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VARIETIES OF IDEMPOTENT GROUPOIDS WITH SMALL CLONES

BY

J. GAŁUSZKA (GLIWICE)

Abstract. We give an equational description of all idempotent groupoids with at most three essentially n-ary term operations.

1. Introduction. The notion of p_n-sequences is connected with the concept of compositions of algebraic operations contained in the papers of W. Sierpiński [22], E. Marczewski [18] and G. Grätzer [12]. The main problem connected with representability of p_n-sequences was formulated by G. Grätzer in [13] (Problem 42). The problem is still open. The most general case was solved by A. Kisielewicz in [17]. G. Grätzer and A. Kisielewicz devoted a considerable part of their survey paper [14] to representability of p_n-sequences. Many authors were interested in p_n-sequences of idempotent algebras with small rates of growth (e.g. [1, 2, 19, 20, 6, 10, 9, 8, 16, 15]).

In [8] a description of idempotent groupoids with $p_2 \leq 2$ is given. In this paper we present a full characterization of idempotent groupoids with $p_2 \leq 3$.

The notations and notions used in this paper are standard and follow [14]. Recall that $p_n = p_n(\boldsymbol{A})$ denotes the number of all essentially n-ary term operations of a given algebra $\boldsymbol{A}$ for $n \geq 1$ and $p_0(\boldsymbol{A})$ is the number of all unary constant term operations in $\boldsymbol{A}$.

A commutative idempotent groupoid $\boldsymbol{G} = (G, \cdot)$ satisfying $xy^2 = x$ is called a *Steiner quasigroup*; if $\boldsymbol{G}$ satisfies $xy^2 = xy$, then $\boldsymbol{G}$ is called a *near-semilattice*. Similarly to [8] we use the following notation: for a given groupoid $\boldsymbol{G}$ we write xy^n instead of $(\ldots(xy)\ldots)y$ and ${}^n yx$ instead of $y(\ldots(yx)\ldots)$ where y appears n times. Recall that $\boldsymbol{G}$ is a *proper* groupoid if $\mathrm{card}(G) \geq 1$ and the operation "$\cdot$" depends on both its variables. (In the whole paper we assume that the groupoids $\boldsymbol{G}$ are proper.) For a given groupoid $\boldsymbol{G} = (G, \cdot)$ with the fundamental operation xy we consider the dual groupoid $\boldsymbol{G}^{\mathrm{d}} = (G, \circ)$ where $x \circ y = yx$. If K is a class of groupoids, then K^{d} denotes the class of all groupoids $\boldsymbol{G}^{\mathrm{d}}$ such that $\boldsymbol{G} \in K$. Following [21], we say that an identity is *regular* if the sets of variables on both sides coincide. Otherwise we say that the identity is *nonregular*.

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Note that to find $A^{(2)}(\boldsymbol{G})$, the set of all binary term operations over $\boldsymbol{G}$, we use the following formula (cf. [18]):

$$A^{(2)}(\boldsymbol{G}) = \bigcup_{k=0}^{\infty} A_k^{(2)}(\boldsymbol{G}), \tag{1}$$

where $A_0^{(2)}(\boldsymbol{G}) = \{x, y\}$ and $A_{k+1}^{(2)}(\boldsymbol{G}) = A_k^{(2)}(\boldsymbol{G}) \cup \{fg \mid f, g \in A_k^{(2)}(\boldsymbol{G})\}$ (we use the convention: $e_1^2(x,y) = x$, $e_2^2(x,y) = y$).

We use also the following notation:

- $\mathcal{G}$ denotes the class of all groupoids,
- $\mathcal{G}_{\mathrm{C}}$ denotes the class of all commutative groupoids,
- $\mathcal{G}_{\mathrm{I}}$ denotes the class of all idempotent groupoids,
- $\mathcal{G}_{\mathrm{IC}}$ denotes the class of all idempotent commutative groupoids,
- $\mathcal{G}_{\mathrm{I\check{C}}}$ denotes the class of all idempotent noncommutative groupoids.

If $\mathcal{K}$ is a subclass of $\mathcal{G}$ then $\mathcal{K}_{(p_n=m)}$ denotes the subclass of $\mathcal{K}$ defined by the condition $p_n = m$. For example $\mathcal{G}_{\mathrm{I}(p_2=2)}$ denotes the class of all idempotent groupoids having exactly two essentially binary term operations. Similarly $\mathcal{G}_{(p_n \le m)}$ denotes the class of all groupoids with no more than m essentially n-ary term operations. The *clone* of an algebra $\boldsymbol{A}$, denoted by $\mathrm{cl}(\boldsymbol{A})$, is the set of all term operations of $\boldsymbol{A}$. *Minimal clones* are atoms in the lattice of all clones on a set A having more than one element (cf. [14]).

2. Main result. Let us recall the results of J. Dudek summarized in [8].

THEOREM 2.1 ([8]). *Let $\boldsymbol{G} \in \mathcal{G}_{\mathrm{I}}$ (i.e. $\boldsymbol{G}$ is idempotent (and proper)). Then $p_2(\boldsymbol{G}) \le 1$ if and only if $\boldsymbol{G}$ belongs to one of the following varieties*:

- $\mathcal{G}_1^1$: $xy = yx,\ xy^2 = x$ (*the variety of Steiner quasigroups*);
- $\mathcal{G}_2^1$: $xy = yx,\ xy^2 = xy$ (*the variety of near-semilattices*).

