A$ and we are done. Thus by letting $C_{n+1} = D_t$ the induction step is proven.

In order to complete the proof we assume that for some vertex $x \in Q$ condition (2) is not satisfied, hence there exists a nondirecting predecessor of $\operatorname{rad} P_x$. By hypothesis, condition (1) is satisfied for the vertex x, which we may also assume to be a source. Thus we conclude that $\mathcal{P}$ is a preprojective component of Γ_A. ■

4. The main result. We now prove that if Λ is a quasitilted algebra, then the Auslander–Reiten quiver of Λ contains a preprojective component.

We first provide a generalization of a result by Coelho–Happel [3, Lemma 2.1].

PROPOSITION 15. *Let Λ be a quasitilted algebra, and $M = M_1 \oplus M_2$ a Λ-module such that $\Gamma = \left(\begin{smallmatrix} F & 0 \\ M & \Lambda \end{smallmatrix}\right)$ is a quasitilted algebra, where $F \subseteq \mathrm{End}_\Lambda(M)^{\mathrm{op}}$ is a division algebra. Then either each indecomposable summand of M_1 is contained in $\mathcal{R}_\Lambda$ or M_2 is projective.*

Proof. Assume that there exists an indecomposable direct summand M_1' of M_1 with $M_1' \notin \mathcal{R}_\Lambda$ and that M_2 is not projective. Consider the Γ-module $Y = (F, M_1', (\pi_1', 0))$ where $\pi_1' : M_1 \to M_1'$ is the projection according to a chosen decomposition of M. By Lemma 9, Y is indecomposable, and since M_2 is a direct summand of $\mathrm{Ker}(\pi_1', 0)$, we find by Lemma 5 that $\mathrm{pd}_\Gamma Y = 2$. Thus there exists an indecomposable injective Γ-module I such that $\operatorname{Hom}_\Gamma(I, \mathrm{DTr}\, Y) \neq 0$. Therefore there exists a path $I \to \mathrm{DTr}\, Y \to E \to Y$ where E is an indecomposable direct summand of the middle term in the almost split sequence ending in Y. Since $M_1' \notin \mathcal{R}_\Lambda$, there is a path from M_1' to an indecomposable Λ-module X with $\operatorname{id}_\Lambda X = 2$. In particular, $X \in \mathcal{L}_\Lambda$. By Lemma 3 there is a path $M_1' \xrightarrow{f} N \xrightarrow{g} X$. If $gf \neq 0$, then by Lemma 5 the indecomposable Γ-module $(F, X, (gf\pi_1', 0))$ has both projective and injective dimension equal to two, a contradiction. Thus $gf = 0$. We then obtain the diagram

$$\begin{tikzcd}
M_1 \oplus M_2 \arrow[r, "\mathrm{id}"] \arrow[d, "\pi_1'"'] & M_1 \oplus M_2 \arrow[r] \arrow[d, "(f\pi_1', 0)"] & 0 \arrow[d] \\
M_1' \arrow[r, "f"] & N \arrow[r, "g"] & X
\end{tikzcd}$$

which commutes. Since $\operatorname{id}_\Lambda X = 2$ there exists an indecomposable projective Λ-module P and a nonzero Λ-morphism $h : \mathrm{TrD}\, X \to P$ [1, Proposition IX.1.7]. Thus we obtain a path

$$Y \to (F, N, (f\pi_1', 0)) \xrightarrow{(0,g)} (0, X, 0) \to (0, Z, 0) \to (0, \mathrm{TrD}\, X, 0) \xrightarrow{(0,h)} (0, P, 0)$$

in $\operatorname{ind} \Gamma$. Since $(0, P, 0)$ is an indecomposable projective Γ-module and $\operatorname{pd}_\Gamma Y = 2$, we see that $(0, P, 0) \notin \mathcal{L}_\Gamma$, which contradicts Theorem 2. ∎

We now have the main result.

THEOREM 16. *The Auslander–Reiten quiver of any quasitilted algebra has a preprojective component.*

Proof. The proof is by induction on the number n of isomorphism classes of simple Λ-modules. Assume Λ is quasitilted with $n = 1$ isomorphism class of simple modules. Since the valued quiver of Λ contains no loops, the Auslander–Reiten quiver of Λ consists of one point with no arrows, thus Λ is a finite-dimensional k-division algebra.

