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SYMMETRIC HOCHSCHILD EXTENSION ALGEBRAS

BY

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Dedicated to Helmut Lenzing on the occasion of his 60th birthday

Abstract. By an extension algebra of a finite-dimensional K-algebra A we mean a Hochschild extension algebra of A by the dual A-bimodule $\mathrm{Hom}_K(A,K)$. We study the problem of when extension algebras of a K-algebra A are symmetric. (1) For an algebra $A = KQ/I$ with an arbitrary finite quiver Q, we show a sufficient condition in terms of a 2-cocycle for an extension algebra to be symmetric. (2) Let L be a finite extension field of K. By using a given 2-cocycle of the K-algebra L, we construct a 2-cocycle of the K-algebra LQ for an arbitrary finite quiver Q without oriented cycles. Then we show a criterion on L for all those K-algebras LQ to have symmetric non-splittable extension algebras defined by the 2-cocycles.

Introduction. Throughout the paper, an *algebra* means a finite-dimensional K-algebra over a field K. Frobenius algebras, weakly symmetric algebras and almost symmetric algebras are defined for rings with minimum condition, but no structural characterization of symmetric algebras is known. G. Azumaya and T. Nakayama suggested that the notion of symmetric algebra depends on the base field [1]. The aim of the present paper is to show sufficient conditions related to 2-cocycles for self-injective Hochschild extension algebras to be symmetric, and to present a construction of symmetric algebras by using finite extension fields of the base field.

The trivial extension algebras of a K-algebra A by the standard duality module $\mathrm{Hom}(A,K)$ are very important in the representation theory of self-injective algebras. They are symmetric algebras and correspond to zero in the second cohomology groups $H^2(A,\mathrm{Hom}(A,K))$. For an algebra A without oriented cycles in its ordinary quiver, any Hochschild extension algebra of A by a duality module, say M, is stably equivalent to the split extension algebra of A by M. Moreover, if there is a symmetric extension of A by

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M, the duality module M is isomorphic to the standard duality module $\mathrm{Hom}(A,K)$ as A-bimodules [12]. Thus we restrict ourselves to Hochschild extension algebras of a given algebra by its standard duality module. It should be noted that the extension algebras by duality modules are always self-injective [11].

Let A be a K-algebra and DA the standard duality module $\mathrm{Hom}_K(A,K)$. The Hochschild extension algebras form the second cohomology groups $H^2(A,DA)$, and an extension algebra is defined by a 2-cocycle $\alpha : A\times A \to DA$. Let L be a field in the centre of A which contains K as a subfield. Then $DA \cong \mathrm{Hom}_L(A,L)$ as A-bimodules and hence the composite $A\times A \xrightarrow{\alpha} DA \xrightarrow{\sim} \mathrm{Hom}_L(A,L)$ is also a 2-cocycle of the K-algebra A which defines an extension algebra isomorphic to the extension algebra defined by α. Thus by a 2-cocycle α of A we understand a 2-cocycle of the K-algebra A to the duality module $\mathrm{Hom}_L(A,L)$.

In the first section we recall some definitions and elementary facts about Hochschild extensions, and in the second section we consider split algebras A, i.e. factor algebras of path algebras KQ of finite quivers Q (cf. [3]). We show a sufficient condition on a given 2-cocycle which yields the symmetry of the extension algebra of A defined by the 2-cocycle, where we do not assume that Q does not contain oriented cycles. A *path* is an element x of A represented by a path of Q, and $s(x), t(x)$ denote primitive idempotents with $x = t(x)xs(x)$. The main theorem in Section 2 is

THEOREM 1. *Let Q be a finite quiver and $A = KQ/I$ for an admissible ideal I. Let T be an extension algebra of A by $DA = \mathrm{Hom}_K(A,K)$, which is defined by a 2-cocycle $\alpha : A\times A\to DA$. Then T is a symmetric algebra if*

$$\alpha(x,y)(s(y)) = \alpha(y,x)(s(x))$$

for non-zero paths x, y with $s(x) = t(y)$ and $s(y) = t(x)$.

As an example we show that any proper factor algebra of the polynomial ring $K[X]$ has the property that all extension algebras are symmetric, which is a special case of a fact proved as an application of Theorem 1.

In the third section, we consider algebras A whose quivers have no oriented cycles. In this case, since $H^2(A,DA) = 0$ if A is a split algebra, we assume that for a finite extension field L of K the K-algebra A is a path algebra LQ, where Q is a finite quiver without oriented cycles. By $T(A,\beta)$ we understand the extension algebra of $A = LQ$ by $DA = \mathrm{Hom}_L(A,L)$, whose multiplication is defined by a 2-cocycle $\beta : A\times A\to DA$. Now let $\alpha : L\times L \to L$ be a 2-cocycle of the K-algebra L, and define a 2-cocycle $\widehat{\alpha} : A\times A \to DA$ by

$$\widehat{\alpha}(x,y) = \sum_{i\in Q_0} \alpha(e(i)xe(i), e(i)ye(i))e(i)^*$$

for $x, y \in A$ and $e(i)$ the idempotent corresponding to the vertex $i \in Q_0$. This 2-cocycle was first introduced in [9]. The following theorem is proved.

THEOREM 2. *The following conditions are equivalent*:

(1) $T(L, \alpha)$ *is symmetric.*
(2) $T(LQ, \widehat{\alpha})$ *is symmetric for some quiver* Q *without oriented cycles.*
(3) $T(LQ, \widehat{\alpha})$ *is symmetric for any quiver* Q *without oriented cycles.*

In the final section, we exhibit some examples to clarify the theorems. We show that, if L is a field generated by at most two elements over K, then any extension algebra of the K-algebra L is symmetric.

The main results of Sections 3 and 4 were announced by the third named author during the ICRTA 8.5 at Bielefeld (September 1998).

1. Hochschild extension algebras. In this section we recall from [2], [13] some properties of Hochschild extensions of a K-algebra A by a duality bimodule, and show some preliminary results about 2-cocycles.

Let D be a duality between A-mod and A^{op}-mod. Then, by the Morita duality theorem, there is an A-bimodule M with $D \cong \mathrm{Hom}_A(-, M)$ as functors, and such a module M is characterized by the property that it is an injective cogenerator as a left and a right A-module. Moreover, in this case, $M \cong DA$ as A-bimodules. Such a module DA is called a *duality module*. By *extensions* or *extension algebras* in this paper we understand Hochschild extensions or Hochschild extension algebras by duality modules, respectively.

Now let A be a K-algebra and DA a duality module. Then any extension algebra T of A by DA is self-injective [11]. Let $0 \to DA \to T \to A \to 0$ be an extension of A by DA. The extension algebra T is defined by a 2-*cocycle*, say $\alpha : A \times A \to DA$, which is a K-bilinear map with the 2-*cocycle condition*

$$(a, b, c)_\alpha := a\alpha(b, c) - \alpha(ab, c) + \alpha(a, bc) - \alpha(a, b)c = 0$$

for any triplet (a, b, c) from A. Then T is the K-vector space $A \oplus DA$ with multiplication

$$(a, u)(b, v) = (ab, av + ub + \alpha(a, b))$$

for $(a, u), (b, v) \in A \oplus DA$. We denote by $T(A, \alpha)$ the extension K-algebra T of A. Note that $T(A, 0)$ is the trivial extension algebra of A by DA for $\alpha = 0$ (the zero map).

LEMMA 1.1 [12]. *There is a symmetric extension algebra of* A *by a duality module* DA *if and only if* DA *is isomorphic to* $\mathrm{Hom}_K(A, K)$ *as an* A-*bimodule.*

Here a K-algebra A is said to be *symmetric* if $A \cong \mathrm{Hom}_K(A,K)$ as A-bimodules, or equivalently, there is a *symmetric regular* K-linear map $\lambda : A \to K$, which by definition is a K-linear map satisfying

(S1) *Regularity*: $\lambda(Ax) \neq 0$ for $0 \neq x \in A$,
(S2) *Symmetry*: $\lambda(xy) = \lambda(yx)$ for $x, y \in A$.

Let $\alpha : A \times A \to M$ be a 2-cocycle of a K-algebra A with an A-bimodule M. For an A-bimodule N isomorphic to M, the composite

$$\beta : A \times A \xrightarrow{\alpha} M \xrightarrow{\sim} N$$

is also a 2-cocycle of the K-algebra A, and the extension K-algebras T and T' of A defined by α and β respectively are canonically isomorphic.

Let L be a finite extension field of K, and let A be a finite-dimensional L-algebra. We need the following well known fact.