Thus $\mathcal{G}_{\mathrm{I}(p_2 \le 1)} = \mathcal{G}_1^1 \cup \mathcal{G}_2^1$.

THEOREM 2.2 ([8]). *Let $\boldsymbol{G} \in \mathcal{G}_{\mathrm{I}}$. Then $p_2(\boldsymbol{G}) \le 2$ if and only if $\boldsymbol{G}$ belongs to one of the following varieties*:

- $\mathcal{G}_1^2$: $xy^2 = x,\ xy = (xy)x = x(yx),\ {}^2xy = (xy)(yx) = x$;
- $\mathcal{G}_2^2$: $xy^2 = y,\ (xy)(yx) = (xy)x = x,\ xy = {}^2xy = y(xy)$;
- $\mathcal{G}_3^2$: $xy^2 = y,\ (xy)x = x,\ xy = {}^2xy = y(xy) = (yx)(xy)$;
- $\mathcal{G}_4^2$: $xy^2 = y,\ xy = (yx)y = y(xy) = {}^2xy = (yx)(xy)$;
- $\mathcal{G}_5^2$: $xy^2 = xy,\ (xy)x = x(yx) = (xy)(yx) = x$;
- $\mathcal{G}_6^2$: $xy^2 = xy = (xy)x = x(yx) = {}^2xy = (xy)(yx)$;
- $\mathcal{G}_7^2$: $xy^2 = yx,\ (xy)x = x(yx) = y,\ {}^2xy = yx,\ (xy)(yx) = x$;
- $\mathcal{G}_8^2$: $xy^2 = x,\ xy = yx$ (*the variety of Steiner quasigroups*);
- $\mathcal{G}_9^2$: $xy^2 = yx^2,\ xy = yx,\ xy^2 = xy^3$ (*the variety* $\mathbf{N}_2$);

or to one of the varieties $\mathcal{G}_i^{2\mathrm{d}}$ ($i = 1, \ldots, 9$).

From this result it is not difficult to infer that

$$\begin{aligned}\mathcal{G}_{\mathrm{I}\,(p_2\le 2)} &= \mathcal{G}_1^2 \cup \ldots \cup \mathcal{G}_9^2 \cup \mathcal{G}_1^{2\mathrm{d}} \cup \ldots \cup \mathcal{G}_9^{2\mathrm{d}},\\ \mathcal{G}_{\mathrm{IC}\,(p_2\le 2)} &= \mathcal{G}_8^2 \cup \mathcal{G}_9^2,\\ \mathcal{G}_{\mathrm{I\check{C}}\,(p_2\le 2)} &= (\mathcal{G}_1^2 \cup \ldots \cup \mathcal{G}_7^2 \cup \mathcal{G}_1^{2\mathrm{d}} \cup \ldots \cup \mathcal{G}_7^{2\mathrm{d}}) - \mathcal{G}_2^1.\end{aligned}$$

Now we formulate our main result.

THEOREM 2.3. *Let $\boldsymbol{G} \in \mathcal{G}_{\mathrm{I}}$. Then $p_2(\boldsymbol{G}) \le 3$ if and only if $\boldsymbol{G}$ belongs to $\mathcal{G}_{\mathrm{I}\,(p_2\le 2)}$ or to one of the following varieties*:

(*commutative case*:)

- $\mathcal{G}_1^3$: $xy = yx,\ xy^2 = yx^2,\ xy^3 = xy^4$;
- $\mathcal{G}_2^3$: $xy = yx,\ xy = xy^3,\ xy^2 = (xy^2)x = x(xy)^2$;
- $\mathcal{G}_3^3$: $xy = yx,\ xy = (xy^2)(yx^2) = (xy^2)x,\ xy^2 = xy^3$;
- $\mathcal{G}_4^3$: $xy = yx,\ xy^2 = yx^3,\ xy^4 = x$;

(*noncommutative case*:)

- $\mathcal{G}_5^3$: $xy = (xy)x = x(yx) = {}^2xy,\ xy^2 = yx^2$;
- $\mathcal{G}_6^3$: $xy = (xy)x = y(xy) = {}^2xy$,
  $(xy)(yx) = (yx)(xy) = x((xy)(yx)) = ((xy)(yx))x$;
- $\mathcal{G}_7^3$: $xy = (yx)y = y(xy) = xy^2,\ {}^2xy = {}^2yx$;
- $\mathcal{G}_8^3$: $xy = y(xy) = xy^2 = {}^2xy,\ (xy)x = (yx)y = (xy)(yx)$;
- $\mathcal{G}_9^3$: $xy = y(xy) = xy^2,\ {}^2xy = {}^2yx = (xy)x = (yx)y = (xy)(yx)$;
- $\mathcal{G}_{10}^3$: $(xy)x = (yx)y = x(yx),\ xy^2 = x$;
- $\mathcal{G}_{11}^3$: $(xy)x = (yx)y = x(yx) = (xy)(yx) = {}^2xy = xy^2 = xy^3$;

or to one of the varieties $\mathcal{G}_i^{3\mathrm{d}}$ $(i = 5, \ldots, 11)$.

