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FUNDAMENTAL SOLUTIONS FOR TRANSLATION AND ROTATION INVARIANT DIFFERENTIAL OPERATORS ON THE HEISENBERG GROUP

BY

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Abstract. Let H_1 be the three-dimensional Heisenberg group. Consider the left invariant differential operators of the form $D = P(-iT, -L)$, where P is a polynomial in two variables with complex coefficients, L is the sublaplacian on H_1 and T is the derivative with respect to the central direction. We find a fundamental solution of D, whose definition is related to the way the plane curve defined by $P(x, y) = 0$ intersects the Heisenberg fan $F = A \cup B$, $A = \{(x, y) \in \mathbb{R}^2 : y = (2m + 1)|x|,\ m \in \mathbb{N}\}$, $B = \{(x, y) \in \mathbb{R}^2 : x = 0,\ y > 0\}$. We can write an explicit expression of such a fundamental solution when the curve $P(x, y) = 0$ intersects F at finitely many points, all belonging to A and, if one of them is the origin, the monomial y^k has a nonzero coefficient, where k is the order of zero at the origin. As a consequence, such operators are globally solvable on H_1.

1. Introduction. In this paper we study problems of solvability of left invariant differential operators on the three-dimensional Heisenberg group H_1.

Let Ω be an open set in a Lie group G. A left-invariant differential operator P on G is *locally solvable* at $x_0 \in \Omega$ if there exists a neighborhood U of x_0 in Ω such that for all $f \in C^\infty(\overline{U})$ there exists a distribution u on U that satisfies $Pu = f$ on U.

P is *semiglobally solvable* in Ω if for all $f \in \mathcal{D}(\Omega)$ and for all open sets U relatively compact in Ω there exists $u \in C^\infty$ such that $Pu = f$ on U.

Finally, P is *globally solvable* in Ω if $PC^\infty(\Omega) = C^\infty(\Omega)$. Global solvability is stronger than semiglobal solvability, and the latter implies local solvability.

We shall consider those differential operators that are expressed as polynomials with complex coefficients in L and T, L being the sublaplacian and T the derivative with respect to the central direction. L and T commute and generate the algebra of differential operators on H_1 which are invariant with respect to both left translations and rotations.

Such a problem has already been solved for operators represented by polynomials of degree one. In [9] and [6] it is shown that the operator $-L +$

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$i\alpha T + c$, $\alpha, c \in \mathbb{C}$, is locally solvable unless $c = 0$ and $\alpha = 2m + 1$, for some integer m. As we shall see, it is natural to formulate the following conjecture: the operator $D = P(-iT, -L)$, where P is a polynomial with complex coefficients, is locally solvable on the Heisenberg group H_1 if and only if $P(\lambda, \xi)$ is not divisible by $\xi - (2m + 1)\lambda$, for some $m \in \mathbb{Z}$.

In this work, we show that the above conjecture is correct with certain restrictions on P. In the solvable case we in fact construct a fundamental solution. If G is a Lie group, a distribution $E \in \mathcal{D}'(G)$ is a *fundamental solution* of an invariant operator P if $PE = \delta_0$, δ_0 being the Dirac delta at the identity. The existence of a fundamental solution implies semiglobal solvability. Moreover, if G is P-convex, then the semiglobal solvability of P implies its global solvability. The Heisenberg group is P-convex with respect to all nonzero invariant differential operators (see [4]).

2. Preliminaries. The $(2n + 1)$-dimensional Heisenberg group H_n is the Lie group, diffeomorphic to $\mathbb{R}^{2n+1}$, whose multiplication law is defined as

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' + 2(x' \cdot y - x \cdot y')), \tag{1}$$

where $x, y, x', y' \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $x \cdot y$ is the usual inner product on $\mathbb{R}^n$.

A base for its Lie algebra $\mathbf{h}_n$ consists of the left invariant vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \qquad T = \frac{\partial}{\partial t}$$

where $j = 1, \ldots, n$. The commutation relations are $[X_j, T] = [Y_j, T] = 0$, $[X_j, Y_k] = -4\delta_{j,k}T$, for all $j, k = 1, \ldots, n$.

The *sublaplacian* is the left invariant operator on H_n defined by

$$L = \frac{1}{4} \sum_{j=1}^{n} (X_j^2 + Y_j^2).$$

If $n = 1$, then $L = \frac{1}{4}(X^2 + Y^2)$. It is a homogeneous operator of degree two with respect to the dilations δ_r on H_1, induced by the automorphisms of $\mathbf{h}_1$ defined by

$$\delta_r X = rX, \qquad \delta_r Y = rY, \qquad \delta_r T = r^2 T.$$

Indeed,

$$\delta_r L = \frac{1}{4}(\delta_r X^2 + \delta_r Y^2) = r^2 L.$$

Note that also $T = \partial/\partial t$ is homogeneous of degree two.

Consider the spherical functions

$$\varphi_{\lambda,m}(x, y, t) = e^{-i\lambda t} l_m(2|\lambda|(x^2 + y^2)),$$

where $l_m(x) = e^{-x/2}L_m(x)$ and $L_m(x) = L_m^{(0)}(x)$ is the mth Laguerre polynomial of index $\alpha = 0$, defined by

$$L_m^{(\alpha)}(x) = \sum_{k=0}^{m} \binom{m+\alpha}{m-k} \frac{(-x)^k}{k!}.$$

The $\varphi_{\lambda,m}$ are joint bounded radial eigenfunctions of L and T, and

$$T\varphi_{\lambda,m} = -i\lambda\varphi_{\lambda,m}, \tag{2}$$

$$L\varphi_{\lambda,m} = -|\lambda|(2m+1)\varphi_{\lambda,m}. \tag{3}$$

Let Δ be the Gelfand spectrum of the Banach algebra $L^1_{\mathrm{rad}}(H_1)$ of integrable radial functions on H_1. Then

$$\Delta = \{\varphi_{\lambda,m} : \lambda \neq 0,\ m \in \mathbb{N}\} \cup \{\varphi_{0,\xi} : \xi \geq 0\}$$

where

$$\varphi_{0,\xi}(x,y,t) = J_0(2\sqrt{\xi(x^2+y^2)})$$

and

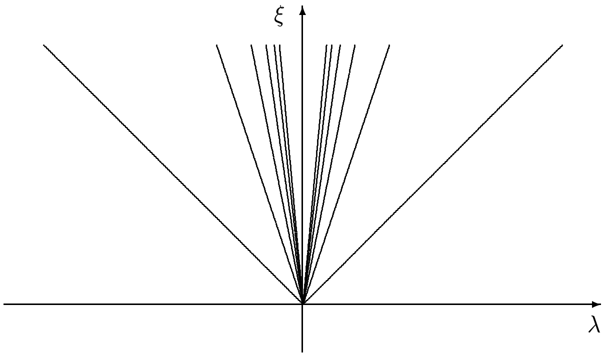
$$J_0(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{it\sin\theta}\, d\theta$$

is the Bessel function of order 0.