The dual space $\mathrm{Hom}_L(A,L)$ *is isomorphic to the space* $\mathrm{Hom}_K(A,K)$ *as an A-bimodule.*

We recall the corresponding isomorphism. Take any non-zero element u of $L^* := \mathrm{Hom}_K(L,K)$ and the K-linear map $f_1 : L \to L^*$ with $f_1(x) = xu$. Then f_1 is an L-bimodule isomorphism. Consider the maps $f_2 : \mathrm{Hom}_L(A,L) \to \mathrm{Hom}_L(A,L^*)$ and $f_3 : \mathrm{Hom}_L(A,L^*) \to \mathrm{Hom}_K(A,K)$ defined by $f_2(v) = f_1 v$, $f_3(w)(a) = w(a)(1_L)$ for $v \in \mathrm{Hom}_L(A,L)$, $w \in \mathrm{Hom}_L(A,L^*)$ and $a \in A$. It is then easily seen that f_2 and f_3 are A-bimodule isomorphisms, and the composite $f_3 f_2$ is the required isomorphism.

Thus, for such an algebra A, by 2-cocycles we understand K-bilinear maps $A \times A \to \mathrm{Hom}_L(A,L)$ satisfying the 2-cocycle condition.

For a 2-cocycle $\alpha : A \times A \to DA = \mathrm{Hom}_L(A,L)$, we define two K-bilinear maps

$$[\ ,\ ] : A \times A \to \mathrm{Hom}_L(A,L), \qquad \langle\ ,\ \rangle : A \times A \to L$$

such that $[a,b] = \alpha(a,b) - \alpha(b,a)$ and $\langle a,b\rangle = [a,b](1_A)$ for $a, b \in A$.

LEMMA 1.2. *For $a, b, c \in A$, we have*

$$\langle a,b\rangle + \langle b,a\rangle = 0, \qquad \langle a,bc\rangle + \langle b,ca\rangle + \langle c,ab\rangle = 0.$$

Proof. The first property is trivial. For the second, note that

$$(z\alpha(x,y))(1_A) = (\alpha(x,y)z)(1_A)$$

for $x, y, z \in A$. Then

$$\begin{aligned}
\langle a,bc\rangle &= (\alpha(a,bc) - \alpha(bc,a))(1_A)\\
&= (\alpha(ab,c) + \alpha(a,b)c - \alpha(b,ca) - b\alpha(c,a))(1_A)\\
&= (\alpha(ab,c) + c\alpha(a,b) - \alpha(b,ca) - \alpha(c,a)b)(1_A)\\
&= (\alpha(ab,c) - \alpha(c,ab) + \alpha(ca,b) - \alpha(b,ca))(1_A)\\
&= -\langle c,ab\rangle - \langle b,ca\rangle.
\end{aligned}$$

Since our concern in this paper is non-split extension algebras, the following lemma may be applicable to concrete examples.

LEMMA 1.3. *Let L be a commutative K-algebra and let $\alpha : L \times L \to \mathrm{Hom}_K(L, K)$ be a 2-cocycle. If $\alpha(a, b) \neq \alpha(b, a)$ for some $a, b \in L$, then $T(L, \alpha)$ is a non-split extension algebra.*

Proof. Let $DL = \mathrm{Hom}_K(L, K)$ and $0 \to DL \to T(L, \alpha) \xrightarrow{f} L \to 0$ be a Hochschild extension. To show that it is not splittable, suppose that there is a K-algebra morphism $g : L \to T(L, \alpha)$ such that $fg = 1_L$. Let $g(x) = (x, \widetilde{x})$ for an element $x \in L$. Then, for any $x, y \in L$, we have $g(xy) = g(x)g(y) = (x, \widetilde{x})(y, \widetilde{y}) = (xy, x\widetilde{y} + \widetilde{x}y + \alpha(x, y))$, and similarly, $g(yx) = (yx, y\widetilde{x} + \widetilde{y}x + \alpha(y, x))$. Since L is commutative by assumption, we have $g(xy) = g(yx)$ and so $\alpha(x, y) = \alpha(y, x)$ for any $x, y \in L$, a contradiction.

2. Symmetric extension algebras of split algebras. In this section we do not assume that the ordinary quiver of an algebra has no oriented cycles, but consider a split algebra [3, 3.6]. Since we show a sufficient condition for a given extension algebra to be symmetric, taking account of Lemma 1.1 we may consider only extension algebras by the standard modules over the base field K. Thus, throughout this section, we assume that $DA = \mathrm{Hom}_K(A, K)$. Let Q be a finite quiver and $A = KQ/I$, where I is an admissible ideal. We denote by Q_0, Q_1 and Q_+ the set of vertices (paths of length zero), the set of arrows and the set of paths of positive length, respectively, and put $Q_{\geq 0} = Q_+ \sqcup Q_0$ (disjoint union). Let $Q_0 = \{1, \ldots, n\}$ and $e(i)$ be the primitive idempotent corresponding to $i \in Q_0$. For a non-zero element x of A with $x = e(j)xe(i)$ for some i, j, we denote the $e(i)$ and $e(j)$ by $s(x)$ and $t(x)$, and so $x = t(x)xs(x)$. Two idempotents e and f are said to be *orthogonal* if $ef = 0 = fe$.

LEMMA 2.1. *For $x = t(x)xs(x), y = t(y)ys(y) \in A$, we have*

(1) *$\langle x, e\rangle = 0$ for an idempotent e with $x = ex = xe$. In particular, $\langle x, 1_A\rangle = 0$, and $\langle x, s(x) + t(x)\rangle = 0$.*

(2) *If $s(x) = t(x)$, then $\langle x, e\rangle = 0$ for any idempotent e.*

(3) *If $s(y) \neq t(x)$, then $\langle x, y\rangle = \langle xy, s(y)\rangle$.*

(4) *If $s(y) = t(x)$; then $\alpha(x, y)(e) = \alpha(xy, s(y))(e)$ for any idempotent e orthogonal to $s(y)$. In particular, $\alpha(x, y)(1 - s(y)) = \alpha(xy, s(y))(1 - s(y))$.*

Proof. (1) By Lemma 1.2, we have

$$(*) \qquad \langle x, e^2\rangle + \langle e, ex\rangle + \langle e, xe\rangle = 0$$

and so $\langle x, e\rangle = 0$, which implies the result.

(2) Suppose that $s(x) = t(x)$. Then, because of (1), it is enough to show that $\langle x, e\rangle = 0$ for $e^2 = e$ orthogonal to $s(x)$; but this follows from $(*)$ because $xe = ex = 0$.

(3) This follows from Lemma 1.2 for the triplet $(x, y, s(y))$.

(4) Let $f = s(y)$; then $y = yf$. Hence, for an idempotent e orthogonal to f, we have $(\alpha(x,y)f)(e) = \alpha(x,y)(fe) = 0$ and $(x\alpha(y,f))(e) = \alpha(y,f)(ex) = \alpha(y,f)(et(x)x) = \alpha(y,f)(efx) = 0$ because $t(x) = s(y) = f$ by assumption. It then follows from the 2-cocycle condition $(x,y,f)_\alpha$ that $\alpha(x,y)(e) = (\alpha(xy,f) + \alpha(x,y)f - x\alpha(y,f))(e) = \alpha(xy,f)(e)$.

For a path $p \in Q_{\geq 0}$, we denote by p again the image of p under the canonical map $KQ \to A = KQ/I$, if there is no confusion. Moreover, for simplicity, $x \in Q_{\geq 0}$ for an element x of A means that x is the image of a path in $Q_{\geq 0}$, and so $x = t(x)xs(x)$ for some primitive idempotents $s(x)$ and $t(x)$. The path algebra KQ has a K-basis $\{e(i), p \mid i \in Q_0,\ p \in Q_+\}$ and $\mathrm{Hom}_K(KQ, K)$ has the dual basis $\{e(i)^*, p^* \mid i \in Q_0,\ p \in Q_+\}$. The aim of this section is to prove the following theorem.

THEOREM 2.2. *Let Q be a finite quiver and $A = KQ/I$ for an admissible ideal I. Let T be an extension algebra of A by $DA = \mathrm{Hom}_K(A,K)$, which is defined by a 2-cocycle $\alpha : A \times A \to DA$. Then T is a symmetric algebra if*

$$\alpha(x,y)(s(y)) = \alpha(y,x)(s(x))$$

for any non-zero paths $x, y \in Q_+$ with $s(x) = t(y)$ and $s(y) = t(x)$.

As a special case of the theorem, for a finite quiver Q without oriented cycles we know that all extension algebras of $A = KQ/I$ are symmetric. But this also follows from the fact that $H^2(A, \mathrm{Hom}_K(A,K)) = 0$, i.e. any extension algebra of A is splittable [6], [9].