Assume that all quasitilted algebras with less than n isomorphism classes of simple modules have a preprojective component, and let Λ be a quasitilted algebra with $n \geq 2$ isomorphism classes of simple modules. Let Q be the valued quiver of Λ. Let a be a vertex in Q. We want to prove that a satisfies either condition (1) or (2) in Theorem 14. First we consider the case when a is not a source in Q.

If a is not a source in Q, there exists a source ω and a path from ω to a in Q. Let $M = \operatorname{rad}_\Lambda P_\omega$. Then there exists a quasitilted algebra A such that $\Lambda = \left(\begin{smallmatrix} F & 0 \\ M & A \end{smallmatrix}\right)$, where $F \subseteq \operatorname{End}_A(M)^{\mathrm{op}}$ is a division algebra. Also, $\Lambda^a = A^a$. By induction the Auslander–Reiten quiver Γ_A of A has a preprojective component, so the vertex a satisfies one of the conditions of Theorem 14.

Thus we are left with the case where $a = \omega$ is a source. As noted before, we can write $\Lambda = \left(\begin{smallmatrix} F & 0 \\ M & A \end{smallmatrix}\right)$ for a quasitilted algebra A and an A-module $M =$

$\operatorname{rad}_\Lambda P_\omega$, where $F \subseteq \operatorname{End}_\Lambda(M)^{\mathrm{op}}$ is a division algebra. By induction Γ_A has a preprojective component. Let M_1 be the direct sum of all indecomposable direct summands of M that are contained in some preprojective component of Γ_A. Then $M = M_1 \oplus M_2$ for some direct summand M_2 of M. If $\mathcal{P}$ is a preprojective component of Γ_A, then we may assume that $\mathcal{P}$ contains an indecomposable direct summand of M_1. Otherwise the vertex ω satisfies condition (1) of Theorem 14, and $\mathcal{P}$ is a preprojective component of Γ_Λ. So from now on we assume that $M_1 \neq 0$ and that ω does not satisfy condition (1) of Theorem 14. We show that ω satisfies condition (2) of Theorem 14. The proof is divided into several steps. Our main aim is to show that M_2 is hereditary projective. By doing this we also show that there is no path from an indecomposable direct summand of M_1 to an indecomposable direct summand of M_2. Then it is straightforward to show that M is directing is $\operatorname{mod} A$, and hence that ω satisfies condition (2) of Theorem 14. In order to show that M_2 is hereditary projective we need two preliminary steps.

STEP 1. We show that M_2 is projective. Assume it is not, and let M_2' be its nonprojective indecomposable direct summand. Let A_2 be the block of A supporting M_2'. We consider two cases, according to whether or not all projective A_2-modules are contained in preprojective components of Γ_{A_2}.

STEP 1a. Assume that all projective A_2-modules are contained in preprojective components of Γ_{A_2}. Let P be an indecomposable projective A_2-module with $\operatorname{Hom}_{A_2}(P, \operatorname{DTr}_{A_2} M_2') \neq 0$. By assumption P is contained in a preprojective component $\mathcal{P}$ of Γ_{A_2} which also contains an indecomposable direct summand M_1' of M_1. We show that this contradicts Λ being quasitilted.

If $\mathcal{P}$ contains no injective modules, then by Lemma 11, $\mathcal{P} \setminus \mathcal{P}(M_1'{\rightsquigarrow})$ is finite, where $\mathcal{P}(M_1'{\rightsquigarrow}) = \{X \in \mathcal{P} \mid M_1' \rightsquigarrow X\}$. Now for all nonzero $f : P \to \operatorname{DTr} M_2'$ and $m \in \mathbb{N}$, there is a direct sum of modules of the form $(\mathrm{TrD})^j X_i \in \mathcal{P}$ with $j \geq m$ such that f factors through $\bigoplus_i (\mathrm{TrD})^j X_i$. Choose f and m as above such that there is a path $M_1' \rightsquigarrow (\mathrm{TrD})^j X_i \rightsquigarrow \operatorname{DTr} M_2' \rightsquigarrow M_2'$. By Theorem 10, Λ is not quasitilted.