Thus $\mathcal{G}_{\mathrm{I}\,(p_2\le 3)} = \mathcal{G}_{\mathrm{I}\,(p_2\le 2)} \cup \mathcal{G}_1^3 \cup \ldots \cup \mathcal{G}_{11}^3 \cup \mathcal{G}_1^{3\mathrm{d}} \cup \ldots \cup \mathcal{G}_{11}^{3\mathrm{d}}$.

Note that $\mathcal{G}_1^3$ is a subvariety of the variety of all totally commutative groupoids. Such groupoids were considered e.g. in [5]. (Recall that a groupoid $\boldsymbol{G}$ is *totally commutative* if every essentially binary term operation f over $\boldsymbol{G}$ is commutative, i.e. $f(x,y) = f(y,x)$ for all x, y from G.) It is clear that the variety of affine spaces over GF(5) is a subvariety of $\mathcal{G}_4^3$. We can easily check that the varieties $\mathcal{G}_4^3$ and $\mathcal{G}_{10}^3$ are *polynomially equivalent*, i.e. there exists a bijection $\varphi : \mathcal{G}_4^3 \to \mathcal{G}_{10}^3$ such that $(G, \cdot)$ and $\varphi((G, \cdot))$ are polynomially equivalent in the sense of [13]. From the proof of Theorem 2.3 we get

THEOREM 2.4. *Let $\boldsymbol{G}$ be an idempotent groupoid such that $p_2(\boldsymbol{G}) = 3$. Then the following conditions are equivalent*:

(i) *$\boldsymbol{G}$ satisfies a nonregular identity.*
(ii) *The clone of $\boldsymbol{G}$ is minimal.*
(iii) *Every two-generated subgroupoid of $\boldsymbol{G}$ is an affine space over* GF(5).

Theorem 2.3 is proved in Sections 3–13.

COMMUTATIVE CASE

3. The term operation xy^3. According to Theorem 1 of [5] in any proper commutative idempotent groupoid $\boldsymbol{G}$ we have $xy^n \neq y$ for all n.

We start with the following obvious

LEMMA 3.1 (cf. [4], Theorem 2.1). *Let $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}}$. If $\boldsymbol{G}$ is a totally commutative groupoid satisfying $xy = xy^n$ for some $n \geq 2$ then $\boldsymbol{G}$ is a near-semilattice.*

Proof. Since $xy = xy^n$ we have $xy^2 = xy^{n+1}$. Hence $xy^2 = y(xy)^n = (y(xy)^{n-1})(xy) = (xy^n)(xy) = (xy)(xy) = xy$. So $\boldsymbol{G}$ is a near-semilattice.

LEMMA 3.2. *Let $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}}$. Then*:

(i) *If $p_2(\boldsymbol{G}) = 3$ then the term operations xy^k for $k = 1, 2, 3$ are essentially binary.*

(ii) *If $p_2(\boldsymbol{G}) \leq 3$, xy^3 is commutative and $xy^3 \notin \{xy, xy^2, yx^2\}$ then xy^2 is commutative.*

Proof. (i) For $k = 1$ the statement is obvious. If xy^2 is not essentially binary, then $p_2(\boldsymbol{G}) < 3$. If xy^3 is not essentially binary then Theorem 3 of [5] shows that $p_2(\boldsymbol{G}) \geq 5$, a contradiction.

(ii) Obvious.

LEMMA 3.3. *Let $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}}$ satisfy $xy^2 = yx^2$. Then*:

(i) *$p_2(\boldsymbol{G}) = 3$ if and only if $\boldsymbol{G}$ satisfies $xy^3 = xy^4$ but not $xy^2 = xy^3$.*

(ii) *$p_2(\boldsymbol{G}) \leq 3$ if and only if $\boldsymbol{G}$ satisfies $xy^3 = xy^4$.*

Proof. If $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}}$ and $xy^2 = yx^2$, then by Theorem 4 of [5], $\boldsymbol{G}$ is totally commutative.

(i) Assume that $p_2(\boldsymbol{G}) = 3$. By the preceding lemma the term operations xy, xy^2 and xy^3 are essentially binary. If $xy = xy^3$ then $\boldsymbol{G}$ is a near-semilattice (by Lemma 3.1) and so $p_2(\boldsymbol{G}) = 1$, a contradiction. If $xy^2 = xy^3$ and $xy \neq xy^2$, then one can check that $p_2(\boldsymbol{G}) = 2$, a contradiction. Thus xy, xy^2, xy^3 are the only essentially binary term operations over $\boldsymbol{G}$.