To prove the theorem, we define a K-linear map $T \to K$ with the properties (S1), (S2) in Section 1, and we need some lemmas.

DEFINITION. A K-linear map $\lambda : KQ \oplus \mathrm{Hom}_K(KQ,K) \to K$ is defined in the following way:

$$\lambda(e(i), 0) = 0 \quad \text{for } i \in Q_0,$$

$$\lambda(x,0) = \begin{cases} \langle s(x), x\rangle & \text{if } s(x) \neq t(x) \text{ for } x \in Q_+, \\ -\alpha(x, s(x))(1_A - s(x)) & \text{if } s(x) = t(x) \text{ for } x \in Q_+, \end{cases}$$

$$\lambda(0, x^*) = x^*(1_A) \quad \text{for } x \in Q_{\geq 0}.$$

Note that $\lambda(0, e(i)^*) = 1_K$ and $\lambda(0, x^*) = 0$ for $i \in Q_0$ and $x \in Q_+$.

Let $p = c_1x_1 + \ldots + c_nx_n$ $(c_i \in K,\ x_i \in Q_+)$ be a minimal relation in I. Then, by definition, $e := s(x_1) = \ldots = s(x_n)$ and $f := t(x_1) = \ldots = t(x_n)$. In the case when $e \neq f$, $\lambda(p,0) = \sum_i c_i\lambda(x_i,0) = \sum_i c_i\langle e, x_i\rangle$ by definition and hence $\lambda(p,0) = \langle e, \sum_i c_ix_i\rangle = \langle e, p\rangle = 0$, because $p \in I$ and so p

represents the zero in A. In the other case, i.e. $e = f$,

$$\begin{aligned}\lambda(p,0) &= -\sum_i c_i\alpha(x_i,e)(1-e)\\ &= -\alpha\Big(\sum_i c_i x_i, e\Big)(1-e) = -\alpha(p,e)(1-e) = 0,\end{aligned}$$

because p is zero in A. Moreover, for $u = c_1x_1^* + \ldots + c_nx_n^* \in DA$, we have $\lambda(0,u) = \sum_i c_i\lambda(0,x^*) = 0$. Thus λ naturally induces a K-linear map $T \to K$, which will be denoted by λ again. We note that $\lambda(0,u) = u(1_A)$ for any $u \in DA$.

LEMMA 2.3. *$\lambda(T(a,x)) \neq 0$ for any $0 \neq (a,x) \in T$.*

Proof. Suppose that $\lambda(T(a,x))=0$; we show that $a=0$ and $x=0$. For any $y \in DA$, $\lambda((0,y)(a,x)) = \lambda(0,ya) = y(a)$ by the definition of λ, which implies that $(DA)(a) = 0$ and so $a = 0$. Moreover, for $b\in A$, $\lambda((b,0)(0,x)) = \lambda(0,bx) = x(b)$, hence $x(A) = 0$ and so $x = 0$.

LEMMA 2.4. *For $x = (a,0)$, $y = (b,0)$, $a,b \in Q_{\geq 0}$, we have*

(1) *$\lambda(xy) - \lambda(yx) = \alpha(a,b)(s(b)) - \alpha(b,a)(s(a))$ if $s(a) = t(b)$ and $t(a) = s(b)$.*

(2) *$\lambda(xy) = \lambda(yx)$ if $s(a) \neq t(b)$ or $t(a) \neq s(b)$.*

Proof. Let $\varphi(x,y) = \lambda(xy)-\lambda(yx)$. Then $\varphi(x,y) = \lambda(ab,0)-\lambda(ba,0)+\langle a,b\rangle$, because $\lambda(0,\alpha(a,b)) = \alpha(a,b)(1_A)$ and $\lambda(0,\alpha(b,a)) = \alpha(b,a)(1_A)$.

(1) Assume that $s(a) = t(b)$ and $t(a) = s(b)$. It follows from the definition of λ and Lemma 2.1(4) that $\lambda(ab,0) = -\alpha(ab,s(b))(1-s(b)) = -\alpha(a,b)(1-s(b))$, and $\lambda(ba,0) = -\alpha(b,a)(1-s(a))$ similarly. Hence, by Lemma 1.2,

$$\begin{aligned}\varphi(x,y) &= \langle b,a\rangle + \alpha(a,b)(s(b)) - \alpha(b,a)(s(a)) + \langle a,b\rangle\\ &= \alpha(a,b)(s(b)) - \alpha(b,a)(s(a)).\end{aligned}$$

(2) In the case when $s(a) \neq t(b)$, we have $ab = 0$ and so $\varphi(x,y) = -\lambda(ba,0) + \langle a,b\rangle$, where $\langle a,b\rangle = -\langle b,a\rangle = -\langle ba,s(a)\rangle = \langle s(a),ba\rangle$ by Lemma 2.1(3). Thus $\varphi(x,y) = -\lambda(ba,0) + \langle s(a),ba\rangle$. To show that $\varphi(x,y) = 0$, it is obviously enough to consider the case when $ba \neq 0$. If a or $b \in Q_+$, and so $ba \in Q_+$, then $\lambda(ba,0) = \langle s(a),ba\rangle$ by definition and hence $\varphi(x,y) = 0$. If $a,b \in Q_0$, then $a = b = e(i)$ for some $i \in Q_0$. Hence $\lambda(ba,0) = \lambda(e(i),0) = 0$ by definition, and clearly $\langle s(a),ba\rangle = \langle e(i),e(i)\rangle = 0$. Thus in any case $\varphi(x,y) = 0$.

(2.5) *Proof of Theorem 2.2.* The algebra A is spanned by $\{x \mid x \in Q_{\geq 0}\}$ over K, and DA is spanned by $\{x^* \mid x \in Q_{\geq 0}\}$. Since the K-linear map $\lambda : T \to K$ defined in 2.2 satisfies the regularity condition (S1) in Section 1 by Lemma 2.3, we only have to show the symmetry condition (S2). This follows for $x = (a, 0), y = (b, 0)$ with $a, b \in Q_{\geq 0}$ from Lemma 2.4 and the assumption on α. For $x = (a, 0)$ and $y = (0, b^*)$ with $a, b \in Q_{\geq 0}$, this follows from $\lambda(xy) = \lambda(0, ab^*) = b^*(a)$ and $\lambda(yx) = \lambda(0, b^*a) = b^*(a)$. Thus we have proved the theorem.

Let Q be a finite quiver without oriented cycles, and let $Q[X_1, \ldots, X_m]$ be the finite quiver whose vertices are the vertices of Q_0. The set of arrows is the disjoint union of Q_1 and additional arrows $X_1, \ldots, X_m$, where each X_i is an arrow $v_i \to v_i$ for some different vertices $v_1, \ldots, v_m$ of Q. Let A be a K-algebra $KQ[X_1, \ldots, X_m]/I$ where I is an admissible ideal. Observe that the admissible ideal is nothing else than an ideal containing non-constant polynomials $f_i(X_i)$ over K for all i.

PROPOSITION 2.6. *Any extension algebra of A by $\mathrm{Hom}_K(A, K)$ is symmetric.*

Proof. Let α be a 2-cocycle $A \times A \to \mathrm{Hom}_K(A, K)$, and $Q(A)$ the quiver of the algebra A. Only the pairs (X_i^s, X_i^t) for $1 \leq i \leq m$ and $s, t > 0$ are pairs of paths $p, q \in Q_+$ with $s(p) = t(q)$ and $s(q) = t(p)$. Hence, by Theorem 2.2, it is enough to show that $\alpha(x_i^s, x_i^t)(e(v_i)) = \alpha(x_i^t, x_i^s)(e(v_i))$ for any i and $s, t > 0$, where x_i is the residue class of X_i. We fix any one of $\{x_1, \ldots, x_m\}$, denote it by x, and more generally we show

$$\alpha(x^i, x^j)(x^k) = \alpha(x^j, x^i)(x^k) \tag{$**$}$$

for any integers $i, j, k \geq 0$. (This implies the proposition in case $k = 0$.)

Let $e = s(x) = t(x)$. If one of i and j is zero, say $i = 0$, then $(**)$ follows from the 2-cocycle condition $(e, e, x^j)_\alpha = 0$, because $e\alpha(e, x^j) = \alpha(e, e)x^j$ and so $\alpha(e, x^j)(x^k) = \alpha(e, e)(x^{j+k})$. Similarly, from $(x^j, e, e)_\alpha = 0$ we get $\alpha(x^j, e)(x^k) = \alpha(e, e)(x^{j+k})$. Next, we consider the case when $i > 0$ and $j > 0$, and we show $(**)$ by induction on i.