If $\mathcal{P}$ contains some indecomposable injective modules, let J be the direct sum of one copy of each. By Lemma 11, $\mathcal{P} \setminus \mathcal{P}(J{\rightsquigarrow})$ is finite, where $\mathcal{P}(J{\rightsquigarrow}) = \{X \in \mathcal{P} \mid I \rightsquigarrow X$ for some indecomposable direct summand I of $J\}$. Again, for all nonzero $f : P \to \operatorname{DTr} M_2'$ and $m \in \mathbb{N}$, there is a direct sum of modules of the form $(\mathrm{TrD})^j X_i \in \mathcal{P}$ with $j \geq m$ such that f factors through $\bigoplus_i (\mathrm{TrD})^j X_i$. Choose f and m as above such that there is a path $I \rightsquigarrow (\mathrm{TrD})^j X_i \rightsquigarrow \operatorname{DTr} M_2' \rightsquigarrow M_2'$, where I is an indecomposable direct summand of J. If $\operatorname{Hom}_{A_2}(M_1, I) \neq 0$, then we obtain a path $M_1 \rightsquigarrow I \rightsquigarrow (\mathrm{TrD})^j X_i \rightsquigarrow \operatorname{DTr} M_2' \rightsquigarrow M_2'$ in $\operatorname{ind} A$. By Theorem 10, Λ is not quasitilted. If $\operatorname{Hom}_{A_2}(M_1, I) = 0$, then $(0, I, 0)$ is an indecomposable injective Λ-module. We obtain a commutative diagram

$$\begin{tikzcd} & (0, \mathrm{DTr}_A M_2', 0) \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & (0, M_2', 0) \arrow[d, equal] \\ 0 \arrow[r] & \mathrm{DTr}_\Lambda(0, M_2', 0) \arrow[r] & E \arrow[r] & (0, M_2', 0) \end{tikzcd}$$

Thus we get a nonsectional path

$$(0, I, 0) \rightsquigarrow (0, (\mathrm{TrD}_A)^j X_i, 0) \to (0, \mathrm{DTr}_A M_2', 0) \\ \to \mathrm{DTr}_\Lambda(0, M_2', 0) \to E' \to (0, M_2', 0) \to P_\omega$$

in $\operatorname{ind} \Lambda$. By Theorem 2, Λ is not quasitilted.

STEP 1b. Assume that there exists an indecomposable projective A_2-module which is not contained in a preprojective component of Γ_{A_2}. Since A_2 is an indecomposable algebra there exists an indecomposable projective A_2-module P contained in a preprojective component $\mathcal{P}$ of Γ_{A_2}, and a projective A_2-module P' which is not contained in a preprojective component of Γ_{A_2}, such that there exists a nonzero morphism $f : P \to P'$. By the choice of P and P', we have $f \in \operatorname{rad}^\infty_{A_2}(P, P')$. Thus for each $r \geq 1$ there exists a chain of irreducible morphisms $P = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \ldots \xrightarrow{f_r} X_r$ and a morphism $g_r : X_r \to P'$ such that $g_r f_r \ldots f_1 f_0 \neq 0$. By Lemma 12, $\mathcal{P}$ contains no injective modules. Then choose r such that $\mathrm{DTr}\, X_r$ is a successor of M_1', where M_1' is as in Step 1a. Since $\operatorname{Hom}_A(X_r, P') \neq 0$, we have $\operatorname{id}_A \mathrm{DTr}\, X_r = 2$ [1, Proposition IX.1.7]. Now M_2 is not projective, hence by Proposition 15, M_1' is in $\mathcal{R}_A$. The subclass $\mathcal{R}_A$ is closed under successors, hence $\mathrm{DTr}\, X_r \in \mathcal{R}_A$, contrary to $\operatorname{id}_A \mathrm{DTr}\, X_r = 2$. We conclude that M_2 is projective.

STEP 2. Now assume $M_2 \neq 0$. We show that in this case there exists an indecomposable A-module X with $\operatorname{id}_A X = 2$ and $\operatorname{Hom}_A(M_1, X) \neq 0$.