It is shown in [2] that the Gelfand topology on Δ coincides with the topology on

$$F = \{(\lambda, |\lambda|(2m+1)) \in \mathbb{R}^2 :\ \lambda \neq 0,\ m \in \mathbb{N}\} \cup \{(0,\xi) \in \mathbb{R}^2 :\ \xi \geq 0\}$$

induced from the Euclidean topology of $\mathbb{R}^2$.



The set F is usually called the *Heisenberg fan*.

We now state two technical lemmas involving Laguerre functions, which will be useful later on.

LEMMA 2.1. *The Laguerre functions $l_m^{(\alpha)}(x) = e^{-x/2}L_m^{(\alpha)}(x)$ satisfy the following estimates*:

$$|l_m^{(\alpha)}(x)| \leq 1, \tag{4}$$

$$\left|\frac{d^j}{dx^j} l_m^{(\alpha)}(x)\right| \le C_{\alpha j}(m+1)^j, \qquad j \ge 1. \tag{5}$$

P r o o f. Estimate (4) follows from the properties of Laguerre polynomials (see, for instance, Section 10.12 in [5]), while (5) is an immediate consequence of the following property:

$$\frac{d^j}{dx^j} l_m^{(\alpha)}(x) = \sum_{h=0}^{j} c_h m(m-1)\ldots(m-h+1) l_{m-h}^{(\alpha+h)}(x), \tag{6}$$

which can be proved by induction from the identity

$$\frac{d}{dx} l_m^{(\alpha)}(x) = -\frac{1}{2} l_m^{(\alpha)}(x) + \frac{m}{\alpha+1} l_{m-1}^{(\alpha+1)}(x)$$

(see [5], formula (15) of Section 10.12). ■

LEMMA 2.2. *For all* $\lambda \neq 0$,

$$\left|\frac{\partial^j}{\partial\lambda^j} \varphi_{-\lambda,m}(x,y,t)\right| \le C_j[|t| + (m+1)(x^2+y^2)]^j. \tag{7}$$

P r o o f. By the estimates of Lemma 2.1 we have

$$\begin{aligned}
\left|\frac{\partial^j}{\partial\lambda^j} \varphi_{-\lambda,m}(x,y,t)\right| &= \left|\frac{\partial^j}{\partial\lambda^j}(e^{i\lambda t} l_m^{(0)}(2|\lambda|(x^2+y^2)))\right| \\
&\le \sum_{h=0}^{j} \binom{j}{h} \left|\frac{\partial^{j-h}}{\partial\lambda^{j-h}} e^{i\lambda t}\right| \left|\frac{\partial^h}{\partial\lambda^h} l_m^{(0)}(2|\lambda|(x^2+y^2))\right| \\
&= \sum_{h=0}^{j} \binom{j}{h} |t|^{j-h}(2(x^2+y^2))^h \left|\frac{\partial^h}{\partial\eta^h} l_m^{(0)}(\eta)\right|_{\eta=2|\lambda|(x^2+y^2)} \\
&\le \sum_{h=0}^{j} \binom{j}{h} |t|^{j-h} C_h (m+1)^h (2(x^2+y^2))^h \\
&\le C_j \sum_{h=0}^{j} \binom{j}{h} |t|^{j-h}[(m+1)(x^2+y^2)]^h \\
&= C_j[|t| + (m+1)(x^2+y^2)]^j. \ ■
\end{aligned}$$

3. Solvability of polynomials in L and T. We will give some techniques that enable us to find a fundamental solution of operators of the form

$$D = P(-iT, -L), \tag{8}$$

where P is a polynomial in two variables with complex coefficients, L is the sublaplacian, T is the derivative with respect to t.

PROPOSITION 3.1. *Let* D_1 *and* D_2 *be operators of the form* (8). *Then* $D = D_1 D_2$ *is locally solvable if and only if* D_1 *and* D_2 *are locally solvable.*

Proof. Suppose D is locally solvable; then there exist a neighborhood U and a distribution $u \in \mathcal{D}(U)$ such that for all $f \in C^\infty(\overline{U})$ one has $Du = f$ in U. Since D_1 and D_2 commute, we have

$$D_1(D_2 u) = f = D_2(D_1 u)$$

on U, that is, D_1 and D_2 are locally solvable. Let us see that the converse is also true.

If D_1 is locally solvable, then there exists an open set U_1 such that for all $f \in C^\infty(\overline{U}_1)$ (in particular $f \in \mathcal{S}(\overline{U}_1)$) there exists $u \in \mathcal{D}'(U_1)$ which is a solution of $D_1 u = f$ in U_1. Since D_2 is locally solvable, there exist a neighborhood U_2 and a distribution $v \in \mathcal{D}'(U_2)$ such that $D_2 v = u$ in U_2. Therefore D is locally solvable, for $Dv = D_1 D_2 v = D_1 u = f$ in $U_1 \cap U_2$. ∎

COROLLARY 3.2. (a) *If* $P(\lambda, \xi)$ *is identically zero on some oblique ray of the fan, then* D *is not locally solvable.*

(b) *If* $P(\lambda, \xi)$ *is identically zero on the vertical ray of the fan, i.e.* $D = T^h D_1$, *then* D *is locally solvable if and only if* D_1 *is locally solvable.*

Proof. (a) By hypothesis $P(\lambda, \xi)$ is divisible by $\xi - (2m+1)\lambda$, for some $m \in \mathbb{Z}$. Then $D = D_1 D_2$, where $D_1 = -L + i(2m+1)T$. Such an operator is not locally solvable (see [6]). By Proposition 3.1, D is not locally solvable.

(b) T^h is known to be locally solvable. Indeed, solving the problem $T^h w = u$, where $u \in \mathcal{D}'(U)$, is equivalent to finding an hth primitive of u in the variable t. Such a primitive always exists (see Theorem IV, Ch. II, Sec. 5 in [8]). The statement follows from Proposition 3.1. ∎

We will therefore restrict our investigation to those operators such that $P(\lambda, \xi)$ does not vanish identically on any ray of the fan.

THEOREM 3.3. *If* P *is a homogeneous polynomial, then* $D = P(-iT, -L)$ *is solvable if and only if* $P(\lambda, \xi)$ *is not divisible by* $\xi - (2m+1)\lambda$, *for some* $m \in \mathbb{Z}$. *Moreover, in this case* D *is globally solvable.*

Proof. It is well known that if P is a homogeneous polynomial in two variables, then it factors as a product of terms of degree one. Since the operator $-L + i(2m+1)T$, corresponding to the polynomial $\xi - (2m+1)\lambda$, is not locally solvable for all $m \in \mathbb{Z}$, the assertion follows from Proposition 3.1.

The last statement is true because D is homogeneous with respect to the dilations δ_r on H_1 defined before. ∎

Let us describe the irreducible unitary representations of H_n. For every $\lambda \neq 0$, we have the Schrödinger representation π_λ, which is unique up to equivalence and is defined in the following way. Given $f \in L^2(\mathbb{R}^n)$,

$$(\pi_\lambda(x,y,t)f)(\xi) = e^{-i\lambda(t+2x\cdot y-4\xi\cdot y)} f(\xi - x).$$

To the value $\lambda = 0$ there correspond the one-dimensional representations

$$\pi_{\xi,\eta}(x,y,t) = e^{-i(\xi\cdot x+\eta\cdot y)},$$

where $\xi, \eta \in \mathbb{R}^n$.