For $i = 1$, from the 2-cocycle condition $(x, x^{j-1}, x)_\alpha = 0$, it follows that $(\alpha(x, x^j) - \alpha(x^j, x))(x^k) = (\alpha(x, x^{j-1}) - \alpha(x^{j-1}, x))(x^{k+1})$. Hence, for $i = 1$, $(**)$ follows by induction on j, because it is trivial for $j = 1$.

For $i > 1$, from the 2-cocycle conditions $(x, x^{i-1}, x^j)_\alpha = 0$ and $(x^j, x^{i-1}, x)_\alpha = 0$, we have

$$\begin{aligned}
&(\alpha(x^i, x^j) - \alpha(x^j, x^i))(x^k) \\
&\qquad = (\alpha(xx^{i-1}, x^j) - \alpha(x^j, x^{i-1}x))(x^k) \\
&\qquad = (\alpha(x, x^{i+j-1}) + x\alpha(x^{i-1}, x^j) - \alpha(x, x^{i-1})x^j)(x^k) \\
&\qquad\quad - (\alpha(x^{i+j-1}, x) + \alpha(x^j, x^{i-1})x - x^j\alpha(x^{i-1}, x))(x^k) \\
&\qquad = [x, x^{i+j-1}](x^k) + [x^{i-1}, x^j](x^{k+1}) + [x^{i-1}, x](x^{k+j}) = 0.
\end{aligned}$$

Hence it follows that $[x^i, x^j](x^k) = 0$ by the induction hypothesis on i.

As a corollary of the proof of Proposition 2.6 we have

Corollary 2.7. *Let $K[X]$ be the polynomial ring with indeterminate X, and f a non-constant polynomial in $K[X]$. Let $A = K[X]/(f)$ be the factor algebra by the ideal generated by f. Then any extension algebra of A by $\mathrm{Hom}_K(A, K)$ is commutative, and hence symmetric.*

Proof. The polynomial ring $K[X]$ is a special case of 2.6, $K[X] = K(A_1[X_1])$. The K-space $A{=}K[X]/(f)$ is spanned by the set $\{x^l \mid 0 \le l < \deg f\}$ and hence $(**)$ in 2.6 implies that $\alpha(x^i, x^j) = \alpha(x^j, x^i)$ for all $i, j \ge 0$ and a 2-cocycle α. It therefore follows that $\alpha(a, b) = \alpha(b, a)$, i.e. $[a, b] = 0$, for any $a, b \in A$. Let T be an extension algebra of A and α the corresponding 2-cocycle. Then, for $(a, u^*), (b, v^*) \in T = A \oplus \mathrm{Hom}_K(A, K)$, $(a, u^*)(b, v^*) = (ab, av^* + u^*b + \alpha(a, b))$. Since A is commutative, we have $(a, u^*)(b, v^*) - (b, v^*)(a, u^*) = (0, [a, b])$, which is zero for $a, b \in \{x^i \mid i \ge 0\}$ from the above observation. Thus we know that T is commutative. Symmetry of T is now trivial.

The algebras considered in 2.6 have quivers whose vertices have at most one circle. The following algebra shows that the assertion in 2.6 is not true in general for algebras with a vertex having two circles.

Let $A = K[X, Y]/(X^2, Y^2)$ be the factor algebra of the polynomial ring with two indeterminates. Let $\alpha(a, b) = a_1 b_2 xy$ for $a, b \in A$, where x, y are the residue classes of X, Y respectively and $a = a_0 + a_1 x + a_2 y + a_3 xy$, $b = b_0 + b_1 x + b_2 y + b_3 xy$ for $a_i, b_i \in K$ $(0 \le i \le 3)$. Then it is easy to see that $\alpha : A \times A \to A$ is a 2-cocycle and A is a symmetric algebra. Let T be the extension algebra of A by A defined by α.

Proposition 2.8. *T is a local non-symmetric self-injective K-algebra.*

Proof. Suppose on the contrary that T is symmetric and $\lambda : T \to K$ is a symmetric regular K-linear map. Since $\lambda((x, 0)(y, 0)) = \lambda((y, 0)(x, 0))$ by symmetry of λ, we have $0 = \lambda(xy - yx, \alpha(x, y) - \alpha(y, x)) = \lambda(0, xy)$. Hence $\lambda(0, Kxy) = 0$, which implies that $\lambda(T(0, xy)) = 0$ because $T(0, xy) = (0, Kxy)$. Therefore, by regularity of λ, we conclude that $xy = 0$, a contradiction.

We now exhibit a symmetric extension algebra which does not satisfy the condition in Theorem 2.2.

Let $K[X, Y, Z]$ be the polynomial ring with three indeterminates, and $I = (X, Y, Z)^2$ an ideal of $K[X, Y, Z]$. Let $A = K[X, Y, Z]/I$, and let $x = \overline{X}$, $y = \overline{Y}$ and $z = \overline{Z}$, where $\overline{f}$ denotes the residue class of a polynomial f. Each element $a \in A$ is a linear combination $a = a_0 + a_1 x + a_2 y + a_3 z$, $a_i \in K$ $(0 \le i \le 3)$.

For $a, b \in A$, we put

$$\alpha(a,b) = (a_1b_2 + a_2b_3 + a_3b_1)1^* - (a_2b_3x^* + a_3b_1y^* + a_1b_2z^*),$$

where $\{1^*, x^*, y^*, z^*\}$ is the dual basis of $DA = \mathrm{Hom}_K(A, K)$ corresponding to the basis $\{1, x, y, z\}$. Then $\alpha : A \times A \to DA$ is a K-bilinear map, and we claim that α is a 2-cocycle. For $a, b, c \in A$, first observe that

$$a\alpha(b,c) = \lambda_0 1^* + \lambda_1 x^* + \lambda_2 y^* + \lambda_3 z^*,$$

where $\lambda_0 = a_0(b_1c_2 + b_2c_3 + b_3c_1) - (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2)$, $\lambda_1 = -a_0b_2c_3$, $\lambda_2 = -a_0b_3c_1$ and $\lambda_3 = -a_0b_1c_2$. Similarly, in the K-dual basis $\{1^*, x^*, y^*, z^*\}$, we have the K-linear combinations corresponding to $\alpha(ab, c)$, $\alpha(a, bc)$ and $\alpha(a, b)c$, which implies that α is a 2-cocycle.

Now let T be an extension K-algebra of A by DA defined by α.

PROPOSITION 2.9. *T is a local symmetric algebra, and $\alpha(x, y)(1_A) \neq \alpha(y, x)(1_A)$.*

Proof. Let $\lambda(a, f) = f(1 + x + y + z)$ for $(a, f) \in T = A \oplus DA$. Then $\lambda((a, f)(b, g) - (b, g)(a, f)) = \lambda(0, [a, b]) = [a, b](1 + x + y + z) = 0$, because $\alpha(a, b)(1 + x + y + z) = (a_1b_2 + a_2b_3 + a_3b_1) - (a_2b_3 + a_3b_1 + a_1b_2) = 0$ by definition and so $\alpha(-,-)(1+x+y+z) = 0$. This implies the symmetry of λ. It is also easy to show the regularity of λ. Moreover, $(\alpha(x, y) - \alpha(y, x))(1_A) = 1_K \neq 0$, because $\alpha(x, y) = 1^* - z^*$ and $\alpha(y, x) = 0$ by definition.

Nakayama [7] considered a class of self-injective K-algebras $A(\lambda)$ ($\lambda \in K$) of dimension 4, where $A(\lambda)$ is defined as the K-algebra which has only one vertex and two arrows α, β with $\alpha^2 = \beta^2 = \alpha\beta - \lambda\beta\alpha = 0$. He proved that $A(\lambda)$ is symmetric if and only if $\lambda = 1$. We recall these algebras $A(\lambda)$ from [13].

Let Λ be the factor K-algebra $K[X]/(X^2)$, and let $D(\lambda) = (K[X]/(X^2))_\sigma$, where σ is an automorphism of Λ with $\sigma(x) = \lambda^{-1}x$ for x being the residue class of X. Then $D(\lambda)$ is a duality Λ-bimodule, $A(\lambda)$ is a split extension algebra of Λ by $D(\lambda)$, and $D(\lambda) \cong \mathrm{Hom}_K(\Lambda, K)_\sigma$. Moreover, $D(\lambda) \cong \mathrm{Hom}_K(\Lambda, K)$ if and only if $\lambda = 1$, and then it follows from Lemma 1.1 that $A(\lambda)$ is symmetric if and only if $\lambda = 1$, as proved by Nakayama. In these algebras, the Nakayama automorphism ν is an automorphism for $A(\lambda)$ satisfying $\nu(\alpha) = \lambda^{-1}\alpha$ and $\nu(\beta) = \lambda\beta$, that is, there is an $A(\lambda)$-bimodule isomorphism $A(\lambda) \cong \mathrm{Hom}_K(A(\lambda), K)_\nu$. By Corollary 2.7 we also know that $A(\lambda)$, $\lambda \neq 1$, is not isomorphic to any extension algebra of Λ by $\mathrm{Hom}_K(\Lambda, K)$.