The term operation xy^4 is essentially binary (recall that $\boldsymbol{G}$ is idempotent and totally commutative so every binary term operation is essentially binary) and $xy^4 \notin \{xy, xy^2\}$. Indeed, if $xy = xy^4$, then $\boldsymbol{G}$ is a near-semilattice (cf. Lemma 2.1), a contradiction. If $xy^2 = xy^4$, then $xy^2 = (xy^2)y^2 = y(xy^2)^2 = (xy^3)(xy^2)$ and hence $xy^3 = xy^5 = (xy^3)y^2 = y(xy^3)^2 = (y(xy^3))(xy^3) = (xy^4)(xy^3) = (xy^2)(xy^3) = xy^2$. Thus we get $xy^2 = xy^3$, which gives $p_2(\boldsymbol{G}) \leq 2$, a contradiction. Since $p_2(\boldsymbol{G}) = 3$ we deduce that $\boldsymbol{G}$ satisfies $xy^3 = xy^4$ and $xy^2 \neq xy^3$, as required.

Conversely, if $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}}$, $xy^2 = yx^2$, $xy^3 = xy^4$ and $xy^2 \neq xy^3$, then using (1) and the fact that $\boldsymbol{G}$ is totally commutative we infer that xy, xy^2, xy^3 are the only essentially binary term operations over $\boldsymbol{G}$.

(ii) If $p_2(\boldsymbol{G}) < 3$ then $\boldsymbol{G} \in \mathcal{G}_9^2 \subset \mathcal{G}_1^3$ (recall that $\boldsymbol{G}$ is totally commutative).

If $\boldsymbol{G} \in \mathcal{G}_1^3$ and $xy^2 = xy^3$ in $\boldsymbol{G}$ then $\boldsymbol{G} \in \mathcal{G}_9^2$ and by Theorem 2.2, $p_2(\boldsymbol{G}) \leq 2$.

LEMMA 3.4. *If $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}(p_2 \leq 3)}$ and the term operation xy^2 is noncommutative then $\boldsymbol{G}$ satisfies at least one of the following identities*:

$$xy^3 = yx^2, \tag{3.1}$$

$$xy^3 = xy, \tag{3.2}$$

$$xy^3 = xy^2. \tag{3.2}$$

Proof. Lemma 3.2 shows that xy^3 is essentially binary, hence must be one of xy, xy^2, yx^2.

Further we write $\mathcal{G}_{\mathrm{IC}(i.j)}$ for the subvariety of $\mathcal{G}_{\mathrm{IC}}$ defined by the identity $(i.j)$ above.

The varieties $\mathcal{G}_{\mathrm{IC}(3.2)}$ and $\mathcal{G}_{\mathrm{IC}(3.1)}$ are well known. For example any Steiner quasigroup and any Płonka sum of Steiner quasigroups are members of $\mathcal{G}_{\mathrm{IC}(3.2)}$. Every affine space over GF(5) is a model of the variety $\mathcal{G}_{\mathrm{IC}(3.1)}$. The most complicated variety is $\mathcal{G}_{\mathrm{IC}(3.3)}$. Note that any near-semilattice is a member of $\mathcal{G}_{\mathrm{IC}(3.2)} \cap \mathcal{G}_{\mathrm{IC}(3.3)} \cap \mathcal{G}_{\mathrm{IC}(3.1)}$ but we are interested in models $\boldsymbol{G}$ from these varieties satisfying $p_2(\boldsymbol{G}) = 3$.

4. The identity $xy^3 = yx^2$.

In this section we deal with commutative idempotent groupoids $\boldsymbol{G}$ satisfying $xy^3 = yx^2$. Using Lemmas 3.2 and 3.3 it is easy to prove:

LEMMA 4.1. *If $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}(3.1)}$ and $p_2(\boldsymbol{G}) = 3$, then xy, xy^2 and yx^2 are the only essentially binary term operations over $\boldsymbol{G}$.*

LEMMA 4.2. *Let $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}(3.1)}$. Then*:

(i) *$\boldsymbol{G}$ is a near-semilattice if and only if it satisfies $xy = xy^4$.*

(ii) *The following conditions are equivalent*:

(a) *$\boldsymbol{G}$ satisfies $xy^2 = yx^2$.*
(b) *$\boldsymbol{G}$ satisfies $xy^2 = xy^4$.*
(c) *$\boldsymbol{G}$ satisfies $xy^2 = yx^4$.*

Proof. (i) If $\boldsymbol{G}$ is a near-semilattice, then clearly $\boldsymbol{G}$ satisfies $xy = xy^4$. Assume that $xy = xy^4$ in $\boldsymbol{G}$. Putting xy for x in $xy^3 = yx^2$ we get $xy = xy^4 = y(yx)^2$. The identities $xy = y(yx)^2$ and $xy = xy^4$ give $xy = (xy)(xy)^2 = y(yx)^4 = y(yx) = xy^2$ and therefore $\boldsymbol{G}$ is a near-semilattice.

(ii) Since $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}(3.1)}$, $xy^2 = yx^2$ implies $xy^2 = xy^3$ and hence (a)⇒(b) and (a)⇒(c).

Assume that (b) holds. Then $xy^2 = xy^4 = (xy)y^3 = y(yx)^2$. Thus $y(yx) = y(yx)^2$ and hence $y(yx)^2 = y(yx)^3$. Further $y(yx)^2 = (yx)y^3 = xy^4 = xy^2$ and $y(yx)^3 = (yx)y^2 = xy^3 = yx^2$. This proves (b)⇒(a).