Such representations are pairwise inequivalent and every irreducible unitary representation of H_n is equivalent to one of them.

The *Fourier transform* of a function $f \in L^1(H_n)$ is the collection of all operators

$$\pi(f) = \int_{H_n} f(x,y,t)\pi(x,y,t)\,dx\,dy\,dt$$

where π ranges over the set of unitary irreducible representations of H_n described above. The *inversion formula*

$$f(x,y,t) = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}} \operatorname{tr}(\pi_\lambda(f)\pi_\lambda(x,y,t)^*)|\lambda|^n\,d\lambda \tag{9}$$

holds in a dense subspace of $L^1(H_n)$, in particular for Schwartz functions. If we choose an othonormal basis of $L^2(\mathbb{R}^n)$, we can compute the trace explicitly. Put $n = 1$ and fix the normalized Hermite basis $\{h_m^\lambda\}_{m\in\mathbb{N}}$ of $L^2(\mathbb{R})$, where

$$h_m^\lambda(\xi) = \frac{|\lambda|^{1/4}}{2^{(m-1)/2}\sqrt{m!}\,\pi^{1/4}}\phi_m(\sqrt{|\lambda|}\xi)$$

with $\phi_m(\xi) = D_\lambda^m e^{-2\xi^2}$ and $D_\lambda = \frac{1}{2}(d/d\xi - 4\lambda\xi)$. In this basis, (9) can be rewritten as

$$f(x,y,t) = \frac{1}{(2\pi)^{n+1}} \sum_m \int_{\mathbb{R}} \langle \pi_\lambda(f)h_m^\lambda, \pi_\lambda(x,y,t)h_m^\lambda\rangle |\lambda|^n\,d\lambda, \tag{10}$$

where the inner product is taken in $L^2(\mathbb{R})$.

We have

$$\varphi_{\lambda,m}(v) = \langle \pi_\lambda(v)h_m^\lambda, h_m^\lambda\rangle,$$

where $v = (x,y,t)$. If we put $\widehat{g}(\lambda,m,n) = \langle \pi_\lambda(g)h_m^\lambda, h_n^\lambda\rangle$ for all $g \in \mathcal{S}(H_1)$, then

$$\widehat{g}(\lambda,m,m) = \int_{H_1} \varphi_{\lambda,m}(v)g(v)\,dv.$$

Therefore, since $\|\varphi_{\lambda,m}\|_\infty = 1$ for all m and λ,

$$|\widehat{g}(\lambda,m,m)| \le \|g\|_{L^1(H_1)}. \tag{11}$$

Let f be a Schwartz function on H_1 and assume that $Du = f$. By formally applying the Fourier transform to both sides, we get

$$\pi_\lambda(Du)h_m^\lambda = \pi_\lambda(f)h_m^\lambda. \tag{12}$$

If V is a left invariant vector field, we have

$$\pi_\lambda(Vu) = -\pi_\lambda(u)d\pi_\lambda(V). \tag{13}$$

Take D as in (8). By (13) we get

$$\pi_\lambda(Du) = \pi_\lambda(u)d\pi_\lambda({}^tD). \tag{14}$$

Moreover,

$$d\pi_\lambda(T)h_m^\lambda = -i\lambda h_m^\lambda, \qquad d\pi_\lambda(L)h_m^\lambda = -(2m+1)|\lambda|h_m^\lambda.$$

Therefore

$$\begin{aligned} d\pi_\lambda({}^tD)h_m^\lambda &= d\pi_\lambda(P(iT,-L))h_m^\lambda = P(d\pi_\lambda(iT), d\pi_\lambda(-L))h_m^\lambda \\ &= P(\lambda, |\lambda|(2m+1))h_m^\lambda, \end{aligned}$$

and, by (14),

$$\pi_\lambda(Du)h_m^\lambda = \pi_\lambda(u)P(\lambda, |\lambda|(2m+1))h_m^\lambda. \tag{15}$$

Formula (15) can be viewed as an analogue of the identity

$$(p(-i\partial)u)^\wedge(\xi) = p(\xi)\widehat{u}(\xi), \tag{16}$$

holding on $\mathbb{R}^n$ for a differential operator with constant coefficients. The polynomial $p(\xi)$ appearing in (16) is called the symbol of the operator $p(-i\partial)$. For this reason we call $P(\lambda,\xi)$ the *symbol* of D.

From (12) and (15) it follows that

$$\pi_\lambda(Du)h_m^\lambda = \pi_\lambda(u)P(\lambda, |\lambda|(2m+1))h_m^\lambda = \pi_\lambda(f)h_m^\lambda,$$

therefore

$$\pi_\lambda(u)h_m^\lambda = \frac{\pi_\lambda(f)h_m^\lambda}{P(\lambda, |\lambda|(2m+1))}. \tag{17}$$

From the inversion formula (10) and from (17) we get the following formal expression for u:

$$\begin{aligned} u(v) &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}}\sum_{m=0}^{\infty}\frac{\langle \pi_\lambda(f)h_m^\lambda, \pi_\lambda(v)h_m^\lambda\rangle}{P(\lambda, |\lambda|(2m+1))}|\lambda|\,d\lambda \\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}}\sum_{m=0}^{\infty}\int_{H_1} f(w)\frac{\langle \pi_\lambda(w)h_m^\lambda, \pi_\lambda(v)h_m^\lambda\rangle}{P(\lambda, |\lambda|(2m+1))}\,dw\,|\lambda|\,d\lambda \\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}}\sum_{m=0}^{\infty}\int_{H_1} f(w)\frac{\langle h_m^\lambda, \pi_\lambda(w^{-1}v)h_m^\lambda\rangle}{P(\lambda, |\lambda|(2m+1))}\,dw\,|\lambda|\,d\lambda \\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}}\sum_{m=0}^{\infty}\int_{H_1} \frac{f(w)\varphi_{-\lambda,m}(w^{-1}v)}{P(\lambda, |\lambda|(2m+1))}\,dw\,|\lambda|\,d\lambda \\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}}\sum_{m=0}^{\infty}\frac{f*\varphi_{-\lambda,m}(v)}{P(\lambda, |\lambda|(2m+1))}|\lambda|\,d\lambda. \end{aligned}$$

Therefore, if we can define

$$K(v) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \sum_{m=0}^{\infty} \frac{\varphi_{-\lambda,m}(v)}{P(\lambda, |\lambda|(2m+1))} |\lambda| \, d\lambda$$

as a distribution, it follows that $u = f * K$, so that a fundamental solution of D is the tempered distribution defined by

$$\begin{aligned} \langle K, g\rangle &= \frac{1}{(2\pi)^2} \int_{H_1} \int_{\mathbb{R}} \sum_{m=0}^{\infty} \frac{\varphi_{-\lambda,m}(v) g(v)}{P(\lambda, |\lambda|(2m+1))} |\lambda| \, d\lambda \, dv \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \sum_{m=0}^{\infty} \frac{\widehat{g}(-\lambda, m, m)}{P(\lambda, |\lambda|(2m+1))} |\lambda| \, d\lambda, \end{aligned} \tag{18}$$

for all $g \in \mathcal{S}(H_1)$. Note that only the radial coefficients $\widehat{g}(-\lambda, m, m)$ occur in this formula, so K is radial.