3. Symmetric extension algebras of non-split algebras. In this section we consider extension algebras of K-algebras which are not necessarily split algebras, but whose ordinary quivers do not contain oriented cycles.

Let L be a finite extension field of K, and Q a finite quiver without oriented cycles. Let A be the path algebra LQ considered as a K-algebra. In this section, by DA we mean the L-dual space $\mathrm{Hom}_L(A, L)$. Take an L-basis $\{e(i), p \mid i \in Q_0,\ p \in Q_+\}$ of A and the dual basis $\{e(i)^*, p^* \mid i \in Q_0,\ p \in Q_+\}$ of the L-space $DA = \mathrm{Hom}_L(A, L)$. For $a \in A$ and $u \in DA$, we write the corresponding L-linear combinations as follows:

$$a = \sum_{i \in Q_0} a_i e(i) + \sum_{p \in Q_+} a_p p, \qquad u = \sum_{i \in Q_0} u_i e(i)^* + \sum_{p \in Q_+} u_p p^* \tag{3.1}$$

where $a_i, a_p, u_i = u(e(i))$ and $u_p = u(p)$ belong to L.

A K-bilinear map $\beta : L \times L \to L$ is a 2-cocycle if and only if so is the composite $\gamma\beta : L \times L \to \mathrm{Hom}_K(L, K)$, where $\gamma : L \xrightarrow{\sim} \mathrm{Hom}_K(L, K)$ is an L-bimodule isomorphism. The K-algebras $T(L, \beta)$ and $T(L, \gamma\beta)$ are isomorphic. For this reason, by a 2-*cocycle of the K-algebra L* we understand a K-bilinear 2-cocycle $L \times L \to L$.

With a 2-cocycle $\beta : L \times L \to L$ there is associated the K-bilinear map $[-,-] : L \times L \to L$ with $[a, b] = \beta(a, b) - \beta(b, a)$. (See Section 1.)

Criterion Lemma 3.2. *Let E be an extension K-algebra defined by a* 2-*cocycle $\beta : L \times L \to L$ of the K-algebra L. Then E is symmetric if and only if $[L, L]$ is a proper subset of L.*

Proof. Assume that $E = L \oplus L$ is symmetric, and let λ be a symmetric regular K-linear map. Then, for any $(a, 0), (b, 0) \in E$,

$$\begin{aligned} 0 &= \lambda((a,0)(b,0)) - \lambda((b,0)(a,0)) = \lambda(ab, \beta(a,b)) - \lambda(ba, \beta(b,a)) \\ &= \lambda((ab, \beta(a,b)) - (ba, \beta(b,a))) = \lambda(0, [a,b]). \end{aligned}$$

Hence $\lambda(0, [L, L]) = 0$. On the other hand, $\lambda(0, L) = \lambda(E(0, 1)) \neq 0$ by (S1). Thus $[L, L]$ is properly contained in L.

Conversely, assume that $[L, L]$ is a proper subset. Take an element $c_0 \in L \setminus [L, L]$ and a K-subspace L' such that $[L, L] \subseteq L'$ and $L = Kc_0 \oplus L'$, a direct sum. Now let $\lambda : E \to K$ be a K-linear map with $\lambda(a, 0) = 0$, $\lambda(0, c) = 0$ and $\lambda(0, c_0) = 1_K$ for $a \in L$ and $c \in L'$. We show that λ satisfies the conditions (S1), (S2).

To show (S1), take a non-zero $x = (a, b) \in E$. If $a \neq 0$, then we have $\lambda((0, c_0 a^{-1})(a, b)) = \lambda(0, c_0) = 1 \neq 0$, and if $a = 0$, then $b \neq 0$ and $\lambda((c_0 b^{-1}, 0)(a, b)) = \lambda(0, c_0) = 1 \neq 0$. Thus $\lambda(Ex) \neq 0$. Next, to show (S2), take any $(a, a'), (b, b') \in E$. Then $\lambda((a, a')(b, b')) - \lambda((b, b')(a, a')) = \lambda(0, [a, b]) = 0$ by the convention that $\lambda(0, L') = 0$. Hence (S2) follows.

Let $\alpha_i : L \times L \to L$ be the K-bilinear map defined by

$$\alpha_i(x, y) = \alpha(xe(i), ye(i))(e(i)),$$

where $\alpha : A \times A \to DA = \mathrm{Hom}_L(A, L)$ is a given 2-cocycle of the K-algebra A. It is easily seen that α_i is a 2-cocycle of the K-algebra L. We denote by $T(L, \alpha_i)$ the extension K-algebra of L by α_i, and by $[\,,\,]_i$ the K-bilinear map $L \times L \to L$ associated with α_i.

LEMMA 3.3. *For any* $i, j, k \in Q_0$, $[Le(i), Le(j)](e(k)) = 0$ *unless* $i = j = k$.

Proof. Assume that some two of $\{i, j, k\}$ are different. Consider the 2-cocycle conditions $(e(i), ae(i), be(j))_\alpha = 0$ and $(e(j), be(j), ae(i))_\alpha = 0$. (See Section1.) Then, if $i = j$ (and so $i \neq k$), we have $\alpha(ae(i), be(i))(e(k)) = \alpha(e(i), abe(i))(e(k))$ and $\alpha(be(i), ae(i))(e(k)) = \alpha(e(i), bae(i))(e(k))$. Hence $[ae(i), be(j)](e(k)) = 0$ for $i = j \neq k$. For $i \neq j$, we consider two cases:

(1) $i \neq k$ and $j \neq k$, and
(2) $i \neq k$ and $j = k$; or $i = k$ and $j \neq k$.

In the case (1), $\alpha(ae(i), be(j))(e(k)) = 0$ and $\alpha(be(j), ae(i))(e(k)) = 0$, which obviously implies $[ae(i), be(j)](e(k)) = 0$. In the case (2), assume that $i = k$ and $j \neq k$. Then $\alpha(ae(i), be(j))(e(k)) = \alpha(ae(i), be(j))(e(i))$ and $\alpha(be(j), ae(i))(e(k)) = -\alpha(e(j), be(j))(ae(i))$. On the other hand, $\alpha(ae(i), be(j))(e(i)) + \alpha(e(j), be(j))(ae(i)) = 0$, because of the 2-cocycle condition $(ae(i), e(j), be(j))_\alpha = 0$.

LEMMA 3.4. $\langle \mathrm{rad}\, A, - \rangle = 0$ *if* $\langle \mathrm{rad}\, A, Le(i) \rangle = 0$ *for all* i.

Proof. Since A is spanned by $\{e(i), p \mid i \in Q_0,\ p \in Q_+\}$ over L, it is enough to show that $\langle \mathrm{rad}\, A, ap \rangle = 0$ for any $a \in L$ and $p \in Q_+$. For this, we only have to show that $\langle ap, bq \rangle = 0$ for $a, b \in L$ and $p, q \in Q_+$. First, observe that

$$\langle ap, bq \rangle = \langle abpt(q), q \rangle \quad \text{for any } a, b \in L \text{ and } p, q \in Q_+.$$

In fact, it follows from Lemma 1.2 that for the triplet $(ap, bt(q), q)$,

$$\langle ap, bq \rangle + \langle bt(q), aqp \rangle + \langle q, abpt(q) \rangle = 0,$$

where $\langle bt(q), aqp \rangle = 0$ by assumption. There is nothing to prove in the case when $s(p) \neq t(q)$ because then $pt(q) = 0$. So assume that $s(p) = t(q)$. Then it follows from the above observation that $\langle q, abp \rangle = \langle abqt(p), p \rangle$. On the other hand, $s(q) \neq t(p)$ because $s(p) = t(q)$ by assumption and there are no oriented cycles in Q. Therefore $qt(p) = 0$, and we have $\langle ap, bq \rangle = -\langle q, abpt(q) \rangle = -\langle abqt(p), p \rangle = 0$, as desired.