If $xy^2 = yx^4$, then $xy^2 = yx^3 = (yx)x^2 = x(xy)^4 = (x(xy)^3)(xy) = ((xy)x^2)(xy) = (yx^3)(xy) = (xy^2)(xy) = y(yx)^2 = (yx)y^3 = xy^4 = yx^2$, which proves (c)⇒(a).

LEMMA 4.3. *Let* $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}}$. *Then*:

(i) $\boldsymbol{G}$ *satisfies* $(xy^2)x = x$ *if and only if it is a Steiner quasigroup.* (*Hence* $p_2(\boldsymbol{G}) = 1$.)

(ii) *The following conditions are equivalent*:

(a) $\boldsymbol{G}$ *satisfies* $(xy^2)x = y$.

(b) $\boldsymbol{G}$ *satisfies* (3.1) *and* $xy^4 = x$.

(c) $\boldsymbol{G}$ *satisfies* (3.1) *and* $p_2(\boldsymbol{G}) = 3$.

(d) *For every* $a, b \in G$ *such that* $a \neq b$ *the subgroupoid* $G(a,b)$ *of* $\boldsymbol{G}$ *generated by* $\{a, b\}$ *is a five-element affine space over* GF(5).

Proof. (i) If $\boldsymbol{G}$ is a Steiner quasigroup, then obviously $(xy^2)x = x$ in $\boldsymbol{G}$. Assume conversely that $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}}$ and $\boldsymbol{G}$ satisfies $(xy^2)x = x$. Then $x = x^2 = (xy^2)x^2$ and hence $x = (xy^2)x = ((xy^2)x^2)(xy^2) = xy^2$, which proves that $\boldsymbol{G}$ is a Steiner quasigroup.

(ii) (a)⇒(b). First observe that $(xy^2)x = y$ gives $x = ((xy^2)x^2)(xy^2) = (yx)(xy^2) = y(yx)^2$. Put xy^2 for y in $x = y(yx)^2$ to get $x = (xy^2)((xy^2)x)^2 = xy^4$, as required. Further we have $xy^3 = (xy^2)y = (xy^2)((xy^2)x) = x(xy^2)^2$. Thus $xy^2 = (x(xy^2)^2)x = (xy^3)x$. Hence $y = (xy^2)x = (xy^3)x^2$ and so $yx^2 = (xy^3)x^4 = xy^3$.

(a)⇒(c). First we prove that $\boldsymbol{G}$ satisfies the identity $(xy^2)(yx^2) = xy$. Indeed, $(xy^2)(yx^2) = y(xy^2)^2 = (xy^2)y^3$ (as $xy^2 = yx^3$) and hence $(xy^2)(yx^2)$
$= xy^5 = xy$. Using the identities $y = (xy^2)x = yx^4 = x(xy)^2$, $xy^3 = yx^2$, $xy = (xy^2)(yx^2)$ and (1) one can prove that $p_2(\boldsymbol{G}) = 3$ if $\mathrm{card}(G) > 1$.

(a)⇒(d). If $a \neq b$, then $G(a,b) = \{a, b, ab, ab^2, ba^2\}$, $\mathrm{card}(G(a,b)) = 5$ and the groupoid $G(a,b)$ is isomorphic to $(\{0,1,2,3,4\}, 3x + 3y)$ i.e., to a five-element affine space over GF(5) (for details see [3]).

(b)⇒(a) is obvious.

(c)⇒(a). By Theorem 1 of [5] we see that $xy^4 \neq y$. Lemma 4.2(i) shows that $xy^4 \neq xy$. If $xy^4 \in \{xy^2, yx^2\}$, then by Lemma 4.2(ii) we infer that $\boldsymbol{G}$ is totally commutative with $xy^2 = yx^2$. Since $\boldsymbol{G}$ satisfies $xy^2 = yx^3$ we conclude that $p_2(\boldsymbol{G}) = 2$, a contradiction. Thus $p_2(\boldsymbol{G}) = 3$ implies $xy^4 = x$. Using this identity and $xy^2 = yx^3$ we get $(xy^2)x = yx^4 = y$, as required.

(d)⇒(a). We have to check that $(ab^2)a = b$ for all $a, b \in G$. If $a = b$, then the identity is satisfied. If $a \neq b$, then $G(a, b)$ is an affine space over GF(5) and hence satisfies the identity $(xy^2)x = y$.

As a corollary we get:

PROPOSITION 4.4. *Let* $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}(3.1)}$. *Then* $p_2(\boldsymbol{G}) = 3$ *if and only if* $\boldsymbol{G}$ *is a nontrivial groupoid satisfying* $xy^4 = x$ (*or equivalently* $(xy^2)x = y$).

From Lemma 4.3 and the fact that the clone of a nontrivial affine space over GF(p), where p is a prime number, is minimal we get

PROPOSITION 4.5. *Let* $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}(3.1)}$ *and* $p_2(\boldsymbol{G}) = 3$. *Then the clone of* $\boldsymbol{G}$ *is minimal if and only if* $\boldsymbol{G}$ *is a nontrivial affine space over* GF(5).