For a generic polynomial P, (18) does not converge absolutely in general. The series may not converge, and the integral has singularities when the algebraic curve defined by $P(\lambda, \xi) = 0$ intersects the Heisenberg fan. Thus, we are going to face our problem by considering separately different cases, according to the mutual position of the algebraic curve $P(\lambda, \xi) = 0$ and the fan. For each case, we define a fundamental solution of D, modifying (18) in a suitable way, in order to get a well defined tempered distribution.

As we have already said above in this section, we are reduced to considering algebraic curves $P(\lambda, \xi) = 0$ that intersect each ray of the fan in at most finitely many points.

4. First case: no intersections. The simplest situation occurs when $P(\lambda, \xi)$ is never zero on F. To solve this problem we use the following fact (see [7], Appendix A, Example 2.7).

LEMMA 4.1. *If $P \in \mathbb{R}[x_1, \ldots, x_n]$ and $P(x) > 0$ for all $x \in \mathbb{R}^n$, then there exist $C > 0$ and $N \in \mathbb{N}$ such that*

$$P(x) > C(1 + |x|^2)^{-N} \qquad \forall x \in \mathbb{R}^n.$$

A consequence of this lemma is the following

LEMMA 4.2. *If $P \in \mathbb{C}[x, y]$ and $P(x, y) \neq 0$ in the closed domain of the plane defined by $y \geq |x|(2m+1)$, then in this region we have the estimate*

$$|P(x, y)| > C(1 + x^2 + y^2)^{-N},$$

for some $C > 0$ and $N \in \mathbb{N}$.

Proof. By changing coordinates we can reduce to the case $P(x, y) \neq 0$ in the first quarter of the plane. Therefore assume that for all $x \geq 0$, $y \geq 0$ we have $|P(x, y)| > 0$.

If $P(x,y) = P_1(x,y) + iP_2(x,y)$ with $P_1(x,y), P_2(x,y) \in \mathbb{R}[x,y]$, then $|P(x,y)| = \sqrt{P_1(x,y)^2 + P_2(x,y)^2}$ and $Q(x,y) = P_1(x,y)^2 + P_2(x,y)^2 \in \mathbb{R}[x,y]$. Since Q is positive for all $x \geq 0$ and $y \geq 0$, the polynomial $R(x,y) = Q(x^2,y^2)$ is positive for all $(x,y) \in \mathbb{R}^2$.

By Lemma 4.1 there exist $C_1 > 0$ and $N \in \mathbb{N}$ such that

$$R(x,y) > C_1(1+x^2+y^2)^{-N}.$$

For $x \geq 0$ and $y \geq 0$, $Q(x,y) = R(\sqrt{x}, \sqrt{y})$, therefore

$$Q(x,y) > C_1(1+x+y)^{-N} > C_2(1+x^2+y^2)^{-N/2}$$

and so

$$|P(x,y)| > C(1+x^2+y^2)^{-N/4}. \blacksquare$$

Consider an operator D whose symbol P is such that $P(\lambda,\xi) = 0$ defines an algebraic curve that does not intersect F, i.e. $P(\lambda, |\lambda|(2m+1)) \neq 0$ for all $m \in \mathbb{N}$, $\lambda \in \mathbb{R}$ and $P(0,\xi) \neq 0$ for all $\xi > 0$.

THEOREM 4.3. *Take $D = P(-iT, -L)$ such that $P(\lambda,\xi)$ is not zero on F. Define the distribution K by*

$$\langle K, g\rangle = \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \sum_{m=0}^{\infty} \frac{\widehat{g}(-\lambda, m, m)}{P(\lambda, |\lambda|(2m+1))} |\lambda|\, d\lambda, \quad g \in \mathcal{S}(H_1),$$

where the integral on the right-hand side is absolutely convergent. Then K is a fundamental solution of D.

Proof. The algebraic curve $P(\lambda,\xi) = 0$ in the λ,ξ plane has a finite number of connected components (see [3], Theorems 2.3.6 and 2.4.5). Since it does not intersect F, there exists an integer $k \in \mathbb{N}$ such that $P(\lambda,\xi) \neq 0$ in the closed region defined by $\xi \geq (2k+1)|\lambda|$.

By Lemma 4.2, for all $m \geq k$ one has

$$|P(\lambda, |\lambda|(2m+1))| > C(1+\lambda^2(2m+1)^2)^{-N},$$

for some $C > 0$ and $N \in \mathbb{N}$. Moreover, for fixed $m < k$, we define

$$\mu_m = \min_{\lambda \in \mathbb{R}} |P(\lambda, |\lambda|(2m+1))| > 0.$$

Let M be a positive constant such that $M < \min\{\mu_m : m = 1, \ldots, k-1\}$. Hence $|P(\lambda, |\lambda|(2m+1))| > M$ for $m < k$. Putting these two estimates together shows that there exist a positive constant C and a natural number N such that

$$|P(\lambda, |\lambda|(2m+1))| > C(1+\lambda^2(2m+1)^2)^{-N} \tag{19}$$

for every m.

Since the symbol of ${}^tD$ is $P(-\lambda,|\lambda|(2m+1))$, it follows from (15) that, for all $g\in\mathcal{S}(H_1)$,

$$\pi_\lambda(g)h_m^\lambda=\frac{\pi_\lambda({}^tDg)h_m^\lambda}{P(-\lambda,|\lambda|(2m+1))},$$

whence

$$\pi_{-\lambda}(g)h_m^\lambda=\frac{\pi_{-\lambda}({}^tDg)h_m^\lambda}{P(\lambda,|\lambda|(2m+1))} \tag{20}$$

and, recalling (11),

$$|\widehat{g}(-\lambda,m,m)|=\frac{|({}^tDg)^\wedge(-\lambda,m,m)|}{|P(\lambda,|\lambda|(2m+1))|}\le C\frac{\|{}^tDg\|_{L^1}}{|P(\lambda,|\lambda|(2m+1))|}. \tag{21}$$

Set $A=I+L^2$; then ${}^tA=A$ and, by replacing D with A^{N+2} in (20), we get

$$\begin{aligned}\langle K,g\rangle&=\frac{1}{(2\pi)^2}\sum_{m=0}^{\infty}\int_{\mathbb{R}}\frac{\widehat{g}(-\lambda,m,m)|\lambda|}{P(\lambda,|\lambda|(2m+1))}\,d\lambda\\&=\frac{1}{(2\pi)^2}\sum_{m=0}^{\infty}\int_{\mathbb{R}}\frac{(A^{N+2}g)^\wedge(-\lambda,m,m)}{(1+\lambda^2(2m+1)^2)^{N+2}}\cdot\frac{|\lambda|}{P(\lambda,|\lambda|(2m+1))}\,d\lambda.\end{aligned}$$