From Step 1 we know that M_2 is projective. Let M_2' be an indecomposable direct summand of M_2, and let A_2 be the block of A supporting M_2'. By induction Γ_{A_2} contains a preprojective component $\mathcal{P}$ which contains an indecomposable direct summand M_1' of M_1. Note that not all projective A_2-modules are contained in preprojective components of Γ_{A_2} since M_2' is not in a preprojective component. Then, since A_2 is an indecomposable algebra there exist indecomposable projective A_2-modules P and P' with $P \in \mathcal{P}$ and $P' \notin \mathcal{P}$ such that $\operatorname{Hom}_{A_2}(P, P') \neq 0$. Thus for each $r \geq 1$ there exists a chain of irreducible morphisms $P = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \ldots \xrightarrow{f_r} X_r$ and a morphism $g_r : X_r \to P'$ such that $g_r f_r \ldots f_1 f_0 \neq 0$.

Let $\mathcal{S}(M_1' \!\to) = \{Y \in \mathcal{P} \mid M_1' \rightsquigarrow Y$ and all paths from M_1' to Y are sectional paths of irreducible maps$\}$. We consider two cases, according to whether or not there is a proper projective successor of $\mathcal{S}(M_1' \!\to)$ in $\mathcal{P}$.

STEP 2a. Assume that no proper successor of $\mathcal{S}(M_1'\to)$ in $\mathcal{P}$ is projective. By Lemma 12, $\mathcal{P}$ contains no injective modules. Hence by assumption we may choose the number r above so that $\mathrm{DTr}\, X_r \in \mathcal{S}(M_1'\to)$. Since $\mathrm{Hom}_{A_2}(X_r, P') \neq 0$, and hence $\mathrm{Hom}_A(X_r, P') \neq 0$, we have $\mathrm{id}_A \mathrm{DTr}\, X_r = 2$. Also, $\mathrm{Hom}_A(M_1, \mathrm{DTr}\, X_r) \neq 0$, since we have a sectional path of irreducible morphisms in Γ_{A_2}, and thus in Γ_A, from M_1' to $\mathrm{DTr}\, X_r$ [1, Theorem VII.2.4].

STEP 2b. Assume that there exists a proper successor P of $\mathcal{S}(M_1'\to)$ in $\mathcal{P}$ which is projective. Let $\mathcal{S}(\to P)$ consist of those predecessors Y of P with $Y \in \mathcal{P}$ such that all paths from Y to P are sectional paths of irreducible morphisms. Let $\mathrm{DTr}(\mathcal{S}(\to P)) = \{\mathrm{DTr}\, Y \mid Y \in \mathcal{S}(\to P)\}$. Note that all indecomposable modules in $\mathrm{DTr}(\mathcal{S}(\to P))$ have injective A-dimension two, and that there is a path in $\mathcal{P}$ from M_1' to an indecomposable module in $\mathrm{DTr}(\mathcal{S}(\to P))$. Also note that $\mathrm{DTr}(\mathcal{S}(\to P))$ is a separating subcategory in the sense that each morphism from a predecessor of $\mathrm{DTr}(\mathcal{S}(\to P))$ to a module which is not such a predecessor factors through $\mathrm{DTr}(\mathcal{S}(\to P))$. Let I be an indecomposable injective A_2-module such that there exists a nonzero morphism $g : M_1' \to I$. By Lemma 12, I is a not predecessor of $\mathrm{DTr}(\mathcal{S}(\to P))$. Therefore g factors through $\mathrm{DTr}(\mathcal{S}(\to P))$. In particular, there is a module $X \in \mathrm{DTr}(\mathcal{S}(\to P))$ with $\mathrm{Hom}_A(M_1', X) \neq 0$ and $\mathrm{id}_A X = 2$.