LEMMA 3.5. *Assume that* $T(A, \alpha)$ *is symmetric and* $\lambda : T(A, \alpha) \to K$ *is a symmetric regular* K*-linear map. Then*:

(a) $\lambda(0, xe(i)^*) = \lambda(0, xe(j)^*)$ *for all* $i, j \in Q_0$ *and* $x \in L$.
(b) $\lambda(0, Lp^*) = 0$ *for all* $p \in Q_+$.
(c) $\lambda(0, u) = \lambda(0, u(1_A)e(i)^*)$ *for all* $i \in Q_0$ *and* $u \in DA$.
(d) $\lambda(0, (\sum_i [L, L]_i)e(j)^*) = 0$ *for all* $j \in Q_0$.

(e) $\sum_i [L,L]_i \neq L$.

(f) *The K-algebra $T(L,\alpha_i)$ is symmetric for all $i \in Q_0$.*

P r o o f. (a) and (b) follow from the fact that, for any $i \neq j$ and a path p from i to j and $x \in L$,

$$0 = \lambda((xp,0)(0,p^*) - (0,p^*)(xp,0)) = \lambda(0, xe(j)^* - xe(i)^*)$$

and

$$0 = \lambda((xe(i),0)(0,p^*) - (0,p^*)(xe(i),0)) = \lambda(0, xp^*).$$

(c) Since $u = \sum_i u(e(i))e(i)^* + \sum_p u(p)p^*$, this follows from (a) and (b).

(d) Since $[Le(i), Le(i)](1_A) = [Le(i), Le(i)](e(i)) = [L,L]_i$ by Lemma 3.3, we have $\lambda(0, \sum_i [L,L]_i e(j)^*) = 0$ because

$$\lambda\Big(0, \sum_i [Le(i), Le(i)](1_A)e(j)^*\Big) = 0$$

by the symmetry of λ and (c).

(e) If $\sum_i [L,L]_i = L$, we have $\lambda(0, Le(k)^*) = \lambda(0, \sum_i [L,L]_i e(k)^*) = 0$ by (d). Then $\lambda(T(0, e(k)^*)) = \lambda(0, Le(k)^*) = 0$, a contradiction to the regularity of λ, where $T = T(A,\alpha)$.

(f) This is an immediate consequence of (e) and the Criterion Lemma because $[L,L]_i \neq L$ for any i.

LEMMA 3.6. *Let $T(A,\alpha)$ be an extension K-algebra. Then, if $\sum_i [L,L]_i \neq L$, there exist symmetric regular K-linear maps $\lambda_i : T(L,\alpha_i) \to K$ satisfying $\lambda_i = \lambda_j$ for any i,j.*

P r o o f. Take an element $z_0 \in L \setminus \sum_i [L,L]_i$ and a K-subspace L' such that $\sum_i [L,L]_i \subseteq L'$ and $L = Kz_0 \oplus L'$, a direct sum. Let $\lambda_i : T(L,\alpha_i) \to K$ be a K-linear map with $\lambda_i(x,0) = \lambda_i(0,y) = 0$ and $\lambda_i(0,z_0) = 1_K$ for $x \in L$ and $y \in L'$. Then $\lambda_i = \lambda_j$, and every λ_i is regular and symmetric as in the proof of Lemma 3.2, because $[L,L]_i \subseteq \sum_i [L,L]_i \subset L$.

PROPOSITION 3.7. *Consider the following three conditions for an extension algebra $T(A,\alpha)$:*

(1) $\sum_{i\in Q_0} [L,L]_i \neq L$.

(2) *There are symmetric regular K-linear maps $\lambda_i : T(L,\alpha_i) \to K$ satisfying $\lambda_i(0,-) = \lambda_j(0,-)$ for all $i,j \in Q_0$.*

(3) *$T(A,\alpha)$ is symmetric.*

Then the implications (3)$\Rightarrow$(1)$\Rightarrow$(2) *hold. Moreover*, (2) *implies* (3) *if* $[\operatorname{rad} A, e(i)](1_A) = 0$.

P r o o f. The implications (3)$\Rightarrow$(1) and (1)$\Rightarrow$(2) are proved in Lemmas 3.5 and 3.6, respectively. Now assume that $[\operatorname{rad} A, e(i)](1_A) = 0$, and that there are symmetric regular K-linear maps $\lambda_i : T(L,\alpha_i) \to K$ such that $\lambda_i(0,-) =$

$\lambda_j(0,-)$ for any $i,j\in Q_0$. Fix $k\in Q_0$ and take a K-linear map $\lambda : T(A,\alpha) \to K$ with

$$\lambda(a,u) = \lambda_k(0,u(1_A))$$

for any $a \in A$ and $u \in DA$; we show that λ is regular and symmetric.

Let $a = \sum_{i\in Q_0} a_i e(i) + a'$, where $a_i \in L$ and $a' \in \operatorname{rad} A$, and let $T = T(A,\alpha)$ and $T_i = T(L,\alpha_i)$. Then, for $(a,u),(b,v) \in T$,

$$[a,b] = \sum_{i,j}[a_ie(i), b_je(j)] + \sum_i([a_ie(i), b'] + [a', b_ie(i)]) + [a', b'].$$

Hence, $[a,b](1_A) = \sum_{i,j,s\in Q_0}[a_ie(i), b_je(j)](e(s)) = \sum_i[a_ie(i), b_ie(i)](e(i))$ by Lemmas 3.3 and 3.4 and the assumption. It follows that

$$\begin{aligned}\lambda((a,u)(b,v) - (b,v)(a,u)) &= \lambda_k(0,[a,b](1_A))\\ &= \sum_i \lambda_k(0,[a_ie(i), b_ie(i)](e(i)))\\ &= \sum_i \lambda_i(0,[a_ie(i), b_ie(i)](e(i)))\end{aligned}$$

by the assumption on λ_i's. Since λ_i's are symmetric, this implies that λ is symmetric. Also, the regularity of λ_i's implies the regularity of λ. In fact, $\lambda(T(a,u)) \supseteq \lambda_k(0,L) = \lambda_k(T_i(0,1_L)) \neq 0$ for $(a,u) \neq 0$, because $\operatorname{Hom}_L(A,L)(a) = L$ for $0 \neq a \in A$.

We have considered the symmetry of an extension algebra $T(A,\alpha)$ by making use of local data, namely, symmetry of $T(L,\alpha_i)$. Conversely, we now consider a construction of symmetric extension algebras by using given local data. We use an idea from [9].

Let L be a finite extension field of the field K, and $\beta : L \times L \to L$ be a 2-cocycle of the K-algebra L. For any finite quiver Q without oriented cycles, we define a K-bilinear map $\widehat{\beta} : LQ \times LQ \to \operatorname{Hom}_L(LQ,L)$ by

$$\widehat{\beta}(a,b) = \sum_{i\in Q_0} \beta(e(i)ae(i), e(i)be(i))e(i)^*$$

for $a,b \in LQ$. Then it is easy to see that $\widehat{\beta}$ is a 2-cocycle of the K-algebra LQ.

PROPOSITION 3.8. *The extension algebra $T(LQ,\widehat{\beta})$ is non-splittable for any finite quiver Q without oriented cycles if so is $T(L,\beta)$.*

Proof. This is proved as in [9, 6.1].

THEOREM 3.9. *The following conditions are equivalent*:

(1) *The extension K-algebra $T(L,\beta)$ is symmetric.*

(2) *The extension K-algebra $T(LQ,\widehat{\beta})$ is symmetric for some finite quiver Q without oriented cycles.*

(3) *The extension K-algebra $T(LQ,\widehat{\beta})$ is symmetric for any finite quiver Q without oriented cycles.*

Proof. Consider an extension K-algebra $T(LQ,\widehat{\beta})$, and let $\widehat{\beta}_i : Le(i)\times Le(i)\to \mathrm{Hom}_L(Le(i),L)$ be the restriction of $\widehat{\beta}$. Then $\widehat{\beta}_i(e(i)ae(i),e(i)be(i)) = \beta(a,b)e(i)^*$, so that $T(L,\widehat{\beta}_i)=T(L,\beta)$ for any i, which implies the theorem because the 2-cocycle $\widehat{\beta}$ obviously satisfies the condition in Proposition 3.7.

Finally we exhibit an algebra which shows that the implication (2)⇒(3) in Proposition 3.7 is not true without any additional condition in general.