Moreover, by (19), we get

$$\begin{aligned}|\langle K,g\rangle|&\le\frac{\|A^{N+2}g\|_{L^1}}{(2\pi)^2}\sum_{m=0}^{\infty}\int_{\mathbb{R}}\frac{|\lambda|\cdot|P(\lambda,|\lambda|(2m+1))|^{-1}}{(1+\lambda^2(2m+1)^2)^{N+2}}\,d\lambda\\&\le C\|A^{N+2}g\|_{L^1}\sum_{m=0}^{\infty}\int_{\mathbb{R}}\frac{|\lambda|(1+\lambda^2(2m+1)^2)^N}{(1+\lambda^2(2m+1)^2)^{N+2}}\,d\lambda\\&\le C\|A^{N+2}g\|_{L^1}\sum_{m=0}^{\infty}\int_{\mathbb{R}}\frac{|\lambda|}{(1+\lambda^2(2m+1)^2)^2}\,d\lambda\\&\le 2C\|A^{N+2}g\|_{L^1}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\int_0^\infty\frac{t}{(1+t^2)^2}\,dt\\&\le C'\|A^{N+2}g\|_{L^1}\le C''\|g\|_{(n)}\end{aligned}$$

where $\|\cdot\|_{(n)}$ is a continuous Schwartz norm. Therefore K is a tempered distribution. Let us show that it is a fundamental solution. We verify that $DK=\delta$, by testing both sides of the identity on a Schwartz function f and applying (20):

$$\begin{aligned}\langle DK,f\rangle=\langle K,{}^tDf\rangle&=\frac{1}{(2\pi)^2}\sum_m\int_{\mathbb{R}}\frac{({}^tDf)^\wedge(-\lambda,m,m)|\lambda|}{P(\lambda,|\lambda|(2m+1))}\,d\lambda\\&=\frac{1}{(2\pi)^2}\sum_m\int_{\mathbb{R}}\frac{\langle\pi_{-\lambda}({}^tDf)h_m^\lambda,h_m^\lambda\rangle}{P(\lambda,|\lambda|(2m+1))}|\lambda|\,d\lambda\end{aligned}$$

$$= \frac{1}{(2\pi)^2} \sum_m \int_{\mathbb{R}} \langle \pi_{-\lambda}(f) h_m^\lambda, h_m^\lambda \rangle |\lambda| \, d\lambda$$
$$= \frac{1}{(2\pi)^2} \sum_m \int_{\mathbb{R}} \langle \pi_{-\lambda}(f) h_m^\lambda, \pi_{-\lambda}(0,0,0) h_m^\lambda \rangle |\lambda| \, d\lambda$$
$$= \frac{1}{(2\pi)^2} \sum_m \int_{\mathbb{R}} \langle \pi_{\lambda}(f) h_m^\lambda, \pi_{\lambda}(0,0,0) h_m^\lambda \rangle |\lambda| \, d\lambda$$
$$= \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \operatorname{tr}(\pi_\lambda(f)\pi_\lambda(0,0,0)^*) |\lambda| \, d\lambda = f(0,0,0).$$

Therefore a solution of the problem $Du = f$ is

$$u(x,y,t) = (f * K)(x,y,t). \ \blacksquare$$

5. Second case: a finite number of intersections, all away from the vertical ray. We now turn to the case in which the algebraic curve $P(\lambda,\xi) = 0$ intersects the Heisenberg fan in a finite number of points, all of them belonging to the oblique rays and different from the origin.

Let us begin, for simplicity, by assuming that $\{(\lambda,\xi) \in \mathbb{R}^2 : P(\lambda,\xi) = 0\}$ intersects the fan with multiplicity $h \ge 1$ in one single point, lying on the kth ray. We can assume that this point has the form $(\alpha, |\alpha|(2k+1))$, with $\alpha > 0$. Therefore, there exists a polynomial $Q(\lambda)$ such that, for $\lambda \ge 0$, $P(\lambda, |\lambda|(2k+1)) = (\lambda - \alpha)^h Q(\lambda)$ and $Q(\lambda) \neq 0$; for $\lambda < 0$, $P(\lambda, |\lambda|(2k+1)) \neq 0$. Moreover, $P(\lambda, |\lambda|(2m+1)) \neq 0$ for $m \neq k$ and $\lambda \in \mathbb{R}$.

Given a C^∞ function $\varphi(x)$, define

$$R_{h,\alpha}(\varphi(x)) = \varphi(x) - \sum_{j=0}^{h-1} \frac{\varphi^{(j)}(\alpha)}{j!} (x-\alpha)^j.$$

If $g(x)$ is a rational function with a pole of order h at α and I is an interval containing α, then

$$\varphi \mapsto \int_I R_{h,\alpha}(\varphi) g(x) \, dx$$

is a well defined distribution, which is a modified version of *Hadamard's finite part* (see [8], Ch. 2, Sec. 2, Example 2).

Note that

$$R_{h,\alpha}((x-\alpha)^h g(x)) = (x-\alpha)^h g(x). \tag{22}$$

THEOREM 5.1. *Consider $D = P(-iT, -L)$ and suppose that P is as above. Then D has a fundamental solution $K \in \mathcal{S}'(H_1)$, defined as follows: for all $g \in \mathcal{S}(H_1)$,*

$$\langle K, g \rangle = \frac{1}{(2\pi)^2} \sum_{m=0}^{\infty} \langle K_m, g \rangle$$

where

$$\langle K_m, g\rangle = \int\limits_{\mathbb{R}} \frac{\widehat{g}(-\lambda, m, m)|\lambda|}{P(\lambda, |\lambda|(2m+1))}\, d\lambda \qquad \textit{for } m \neq k,$$

$$\langle K_k, g\rangle = \int\limits_{\mathbb{R}\setminus[0,2\alpha]} \frac{\widehat{g}(-\lambda, k, k)|\lambda|}{P(\lambda, |\lambda|(2k+1))}\, d\lambda + \int\limits_0^{2\alpha} \frac{R_{h,\alpha}(\widehat{g}(-\lambda, k, k))|\lambda|}{P(\lambda, |\lambda|(2k+1))}\, d\lambda.$$

P r o o f. All of the integrals converge absolutely. For all $m \neq k$ and for all $\lambda \in \mathbb{R}$, $P(\lambda, |\lambda|(2m+1)) \neq 0$, so we can argue as in the proof of Theorem 4.3 to show that

$$\Big|\sum_m \langle K_m, g\rangle\Big| \leq C\|g\|_{(N)}, \qquad \text{for some } N \gg 0.$$