STEP 3. Now we can prove that M_2 is a hereditary projective A-module. Assume there exists an indecomposable A-module Y with $\mathrm{pd}_A Y = 2$, and such that we have a nonzero morphism $g : M_2 \to Y$. By Step 2 we know that there exists an A-module X with $\mathrm{Hom}_A(M_1, X) \neq 0$ and $\mathrm{id}_A X = 2$. Choose $0 \neq f \in \mathrm{Hom}_A(M_1, X)$, and consider the Λ-module $Z = \left(F, X \oplus Y, \left(\begin{smallmatrix} f & 0 \\ 0 & g \end{smallmatrix}\right)\right)$. By Lemma 8, Z is indecomposable, and since $\mathrm{id}_A(X \oplus Y) = 2$ we have $\mathrm{id}_\Lambda Z = 2$ by Lemma 5. Now, $\mathrm{pd}_A Y = 2$ implies that $\mathrm{Ker}\, g$ is nonprojective, thus $\mathrm{Ker} \left(\begin{smallmatrix} f & 0 \\ 0 & g \end{smallmatrix}\right)$ is nonprojective, therefore $\mathrm{pd}_\Lambda Z = 2$ by Lemma 5. But this contradicts Λ being quasitilted. We conclude that $\mathrm{Hom}_A(M_2, Y) = 0$ for all $Y \in \mathrm{ind}\, A$ with $\mathrm{pd}_A Y = 2$. Let X be a submodule of M_2, and consider the exact sequence $0 \to X \to M_2 \to M_2/X \to 0$. Since M_2/X has projective dimension less than two, it follows that X is projective. We conclude that M_2 is a hereditary projective A-module.

FINAL STEP. It remains to show that M is directing as an A-module. By Step 3, M_2 is directing and each indecomposable direct summand of M_2 has only finitely many predecessors. Indeed, let M_2' be an indecomposable direct summand of M_2, and let $X \in \mathrm{ind}\, A$ with $\mathrm{Hom}_A(X, M_2') \neq 0$. Let $f : X \to M_2'$, and let $f = \mu\pi$ be the canonical factorization through $\mathrm{Im}\, f$. Then $\mathrm{Im}\, f$ is a submodule of M_2, hence projective, thus X is projective and a submodule of M_2. Also, we infer that there is no path from an indecomposable direct summand of M_1 to a summand of M_2. If M is decomposable, then the conclusion follows from Theorem 10 since M_2 is hereditary projective and all indecomposable direct summands of M_1 are directing since they lie in preprojective components of Γ_A. If M is indecomposable, then $M = M_1$ is contained in the preprojective component of Γ_A, thus M is directing.

This shows that the extension vertex ω in Q satisfies condition (2) of Theorem 14. Indeed, we have $\Lambda^\omega = A$, and we have shown that $M = \operatorname{rad} P_\omega$ is directing in $\operatorname{mod} A$. Also, any indecomposable direct summand of M_2 has only finitely many predecessors, all of which are directing. The indecomposable direct summands of M_1 are all contained in preprojective components of the Auslander–Reiten quiver of A, thus they have only finitely many predecessors, and all predecessors are directing.

We conclude that each quasitilted algebra has a preprojective component. ■

REFERENCES

[1] M. Auslander, I. Reiten and S. O. Smalø, *Representation Theory of Artin Algebras*, Cambridge Univ. Press, 1995.

[2] K. Bongartz, *A criterion for finite representation type*, Math. Ann. 269 (1984), 1–12.

[3] F. Coelho and D. Happel, *Quasitilted algebras admit a preprojective component*, Proc. Amer. Math. Soc. 125 (1997), 1283–1291.

[4] F. Coelho and A. Skowroński, *On Auslander–Reiten components for quasitilted algebras*, Fund. Math. 149 (1996), 67–82.

[5] P. Dräxler and J. A. de la Peña, *On the existence of postprojective components in the Auslander–Reiten quiver of an algebra*, Tsukuba J. Math. 20 (1996), 457–469.

[6] O. Enge, I. H. Slungård and S. O. Smalø, *Quasitilted extensions by nondirecting modules*, preliminary manuscript, 1998.

[7] D. Happel, I. Reiten and S. O. Smalø, *Tilting in abelian categories and quasitilted algebras*, Mem. Amer. Math. Soc. 575 (1996).

[8] —, —, —, *Short cycles and sincere modules*, in: CMS Conf. Proc. 14, Amer. Math. Soc., 1993, 233–237.

[9] D. Happel and C. M. Ringel, *Directing projective modules*, Arch. Math. (Basel) 60 (1993), 237–246.

[10] A. Skowroński and M. Wenderlich, *Artin algebras with directing indecomposable projective modules*, J. Algebra 165 (1994), 507–530.

[11] H. Strauss, *The perpendicular category of a partial tilting module*, ibid. 144 (1991), 43–66.

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Received 28 April 1999;
revised 4 August 1999

(3742)