Example 3.10. Let $K=\mathbb{Z}_2(x,y)$ be a rational function field with two invariants over the prime field $\mathbb{Z}_2$ of characteristic 2, and consider the factor ring $L=K[X,Y]/(X^2-x,Y^2-y)$. Define 2-cocycles $\alpha_1,\alpha_2 : L\times L\to L$ by

$$\alpha_1(\overline{X}^l\overline{Y}^m,\overline{X}^{l'}\overline{Y}^{m'}) = lm'\overline{X}^{l+l'}\overline{Y}^{m+m'},$$
$$\alpha_2(\overline{X}^l\overline{Y}^m,\overline{X}^{l'}\overline{Y}^{m'}) = lm'\overline{X}^{l+l'-1}\overline{Y}^{m+m'},$$

where $\overline{X},\overline{Y}$ denote the residue classes of X,Y respectively. Now let Q be a quiver of Dynkin type A_2 and let $\alpha : A=LQ\to\mathrm{Hom}_L(A,L)$ be a 2-cocycle defined by $\alpha(a,b)=\sum_i\alpha_i(a_i,b_i)e(i)^*$, where a_i and b_i are as in (3.1). Then it is easy to see that $[L,L]_1 = K\overline{X}\oplus K\overline{Y}\oplus K\overline{X}\,\overline{Y}$ and $[L,L]_2=K1_L\oplus K\overline{Y}\oplus K\overline{X}\,\overline{Y}$, which implies that $\sum_i[L,L]_i=L$. It then follows from Proposition 3.7 that $T=T(A,\alpha)$ is not symmetric, but $T_i=T(L,\alpha_i)$ is symmetric by Criterion Lemma 3.2. Moreover, we know that T_i is non-splittable by Lemma 1.3, because $\alpha_1(\overline{X},\overline{Y})=\overline{X}\,\overline{Y}$, $\alpha_1(\overline{Y},\overline{X})=0$, and $\alpha_2(\overline{X},\overline{Y})=\overline{Y}$, $\alpha_2(\overline{Y},\overline{X})=0$.

4. Examples. In the preceding section we considered extension algebras of a K-algebra L which is a finite extension field of K. In this section we exhibit some extension algebras of a specified extension field L of K.

Throughout this section, L is a finite extension field of a fixed field K, and a K-bilinear map $[-,-] : L\times L\to L$ is associated with a fixed 2-cocycle $\alpha : L\times L\to L$ of the K-algebra L (see Section 1). By definition, obviously $[a,a]=0$ and $[a,b]=-[b,a]$ for any $a,b\in L$.

Lemma 4.1. (1) *For $a,b,c\in L$, we have*

$$[a,bc]=[a,b]c+[a,c]b \quad\text{and}\quad [ab,c]=a[b,c]+b[a,c].$$

(2) *For $0\neq a, 0\neq b\in L$ and $l,l',m,m'\in\mathbb{Z}$, we have*

(i) $[a^l,b^m]=lma^{l-1}b^{m-1}[a,b]$ *and* $[a^l,a^m]=0$.
(ii) $[a^lb^m,a^{l'}b^{m'}]=(lm'-ml')a^{l+l'-1}b^{m+m'-1}[a,b]$.

P r o o f. (1) By the definition of 2-cocycle,

$$\begin{aligned}[a,bc] &= \alpha(a,bc)-\alpha(bc,a)\\ &= (\alpha(ab,c)+\alpha(a,b)c-a\alpha(b,c))-(b\alpha(c,a)+\alpha(b,ca)-\alpha(b,c)a)\\ &= (\alpha(ab,c)+\alpha(a,b)c-b\alpha(c,a))-\alpha(b,ca)\\ &= (\alpha(ab,c)+\alpha(a,b)c-b\alpha(c,a))-(\alpha(ab,c)+\alpha(b,a)c-b\alpha(a,c))\\ &= [a,b]c+[a,c]b.\end{aligned}$$

(2) (i) First, we show the case $m=1$ by induction on l. It is obviously true for $l=0$, because $[1,x]=0$ for any $x\in L$ by (1) where $a=x$ and $b=c=1$. Assume that $l>0$; then $[a^l,b]=a[a^{l-1},b]+a^{l-1}[a,b]$ by (1). It follows from the induction hypothesis that

$$[a^l,b]=(l-1)a^{l-1}[a,b]+a^{l-1}[a,b]=la^{l-1}[a,b].$$

Next, assume that $l<0$. Then $[a^l,b]=[(a^{-1})^{-l},b]=(-l)(a^{-1})^{-l-1}[a^{-1},b]$ $=(-l)a^{l+1}[a^{-1},b]$. Moreover, by (1), $[a^{-1},b]=a^{-1}a[a^{-1},b]=a^{-1}([1,b]-a^{-1}[a,b])=-a^{-2}[a,b]$. Hence $[a^l,b]=la^{l-1}[a,b]$.

Now consider $m\in\mathbb{Z}$. By the result just proved above, $[a^l,b^m]=la^{l-1}[a,b^m]=-la^{l-1}[b^m,a]$ and $[b^m,a]=mb^{m-1}[b,a]=-mb^{m-1}[a,b]$. Thus $[a^l,b^m]=lma^{l-1}b^{m-1}[a,b]$ and so $[a^l,a^m]=0$ because $[a,a]=0$.

(ii) By (1) we have

$$\begin{aligned}[a^lb^m,a^{l'}b^{m'}] &= a^l[b^m,a^{l'}b^{m'}]+b^m[a^l,a^{l'}b^{m'}]\\ &= a^{l+l'}[b^m,b^{m'}]+a^lb^{m'}[b^m,a^{l'}]\\ &\quad + a^{l'}b^m[a^l,b^{m'}]+b^{m+m'}[a^l,a^{l'}],\end{aligned}$$

where $[b^m,b^{m'}]=0=[a^l,a^{l'}]$ by (i). Hence, by (i) again,

$$\begin{aligned}[a^lb^m,a^{l'}b^{m'}] &= a^lb^{m'}[b^m,a^{l'}]+a^{l'}b^m[a^l,b^{m'}]\\ &= ml'a^{l+l'-1}b^{m+m'-1}[b,a]+lm'a^{l+l'-1}b^{m+m'-1}[a,b]\\ &= (lm'-ml')a^{l+l'-1}b^{m+m'-1}[a,b].\end{aligned}$$

THEOREM 4.2. *If the field L is generated by at most two elements over K, then any extension algebra of the K-algebra L by the L-bimodule L is symmetric.*

P r o o f. Let $\alpha: L\times L\to L$ be a K-bilinear 2-cocycle of L. In the case when L is a simple extension field of K, we obviously have $[L,L]=0$ by Lemma 4.1(2), and so $[L,L]$ is properly contained in L. Hence our assertion follows from Lemma 3.2.

Next, let $L=K(a,b)$ for some a,b, and $d_1=[K(a):K]$, $d_2=[K(a,b):K(a)]$. Assume that L/K is not a simple extension field. Then the characteristic of K is a non-zero prime, say p; take a K-basis $\{a^lb^m \mid 0\le l<d_1, 0\le m<d_2\}$ of L. Since $[a^lb^m,\ a^{l'}b^{m'}]=(lm'-ml')a^{l+l'-1}b^{m+m'-1}[a,b]$

by Lemma 4.1(2), we have $[L,L] = I[a,b]$, where I is the K-subspace of L spanned by the set

$$\{(lm' - ml')a^{l+l'-1}b^{m+m'-1} \mid 0 \le l, l' < d_1,\ 0 \le m, m' < d_2\}.$$

Thus, by Lemma 3.2, it is enough to show that $I[a,b]$ is properly contained in L for $[a,b] \neq 0$.

Assume that $[a,b] \neq 0$; we use the following lemma, where $r(n)$ denotes the remainder of an integer n divided by p.

Lemma 4.3. *The following conditions hold for any non-negative integers l and m:*

(1) *p is a common divisor of d_1 and d_2.*
(2) *$a^l = \sum_{0\le pi<d_1-r(l)} \lambda_i a^{pi+r(l)}$ for $\lambda_i \in K$.*
(3) *$a^l b^m = \sum_{\substack{0\le pi<d_1-r(l)\\ 0\le pj<d_2-r(m)}} \lambda_{i,j} a^{pi+r(l)} b^{pj+r(m)}$ for some $\lambda_{i,j} \in K$.*

Proof. (1) will be shown in the proofs of (2) and (3).