If $m = k$, we have

$$\begin{aligned}\langle K_k, g\rangle = &\int\limits_{\mathbb{R}\setminus[0,2\alpha]} \frac{\widehat{g}(-\lambda, k, k)|\lambda|}{P(\lambda, |\lambda|(2k+1))}\, d\lambda \\ &+ \int\limits_{H_1}\int\limits_0^{2\alpha} \left[\frac{d^h}{d\lambda^h}\varphi_{-\lambda,k}(v)\right]_{\lambda=\xi} \frac{|\lambda| g(v)}{h!Q(\lambda)}\, d\lambda\, dv,\end{aligned}$$

where ξ is strictly between α and λ. The first term is absolutely convergent because again $P(\lambda, |\lambda|(2k+1)) \neq 0$ in $\mathbb{R} \setminus [0, 2\alpha]$. If we apply estimate (5) to the derivatives of the functions $\varphi_{-\lambda,m}(v)$ we can show that also the second integral is absolutely convergent, so we deduce that K is a tempered distribution. Let us show that it is a fundamental solution of D. Using (22) we have

$$\begin{aligned}\langle K_k, {}^tDf\rangle &= \int\limits_{\mathbb{R}\setminus[0,2\alpha]} \widehat{f}(-\lambda, k, k)|\lambda|\, d\lambda \\ &\quad + \int\limits_0^{2\alpha} \frac{R_{h,\alpha}(P(\lambda, |\lambda|(2k+1))\widehat{f}(-\lambda, k, k))}{P(\lambda, |\lambda|(2k+1))}|\lambda|\, d\lambda \\ &= \int\limits_{\mathbb{R}\setminus[0,2\alpha]} \widehat{f}(-\lambda, k, k)|\lambda|\, d\lambda \\ &\quad + \int\limits_0^{2\alpha} \frac{P(\lambda, |\lambda|(2k+1))\widehat{f}(-\lambda, k, k)}{P(\lambda, |\lambda|(2k+1))}|\lambda|\, d\lambda \\ &= \int\limits_{\mathbb{R}} \widehat{f}(-\lambda, k, k)|\lambda|\, d\lambda.\end{aligned}$$

Therefore

$$\langle K, {}^tDf\rangle = \frac{1}{(2\pi)^2}\sum_{m=0}^{\infty}\int\limits_{\mathbb{R}} \widehat{f}(-\lambda, m, m)|\lambda|\, d\lambda = f(0,0,0). \ \blacksquare$$

This result extends in an obvious way to those operatos $D = P(-iT, -L)$ such that $P(\lambda, \xi) = 0$ intersects the fan with finite multiplicity in finitely many points, all of them belonging to the oblique rays and different from the origin.

COROLLARY 5.2. *Let $D = P(-iT, -L)$ be such that $P(\lambda, |\lambda|(2m+1)) = 0$ only at finitely many points, say $(\lambda_{j,h}, |\lambda_{j,h}|(2m_j + 1))$, $j = 1, \dots, r$, $h = 1, \dots, r_j$, each of them lying on the curve $\xi = |\lambda|(2m_j + 1)$ and having multiplicity $\mu_{j,h}$. Suppose also that $P(0, \xi) \neq 0$ for all $\xi \geq 0$. Let $I_{j,h}$ be intervals centered at $\lambda_{j,h}$ such that $I_{j,h} \cap I_{j,h'} = \emptyset$ if $h \neq h'$. Then D has a fundamental solution*

$$\langle K, g\rangle = \frac{1}{(2\pi)^2}\Big(\sum_{j=1}^{r}\langle K_{m_j}, g\rangle + \sum_{m\notin\{m_1,\dots,m_r\}}\langle K_m, g\rangle\Big), \qquad g \in \mathcal{S}(H_1),$$

where

$$\langle K_m, g\rangle = \int_{\mathbb{R}} \frac{\widehat{g}(-\lambda, m, m)|\lambda|}{P(\lambda, |\lambda|(2m+1))}\, d\lambda \qquad \text{if } m \notin \{m_1, \dots, m_r\},$$

$$\begin{aligned}\langle K_{m_j}, g\rangle = &\int_{\bigcup I_{j,h}} \frac{R_{\mu_{j,h},\lambda_{j,h}}(\widehat{g}(-\lambda, m_j, m_j))|\lambda|}{P(\lambda, |\lambda|(2m_j+1))}\, d\lambda \\ &+ \int_{\mathbb{R}\setminus\bigcup I_{j,h}} \frac{\widehat{g}(-\lambda, m_j, m_j)|\lambda|}{P(\lambda, |\lambda|(2m_j+1))}\, d\lambda.\end{aligned}$$

6. Third case: intersection in the origin. The last case we examine concerns the kind of singularity occurring in the distribution (18) when P vanishes at the origin. We consider an operator $D = P(-iT, -L)$ where P is a polynomial with complex coefficients, having only a finite number of zeros on the fan, one of them being the origin and no other lying on the vertical ray.

We are able to find a fundamental solution of D only if we add a technical hypothesis: we must suppose that the homogeneous part of minimum degree (say k) of D contains a term of the form aL^k, with $a \neq 0$. The symbol of D is therefore a polynomial of the form

$$P(\lambda, \xi) = c_{\bar\alpha}\xi^k + \sum_{|\alpha|=k,\, \alpha\neq\bar\alpha} c_\alpha \xi^{\alpha_1}\lambda^{\alpha_2} + \sum_{|\alpha|>k} c_\alpha \xi^{\alpha_1}\lambda^{\alpha_2}, \tag{23}$$

$\overline{\alpha} = (k, 0)$, $c_\alpha \in \mathbb{C}$ and $c_{\bar\alpha} = -a$. This includes the case $D = L^k$, considered in [1].

We will need the following lemma.

LEMMA 6.1. *Suppose $P \in \mathbb{C}[x, y]$, $P(x, y) \neq 0$ for all $x, y \geq 0$, $(x, y) \neq (0, 0)$. Then there exist a positive constant C and $N \in \mathbb{N}$ such that*

$$(24) \qquad |P(x,y)| > C(1+x^2+y^2)^{-N}$$

for all $x, y \geq 0$ *with* $\sqrt{x^2+y^2} \geq 1$.

Proof. Define $Q(x,y) = P(x+1/\sqrt{2}, y)$. Then $Q(x,y) \neq 0$ for all $x \geq 0$, $y \geq 0$, therefore by Lemma 4.2 we have the estimate

$$|Q(x,y)| > C_1(1+x^2+y^2)^{-N_1}$$

for all $x \geq 0$, $y \geq 0$. Hence

$$|P(x,y)| = |Q(x-1/\sqrt{2}, y)| > C_2(1+x^2+y^2)^{-N_1}$$

for all $x \geq 1/\sqrt{2}$, $y \geq 0$. In the same way we can show that there exist $C_3 > 0$ and $N_2 \in \mathbb{N}$ such that, for all $x \geq 0$, $y \geq 1/\sqrt{2}$,

$$|P(x,y)| > C_3(1+x^2+y^2)^{-N_2}.$$

If we take $C = \max(C_2, C_3)$ and $N = \min(N_1, N_2)$, we get the estimate (24) for all $x, y \geq 0$ with $\sqrt{x^2+y^2} \geq 1$. ■

We begin with the case where the origin is the only zero.