5. The identity $xy^3 = xy$. In this section we deal with groupoids $\boldsymbol{G}$ from $\mathcal{G}_{\mathrm{IC}(3.2)}$. We start with

LEMMA 5.1. *If* $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}(3.2)}$ *and* $p_2(\boldsymbol{G}) > 1$, *then* xy^2 *is essentially binary and noncommutative.*

Proof. If xy^2 is not essentially binary, then $\boldsymbol{G}$ satisfies $xy^2 = x$, $\boldsymbol{G}$ is a Steiner quasigroup and $p_2(\boldsymbol{G}) = 1$, contrary to assumption.

If $xy^2 = yx^2$, then $\boldsymbol{G}$ is totally commutative, Lemma 3.1 shows that $\boldsymbol{G}$ is a near-semilattice and again $p_2(\boldsymbol{G}) = 1$.

LEMMA 5.2. *Let* $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}(3.2)}$. *Then*:

(i) *If* $\boldsymbol{G}$ *is not a Steiner quasigroup*, *then* $(xy^2)x$ *is essentially binary.*

(ii) (Lemma 3.1 of [4]) *The following are equivalent*:

(a) $\boldsymbol{G}$ *is a near-semilattice.*

(b) $(xy^2)x \in \{(yx^2)y, yx^2, (xy^2)(xy)\}$.

(iii) *If* $p_2(\boldsymbol{G}) < 5$, *then* $\boldsymbol{G}$ *satisfies* $(xy^2)x = xy^2$.

Proof. (i) If $(xy^2)x = x$, then Lemma 4.3(i) shows that $\boldsymbol{G}$ is a Steiner quasigroup, a contradiction. If $(xy^2)x = y$, then (ii) of the same lemma gives $xy = xy^3 = yx^2$ and hence $\boldsymbol{G}$ is totally commutative. So $\boldsymbol{G}$ is one-element, a contradiction.

(ii) If $\boldsymbol{G}$ is a near-semilattice, then the assertion is obvious.

Assume that $\boldsymbol{G}$ satisfies $(xy^2)x = (yx^2)y$. Then $xy = (xy^3)(xy) = (y(xy)^2)y = ((xy^2)(xy))y$. Putting xy for x in the identity $xy = ((xy^2)(xy))y$ we get $xy^2 = ((xy^3)(xy^2))y = ((xy)(xy^2))y = xy$, as required.

If $(xy^2)x = yx^2$, then $xy = (xy)(xy) = (xy^3)(xy) = y(xy)^2$ and hence $xy = x(xy)^2$. This gives $xy = (xy)(xy) = x(xy)^3 = x(xy) = yx^2$, which proves that $\boldsymbol{G}$ is a near-semilattice.

Let $(xy^2)x = (xy^2)(xy)$. Then we have $xy = (xy^3)(xy) = ((xy)y^2)(xy) = ((xy)y^2)((xy)y) = (xy^3)(xy^2) = (xy)(xy^2) = (xy^2)(xy) = (xy^2)x$. Putting xy for x in the identity $xy = (xy^2)x$ we conclude that $\boldsymbol{G}$ is a near-semilattice.

(iii) If $\boldsymbol{G}$ is a Steiner quasigroup, then obviously $(xy^2)x = xy^2$. If $\boldsymbol{G}$ is not a Steiner quasigroup, then $(xy^2)x$ is essentially binary by (i). If $\boldsymbol{G}$ is a near-semilattice, then obviously $(xy^2)x = xy^2$. Now assume that $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}(3.2)}$ and $\boldsymbol{G}$ is neither a Steiner quasigroup nor a near-semilattice. Then $xy, xy^2, yx^2, (xy^2)x, (yx^2)y$ are essentially binary by (i). Since $p_2(\boldsymbol{G}) < 5$, $\boldsymbol{G}$ satisfies $(xy^2)x = xy^2$ by (ii).

Lemma 5.3. *Let* $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}(3.2)}$. *Then*:

(i) $x(xy)^2 \neq y$.

(ii) *The following are equivalent*:

(a) $\boldsymbol{G}$ *is a near-semilattice.*

(b) $x(xy)^2 \in \{yx^2, y(yx)^2, (yx^2)y\}$.

(iii) *If* $\boldsymbol{G}$ *satisfies* $xy^2 = (xy^2)x = x(xy)^2$, *then* $p_2(\boldsymbol{G}) \leq 3$.

Proof. (i) If $x(xy)^2 = y$, then $yx^2 = x(xy) = x(xy)^3 = (x(xy)^2)(xy) = y(yx) = xy^2$. Thus $\boldsymbol{G}$ is a totally commutative groupoid satisfying a non-regular identity, a contradiction.

(ii) (a)$\Rightarrow$(b) is obvious.

If $\boldsymbol{G}$ satisfies $x(xy)^2 = yx^2$, then $xy^2 = (xy^2)(xy)$. Putting xy for x in this identity we get $xy^3 = (xy^3)(xy^2)$. Hence $xy = xy^3 = (xy^3)(xy^2) = (xy)(xy^2) = (xy^2)(xy) = xy^2$, which proves that $\boldsymbol{G}$ is a near-semilattice.

If $x(xy)^2 = y(yx)^2$, then by Theorem 2.1 of [4], $\boldsymbol{G}$ is also a near-semilattice.