(2) For an integer l with $0 \le l < d_1$, let $l = pi + r(l)$. Then obviously $a^l = a^{pi+r(l)}$ is of the required form. Now let $a^{d_1} = \sum_{0\le i<d_1} \lambda_i a^i$ for some $\lambda_i \in K$. It follows from Lemma 4.1 that

$$d_1 a^{d_1-1}[a,b] = [a^{d_1}, b] = \Big[\sum_i \lambda_i a^i, b\Big] = \sum_i i\lambda_i a^{i-1}[a,b].$$

Hence $d_1 a^{d_1-1} = \sum_i i\lambda_i a^{i-1}$, which implies that $d_1 1_K = 0$ and $i\lambda_i = 0$ for $0 \le i < d_1$. Hence p divides d_1 and $\lambda_i = 0$ for any i not divided by p. Thus we have

$$a^{d_1} = \sum_{0\le pi<d_1} \lambda_{pi} a^{pi},$$

which is also a required form because $r(d_1) = 0$.

Next, for $l > d_1$, let $l = d_1 + l'$. Then $a^l = a^{d_1}a^{l'} = \sum_{0\le pi<d_1} \lambda_i a^{pi+l'}$, where $r(pi + l') = r(l')$, $r(l') = r(d_1 + l') = r(l)$ because $p \mid d_1$, and so $r(pi+l') = r(l)$. Since $pi+l' < l$, the statement (2) then follows by induction on l.

(3) For a non-negative integer $m < d_2$, let $m = pj + r(m)$ and $b^m = b^{pj+r(m)}$. Then clearly the required statement follows from (2). Next, we show that d_2 is divisible by p and that

$$b^{d_2} = \sum_{\substack{0\le pi<d_1\\ 0\le pj<d_2}} \lambda_{i,j} a^{pi} b^{pj}. \tag{${}^{*}_{**}$}$$

Let $b^{d_2} = \sum_{0\le j<d_2} \xi_j(a) b^j$ with some $\xi_j(a) \in K(a)$. Then, by a similar

argument to the proof of (2), we have

$$d_2 b^{d_2-1}[b,a] = \sum_j j\xi_j(a) b^{j-1}[b,a],$$

and hence it follows from the assumption $[b,a] \neq 0$ that

$$b^{d_2} = \sum_{p|j,\, 0\le j<d_2} \xi_j(a) b^j.$$

Let $\xi_j(a) = \sum_{0\le i<d_1} \xi_{ji} a^i$ for some $\xi_{ji} \in K$. Then, by Lemma 4.1, we have

$$0 = [b^{d_2}, b] = \sum_{p|j,\, 0\le j<d_2} \Big(\sum_{0\le i<d_1} \xi_{ji}[a^i b^j, b] \Big) = \sum_{p\,|\,j} \sum_i i\xi_{ji} a^{i-1} b^j [a,b].$$

Thus $\xi_j(a) = \sum_{p|i,\, 0\le i<d_1} \xi_{ji} a^i$, because $i\xi_{ji} = 0$ for i,j with $p\,|\,j$, which implies $({}^{\ *}_{**})$.

Finally, assume that $m > d_2$ and let $m = d_2 + m'$. It then follows from $({}^{\ *}_{**})$ and (2) that

$$a^l b^m = a^l b^{d_2} b^{m'} = \sum_{\substack{0\le pi<d_1 \\ 0\le pj<d_2}} \lambda_{i,j} a^{pi+r(l)} b^{pj+m'},$$

where $pj + m' < m$. Thus, by (2) and induction on m, we deduce the statement (3).

(4.4) *Proof of Theorem 4.2.* Since $lm' - ml' \equiv 0 \pmod p$ for $l + l'$, $m + m' \in p\mathbb{Z}$, we have

$$(lm' - ml') a^{l+l'-1} b^{m+m'-1} = 0.$$

If $l + l'$ or $m + m'$ does not belong to $p\mathbb{Z}$, then $0 \le r(l + l' - 1) \le p - 2$ or $0 \le r(m + m' - 1) \le p - 2$. Hence $I \subseteq \widetilde{I}$, where $\widetilde{I}$ is a K-subspace spanned by the set $\{a^{pi+r} b^{pj+r'} \mid i,j \in \mathbb{Z},\ 0 \le r \le p-2 \text{ or } 0 \le r' \le p-2\}$. This implies that $a^{pi-1} b^{pj-1}$ does not belong to $\widetilde{I}$, because $r(pi-1) = p-1$ and $r(pj-1) = p-1$. Therefore $\widetilde{I} \subset L$ and so $I \subset L$.

Let $L = K(a_1, \ldots, a_m)$ be generated by m elements a_i over K. Theorem 4.2 shows that any extension algebra of the K-algebra L is symmetric if $m \le 2$. We show below that Theorem 4.2, in general, is not true for $m \ge 3$.

Let $K = \mathbb{Z}_2(a,b,c)$ be the rational function field with three invariants a, b, c over the prime field $\mathbb{Z}_2$, and let L be the factor ring $K[X,Y,Z]/(X^2 - a, Y^2 - b, Z^2 - c)$. Let $x = \overline{X}$, $y = \overline{Y}$ and $z = \overline{Z}$, where $\bar{f}$ denotes the residue class of f in L. We define a K-bilinear form $\alpha : L \times L \to L$ by

$$\alpha(x^l y^m z^n, x^{l'} y^{m'} z^{n'}) = x^{l+l'-1} y^{m+m'-1} z^{n+n'-1}(lm'z + mn'xy)$$

where the numbers $l, m, n,\ l', m', n'$ are 0 or 1.

LEMMA 4.5. *α is a 2-cocycle.*

P r o o f. It is enough to show the relation

$$f\alpha(g,h) - \alpha(fg,h) = \alpha(f,g)h - \alpha(f,gh)$$

for f, g, h from the K-basis $\{x^l y^m z^n \mid l, m, n = 0 \text{ or } 1\}$ of L, and this is a direct consequence of the definition of α.

PROPOSITION 4.6. *The extension algebra $T(L,\alpha)$ of the K-algebra L is not symmetric.*

P r o o f. From the definition of α we have the following data for the K-basis $\{1, x, y, z, yz, zx, xy, xyz\}$ of L: $[x,y] = 1$, $[x,xy] = x$, $[y,z] = y$, $[x,yz] = z$, $[xy,z] = xy$, $[yz,z] = yz$, $[x,xyz] = zx$, $[xyz,z] = xyz$. Thus $[L,L] = L$, which implies that $T(L,\alpha)$ is not symmetric by Criterion Lemma 3.2.

The above construction of the extension algebra $T(L,\alpha)$ also applies to the case of the field K of arbitrary non-zero characteristic p.

Let $K = \mathbb{Z}_p(a,b,c)$ with invariants a, b, c, and let $L = K[X,Y,Z]/(X^p - a, Y^p - b, Z^p - c)$ as before. A 2-cocycle $\alpha : L \times L \to L$ of the K-algebra L is defined as in Lemma 4.5. Then, for the K-basis $\{x^l y^m z^n \mid 0 \le l, m, n < p\}$ of L, for $0 \le n < p$ we have

$$x^l y^m z^n = \begin{cases} \dfrac{1}{l(m+1)}[x^l, xy^{m+1}z^n] & \text{for } 1 \le l \le p-1,\ 0 \le m \le p-2, \\ \dfrac{1}{(l+1)(m+1)}[x^{l+1}, y^{m+1}z^n] & \text{for } 0 \le l, m \le p-2, \\ \dfrac{1}{m}[x^l y^m z^n, z] & \text{for } 0 \le l \le p-1,\ 1 \le m \le p-1. \end{cases}$$

This implies that $[L,L]$ contains all $x^l y^m z^n$, so that $[L,L] = L$. Hence $T(L,\alpha)$ is not symmetric by Criterion Lemma 3.2.

Finally we mention some questions related to extension algebras.

(1) For a simple K-algebra A, it is known that the cohomology group $H^n(A,A)$ is not zero for any integer $n > 0$ (cf. [5]). Since A is a symmetric algebra, $A \cong \mathrm{Hom}_K(A,K)$ as A-bimodules; the above non-split extension algebras of K-algebras L show the fact for $n = 2$, i.e. $H^2(A,A) \neq 0$.

(2) Let L be a finite extension field K with a 2-cocycle $\alpha : L \times L \to L$ of the K-algebra L such that $T(L,\alpha)$ is not symmetric (Proposition 4.6), and let Q be an arbitrary finite quiver without oriented cycles. Then $T(LQ,\alpha)$ is stably equivalent to $T(LQ,0)$ (see [12], [13]). But, by a theorem of Rickard [8], $T(LQ,\alpha)$ and $T(LQ,0)$ are not derived equivalent, because $T(LQ,0)$ is symmetric. Thus in general a self-injective algebra being stably equivalent to a self-injective algebra of tilted type in the sense of [4]

does not imply a derived equivalence between them (cf. [10]). Moreover, it also shows that a self-injective algebra is not necessarily symmetric even if it is stably equivalent to a symmetric algebra.

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