Theorem 6.2. *Suppose that*

$$P(\lambda,\xi) = c_{\bar\alpha}\xi^k + \sum_{|\alpha|=k,\, \alpha\neq\bar\alpha} c_\alpha \xi^{\alpha_1}\lambda^{\alpha_2} + \sum_{|\alpha|>k} c_\alpha \xi^{\alpha_1}\lambda^{\alpha_2}, \qquad c_\alpha \in \mathbb{C},\ c_{\bar\alpha} \neq 0,$$

and that $P(\lambda, |\lambda|(2m+1)) \neq 0$ *for all* $\lambda \neq 0$, $m \in \mathbb{N}$. *Define the distribution* K, *for all* $g \in \mathcal{S}(H_1)$, *by*

$$(25) \qquad \langle K, g\rangle = \frac{1}{(2\pi)^2}\sum_{m=0}^{\infty}\Bigg\{\int\limits_{|\lambda|\geq\delta/(2m+1)} \frac{\widehat{g}(-\lambda,m,m)|\lambda|}{P(\lambda,|\lambda|(2m+1))}\,d\lambda + \int\limits_{|\lambda|<\delta/(2m+1)} \frac{R_{k+N_m-1,0}(\widehat{g}(-\lambda,m,m))|\lambda|}{P(\lambda,|\lambda|(2m+1))}\,d\lambda\Bigg\}$$

where $N_m \in \mathbb{N}$ *is zero except for finitely many* $m \in \mathbb{N}$ *and* δ *is a suitable positive constant. Then* K *is a fundamental solution of* $D = P(-iT, -L)$.

Proof. Let σ be a positive constant. On the line $\xi = \lambda/\sigma$, $P(\lambda,\xi)$ takes the value

$$P_\sigma(\xi) = P(\sigma\xi,\xi) = \xi^k\Big(c_{\bar\alpha} + \sum_{|\alpha|=k,\alpha\neq\bar\alpha} c_\alpha\sigma^{\alpha_2}\Big) + \sum_{|\alpha|>k} c_\alpha\sigma^{\alpha_2}\xi^{|\alpha|}.$$

Note that $c_{\bar\alpha} + \sum_{|\alpha|=k,\,\alpha\neq\bar\alpha} c_\alpha\sigma^{\alpha_2}$ tends to $c_{\bar\alpha}$ as $\sigma \to 0$. Therefore, if σ is small enough, the quantity $|c_{\bar\alpha} + \sum_{|\alpha|=k,\alpha\neq\bar\alpha} c_\alpha\sigma^{\alpha_2}|$ is not zero and can be bounded from below by a positive constant. Thus, there exists σ_0 such that for all $\sigma \leq \sigma_0$, $|c_{\bar\alpha} + \sum_{|\alpha|=k,\,\alpha\neq\bar\alpha} c_\alpha\sigma^{\alpha_2}| \geq C_1 > 0$.

Since

$$\sum_{|\alpha|>k} c_\alpha\sigma^{\alpha_2}\xi^{|\alpha|} = o(\xi^k) \qquad \text{as } \xi \to 0,$$

if ξ is small enough, then $\sum_{|\alpha|>k} c_\alpha \sigma^{\alpha_2} \xi^{|\alpha|}$ is negligible with respect to ξ^k. Therefore there exists $\delta_0 > 0$ such that if $\xi \le \delta_0$ and $\sigma \le \sigma_0$, then

$$|P_\sigma(\xi)| \ge \Big| C_1 |\xi|^k - \Big| \sum_{|\alpha|>k} c_\alpha \sigma^{\alpha_2} \xi^{|\alpha|} \Big| \Big| \ge C_2 |\xi|^k.$$

Thus, in the triangle

$$\mathcal{E} = \{(\lambda, \xi) \in \mathbb{R}^2 : |\lambda|/\sigma_0 \le \xi \le \delta_0\}$$

we have $|P(\lambda, \xi)| \ge C\xi^k$.

Hence,

$$|P(\lambda, |\lambda|(2m+1))| \ge C(2m+1)^k \lambda^k \tag{26}$$

for all $m \ge 1/(2\sigma_0) - 1/2$ and all λ such that $|\lambda| \le \delta_0 \sigma_0$.

For finitely many $m < 1/(2\sigma_0) - 1/2$, it may happen that the sum $c_{\bar\alpha} + \sum_{|\alpha|=k, \alpha \ne \bar\alpha} c_\alpha/(2m+1)^{\alpha_2}$ is zero. Therefore, for all $m < 1/(2\sigma_0) - 1/2$, there exist $N_m \in \mathbb{N}$ and $\delta_1 > 0$ such that, if $|\lambda| < \delta_1$, then

$$|P(\lambda, |\lambda|(2m+1))| > M\lambda^{k+N_m}. \tag{27}$$

Put $\delta = \min(\sigma_0, \delta_0, \delta_1)$ and let us show that K in (25) is a tempered distribution. Note that

$$\langle K, g \rangle = \frac{1}{(2\pi)^2} \sum_{m=0}^{\infty} (I_m^1 + I_m^2),$$

where

$$I_m^1 = \int_{|\lambda| \ge \delta/(2m+1)} \frac{\widehat{g}(-\lambda, m, m)|\lambda|}{P(\lambda, |\lambda|(2m+1))}\, d\lambda$$

$$I_m^2 = \int_{|\lambda| < \delta/(2m+1)} \left[\frac{d^{k+N_m-1}}{d\lambda^{k+N_m-1}} \widehat{g}(-\lambda, m, m) \right]_{\lambda=\lambda_m} \times \frac{\lambda^{k+N_m-1}|\lambda|}{(k+N_m-1)! P(\lambda, |\lambda|(2m+1))}\, d\lambda$$

and λ_m in I_m^2 is a value between 0 and λ, for all m.

Take $m < 1/(2\sigma_0) - 1/2$. Then I_m^1 is absolutely convergent because $P(\lambda, |\lambda|(2m+1)) \ne 0$ for all λ such that $|\lambda| \ge \delta/(2m+1)$. By estimate (27) we get

$$\left| \frac{d^{k+N_m-1}}{d\lambda^{k+N_m-1}} \varphi_{-\lambda,m}(v) \right|_{\lambda=\lambda_m} \frac{|\lambda|^{k+N_m}}{(k+N_m-1)! |P(\lambda, |\lambda|(2m+1))|} \le C_m \left\| \frac{d^{k+N_m-1}}{d\lambda^{k+N_m-1}} \varphi_{-\lambda,m} \right\|_\infty \le C,$$

so

$$\left|\frac{d^{k+N_m-1}}{d\lambda^{k+N_m-1}}\widehat{g}(-\lambda,m,m)\right|_{\lambda=\lambda_m}\frac{|\lambda|^{k+N_m}}{(k+N_m-1)!|P(\lambda,|\lambda|(2m+1))|}\le C\|g\|_1.$$

Therefore the integrals occurring in K, corresponding to $m<1/(2\sigma_0)-1/2$, are absolutely convergent.