Let $x(xy)^2 = (yx^2)y$. Then $(xy^2)x = (xy^2)(xy)$ and so $xy = (xy^3)(xy) = (xy^3)(xy^2) = (xy^2)(xy)$, which proves that $\boldsymbol{G}$ is a near-semilattice.

(iii) Using $xy^2 = x(xy)^2$ we obtain $xy = xy^3 = (xy)y^2 = (xy)(xy^2)^2 = ((xy)(xy^2))(xy^2) = (y(yx)^2)(xy^2) = (yx^2)(xy^2)$. Further (by (1)) we have $A_0^{(2)}(\boldsymbol{G}) = \{x, y\}$, $A_1^{(2)}(\boldsymbol{G}) = \{x, y, xy\}$ and $A_2^{(2)}(\boldsymbol{G}) = \{x, y, xy, xy^2, yx^2\} = A_3^{(2)}(\boldsymbol{G})$, which proves $p_2(\boldsymbol{G}) \leq 3$, as required.

Lemma 5.4. *If* $\boldsymbol{G}$ *is a commutative idempotent groupoid satisfying* $x(xy)^2 = x$, *then* $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}(3.2)}$.

Proof. We have $x = (yx^2)(yx)$ and hence $xy = (y(yx)^2)(y(yx)) = y(y(yx)) = xy^3$, as required.

Now we prove the main result of this section.

Proposition 5.5. *Let* $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}(3.2)}$. *Then*:

(i) $p_2(\boldsymbol{G}) = 3$ *iff* $\boldsymbol{G}$ *is neither a near-semilattice nor a Steiner quasigroup satisfying the following identities*:

$$xy^2 = (xy^2)x, \tag{5.1}$$

$$xy^2 = x(xy)^2. \tag{5.2}$$

(ii) *Any nontrivial Płonka sum* $\boldsymbol{G}$ *of Steiner quasigroups which are not all singletons is a member of the variety* $\mathcal{G}_{\mathrm{IC}(3.2)(5.1)(5.2)}$ *and* $p_2(\boldsymbol{G}) = 3$.

(iii) *If* $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}(3.2)(5.1)(5.2)}$, *then the clone of* $\boldsymbol{G}$ *is minimal iff* $\boldsymbol{G}$ *is either a nontrivial affine space over* GF(3) *or a nontrivial near-semilattice.*

(iv) *If* $1 \le p_2(\boldsymbol{G}) \le 4$, *then the clone of* $\boldsymbol{G}$ *is minimal iff* $\boldsymbol{G}$ *is either a proper near-semilattice or a nontrivial affine space over* GF(3).

Proof. (i) If $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}(3.2)(5.1)(5.2)}$ and $\boldsymbol{G}$ is neither a Steiner quasigroup nor a near-semilattice, then $xy = (xy^2)(yx^2)$. Indeed, $(xy^2)(yx^2) = (xy^2)((xy^2)(xy)) = (xy^2)(yx^2) = (xy)(xy^2)^2 = (xy)((xy)y)^2 = xy^3 = xy$. Hence $A_2^{(2)}(\boldsymbol{G}) = \{x, y, xy, xy^2, yx^2\}$ and $p_2(\boldsymbol{G}) = 3$.

Now let $p_2(\boldsymbol{G}) = 3$. Lemma 5.1 shows that xy^2 is essentially binary and noncommutative. Since $\boldsymbol{G}$ is neither a Steiner quasigroup nor a near-semilattice we have $(xy^2)x = xy^2$ by Lemma 5.2(iii). If $x(xy)^2 = x$, then putting xy^2 for y we obtain $x = x((xy^2)x)^2 = x(xy^2)^2 = ((xy^2)x)(xy^2) = xy^2$. This proves that $\boldsymbol{G}$ is a Steiner quasigroup, a contradiction. Now Lemma 5.3 yields $xy^2 = x(xy)^2$ and therefore $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}(3.2)(5.1)(5.2)}$.

(ii) Any Steiner quasigroup satisfies the identities of $\mathcal{G}_{\mathrm{IC}(3.2)(5.1)(5.2)}$. Since those identities are regular we infer that Płonka's sums of such algebras are also in $\mathcal{G}_{\mathrm{IC}(3.2)(5.1)(5.2)}$ (see [21]).

(iii) Obviously the clones of a proper near-semilattice and of a proper affine space over GF(3) are minimal. Now let $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}(3.2)(5.1)(5.2)}$. In a proper groupoid $\boldsymbol{G}$ we have $xy^2 \neq y$. If $xy^2 = x$, then Proposition of [10] shows that the clone of $\boldsymbol{G}$ is minimal if and only if $\boldsymbol{G}$ is a proper affine space over GF(3). If $xy^2 = xy$ then $\boldsymbol{G}$ is a near-semilattice. Thus further we may assume that $p_2(\boldsymbol{G}) > 1$. By Lemma 5.1, xy^2 is essentially binary and noncommutative. Consider now $(G, \circ)$ where $x \circ y = xy^2$. Using the identities of $\mathcal{G}_{\mathrm{IC}(3.2)(5.1)(5.2)}$ one can check that $x \circ y = (x \circ y) \circ y = (x \circ y) \circ x = x \circ (x \circ y) = x \circ (y \circ x) = (x \circ y) \circ (y \circ x)$ and hence $p_2(G, \circ) = 2$. Thus the clone of $\boldsymbol{G}$ is not minimal since the clone of $(G, \circ)$ is its nontrivial subclone.