Consider now the infinitely many terms in K labeled by $m\ge 1/(2\sigma_0)-1/2$. Recall that $N_m=0$ for such m. By applying (7) to the derivatives of $\varphi_{-\lambda,m}(v)$, and (26) to the polynomial P, we get

$$\begin{aligned}\int\limits_{|\lambda|<\delta/(2m+1)}&\left|\frac{d^{k-1}}{d\lambda^{k-1}}\varphi_{-\lambda,m}(v)\right|_{\lambda=\lambda_m}\frac{|\lambda|^k}{(k-1)!|P(\lambda,|\lambda|(2m+1))|}\,d\lambda\\ &\le\int\limits_{|\lambda|<\delta/(2m+1)}\frac{C_1(m+1)^{k-1}(|t|+(x^2+y^2))^{k-1}}{(2m+1)^k}\,d\lambda\\ &\le\frac{C_2(|t|+(x^2+y^2))^{k-1}}{(m+1)^2}.\end{aligned}$$

It follows that, for all $m\ge 1/(2\sigma_0)-1/2$,

$$\begin{aligned}|I_m^2|&\le\frac{C_2}{(m+1)^2}\int\limits_{H_1}(|t|+(x^2+y^2))^{k-1}|g(x,y,t)|\,dx\,dy\,dt\\ &=\frac{C_2}{(m+1)^2}\|(|t|+(x^2+y^2))^{k-1}g\|_1\le\frac{C_3}{(m+1)^2}\|g\|_{(N)},\end{aligned}$$

for some $N\gg 0$.

By hypothesis, estimate (24) holds for P. Let $h=N$ be the exponent appearing in (24) and $A=-L(I+L^2)^{h+1}$. Then, by (21), we have

$$\begin{aligned}&\left|\int\limits_{|\lambda|\ge\delta/(2m+1)}\frac{\widehat{g}(-\lambda,m,m)|\lambda|}{P(\lambda,|\lambda|(2m+1))}\,d\lambda\right|\\ &\quad\le\int\limits_{|\lambda|\ge\delta/(2m+1)}\frac{\|{}^tAg\|_1|\lambda|}{|\lambda|(2m+1)|1+\lambda^2(2m+1)^2|^{h+1}|P(\lambda,|\lambda|(2m+1))|}\,d\lambda\\ &\quad\le\frac{C_1\|{}^tAg\|_1}{2m+1}\int\limits_{|\lambda|\ge\delta/(2m+1)}\frac{d\lambda}{|1+\lambda^2(2m+1)^2|}\\ &\quad\le\frac{C_1}{(2m+1)^2}\|{}^tAg\|_1\int\limits_{|t|\ge\delta}\frac{dt}{1+t^2}\le\frac{C_2}{(2m+1)^2}\|{}^tAg\|_1.\end{aligned}$$

Therefore $K\in\mathcal{S}'(H_1)$. Let us show that it is a fundamental solution of D:

$$\begin{aligned}\langle K, {}^tDf\rangle &= \langle K, P(-iT,-L)f\rangle \\ &= \frac{1}{(2\pi)^2}\sum_{m=0}^{\infty}\Bigg\{\int\limits_{|\lambda|\ge\delta/(2m+1)} \widehat{f}(-\lambda,m,m)|\lambda|\,d\lambda \\ &\quad + \int\limits_{|\lambda|<\delta/(2m+1)} \frac{R_{k+N_m-1,0}(({}^tDf)^\wedge(-\lambda,m,m))|\lambda|}{P(\lambda,|\lambda|(2m+1))}\,d\lambda\Bigg\} \\ &= \frac{1}{(2\pi)^2}\sum_{m=0}^{\infty}\Bigg\{\int\limits_{|\lambda|\ge\delta/(2m+1)} \widehat{f}(-\lambda,m,m)|\lambda|\,d\lambda \\ &\quad + \int\limits_{|\lambda|<\delta/(2m+1)} \frac{R_{k+N_m-1,0}(P(\lambda,|\lambda|(2m+1))\widehat{f}(-\lambda,m,m))|\lambda|}{P(\lambda,|\lambda|(2m+1))}\,d\lambda\Bigg\} \\ &= \frac{1}{(2\pi)^2}\sum_{m=0}^{\infty}\int_{\mathbb{R}} \widehat{f}(-\lambda,m,m)|\lambda|\,d\lambda = f(0,0,0)\end{aligned}$$

since

$$\begin{aligned}&\frac{R_{k+N_m-1,0}(P(\lambda,|\lambda|(2m+1))\widehat{f}(-\lambda,m,m))}{P(\lambda,|\lambda|(2m+1))} \\ &\qquad\qquad = \frac{P(\lambda,|\lambda|(2m+1))\widehat{f}(-\lambda,m,m)}{P(\lambda,|\lambda|(2m+1))}. \ ■\end{aligned}$$

Putting together the results obtained up to now, we can generalize Theorem 6.2, allowing $P(\lambda,\xi)$ to have zeros on F also outside the origin.

THEOREM 6.3. *Suppose $P(\lambda,\xi)$ has the form* (23) *and let $P(\lambda,\xi)$ vanish on F only at the origin and at finitely many points*, $(\lambda_j, |\lambda_j|(2m_j+1))$, $j = 1,\dots,r$, *with multiplicity μ_j. Choosing a sufficiently small positive constant δ, let I_j be intervals centered at λ_j such that*

$$I_j \cap \left(-\frac{\delta}{2m_j+1}, \frac{\delta}{2m_j+1}\right) = \emptyset$$

and $I_j \cap I_{j'} = \emptyset$ if $m_j = m_{j'}$. Put also

$$B_m = \mathbb{R}\setminus\left[\left(-\frac{\delta}{2m_j+1}, \frac{\delta}{2m_j+1}\right) \cup \bigcup_{m=m_j} I_j\right].$$

Define the distribution K, for all $g \in \mathcal{S}(H_1)$, by

$$\begin{aligned}\langle K, g\rangle = \frac{1}{(2\pi)^2}\sum_{m=0}^{\infty}\Bigg\{&\int\limits_{B_m} \frac{\widehat{g}(-\lambda,m,m)|\lambda|\,d\lambda}{P(\lambda,|\lambda|(2m+1))} \\ &+ \int\limits_{|\lambda|<\delta/(2m+1)} \frac{R_{k+N_m-1,0}(\widehat{g}(-\lambda,m,m))|\lambda|\,d\lambda}{P(\lambda,|\lambda|(2m+1))}\Bigg\}\end{aligned}$$

$$+\frac{1}{(2\pi)^2}\sum_{j=1}^{r}\int_{I_j}\frac{R_{\mu_j,\lambda_j}(\widehat{g}(-\lambda,m_j,m_j))|\lambda|\,d\lambda}{P(\lambda,|\lambda|(2m_j+1))},$$

where $N_m = 0$ *except for finitely many* $m \in \mathbb{N}$. *Then* K *is a fundamental solution of* D.

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