(iv) Now let $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}(3.2)(5.1)(5.2)}$, $1 \le p_2(\boldsymbol{G}) \le 4$ and suppose the clone of $\boldsymbol{G}$ is minimal. If $p_2(\boldsymbol{G}) = 1$ or $p_2(\boldsymbol{G}) = 2$ then the assertion follows e.g. from Theorem 2.3 of [4]. If $p_2(\boldsymbol{G}) = 3$, then $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}(3.2)(5.1)(5.2)}$ and the proof is given above (see also Lemma 2.3 of [10]). Now let $p_2(\boldsymbol{G}) = 4$. Lemma 5.2 shows that $(xy^2)x = xy^2$ and xy^2 is essentially binary and noncommutative. By Lemma 5.3, $x(xy)^2 \notin \{y, yx^2, y(yx)^2\}$. If $x(xy)^2 = x$, then using $(xy^2)x = xy^2$ one proves that $\boldsymbol{G}$ is a Steiner quasigroup. Thus either $x(xy)^2 = xy^2$, or xy, xy^2, yx^2, $x(xy)^2$, $y(yx)^2$ are essentially binary and pairwise distinct. The second case is impossible but in the first case $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}(3.2)(5.1)(5.2)}$ and consequently $p_2(\boldsymbol{G}) = 3$, a contradiction.

6. The identity $xy^3 = xy^2$. In this section we deal with groupoids $\boldsymbol{G}$ from the variety $\mathcal{G}_{\mathrm{IC}(3.3)}$. We start with

LEMMA 6.1. *Let $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}(3.3)}$. Then*:

(i) *The term operations xy^2, $(xy^2)x$ are essentially binary.*

(ii) *If $p_2(\boldsymbol{G}) = 3$, then*:

(a) *The term operation xy^2 is noncommutative.*

(b) $\boldsymbol{G}$ *satisfies*:

$$xy = (xy^2)(yx^2). \tag{6.1}$$

(c) $\boldsymbol{G}$ *satisfies at least one of the following identities*:

$$(xy^2)x = xy, \tag{6.2}$$

$$(xy^2)x = xy^2, \tag{6.3}$$

$$(xy^2)x = yx^2. \tag{6.4}$$

P r o o f. (i) If xy^2 or $(xy^2)x$ is not essentially binary, then Theorem 9 of [11] shows that $\boldsymbol{G}$ is cancellative and hence the identity $xy^2 = xy^3$ gives $\mathrm{card}(G) = 1$, a contradiction.

(ii) If $xy^2 = yx^2$ and $xy \neq xy^2$, then $\boldsymbol{G}$ is a totally commutative groupoid satisfying $xy^2 = xy^3$. It is easy to check that $p_2(\boldsymbol{G}) = 2$, a contradiction. Since $p_2(\boldsymbol{G}) = 3$ and xy, xy^2, yx^2 are the only essentially binary term operations over $\boldsymbol{G}$ we infer that $xy = (xy^2)(yx^2)$, and at least one of (6.2)–(6.4) holds.

LEMMA 6.2. *Let $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}(3.3)}$ satisfy* (6.2) *or* (6.3) (*i.e.* $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}(3.3)(6.2)} \cup \mathcal{G}_{\mathrm{IC}(3.3)(6.3)}$). *If $\boldsymbol{G}$ satisfies* (6.1) *then*:

(i) $\boldsymbol{G}$ *satisfies $x(xy)^2 = yx^2$ and consequently $p_2(\boldsymbol{G}) \leq 3$.*

(ii) *If $\boldsymbol{G}$ is not a near-semilattice then the clone* $\mathrm{cl}(G, \circ)$, *where $x \circ y = xy^2$, is a proper subclone of* $\mathrm{cl}(\boldsymbol{G})$ *and consequently the latter is not minimal.*

(iii) *If $\boldsymbol{G}$ is not a near-semilattice*, *then $p_2(\boldsymbol{G}) = 3$.*

P r o o f. Let $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}(3.3)(6.2)}$.

(i) Putting xy for x in $xy = (xy^2)x$ we obtain $xy^2 = (xy^3)(xy) = (xy^2)(xy) = y(yx)^2$.

(ii) To prove that $p_2(G, \circ) \leq 2$ we use $A^{(2)}(\boldsymbol{G}) = \{x, y, xy, xy^2, yx^2\}$ and the fact that $x \circ x = x$, $x \circ y = (x \circ y) \circ y = (y \circ x) \circ y = x \circ (x \circ y) = (y \circ x) \circ (x \circ y)$. For example, $(x \circ y) \circ (y \circ x) = ((xy^2)(yx^2))(yx^2) = x(xy)^2 = yx^2 = y \circ x$, as required. Since $p_2(G, \circ) = 2$ the clone of $\boldsymbol{G}$ is not minimal (cf. also Lemma 2.5 of [10]).

(iii) Easy.

The proof for $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}(3.3)(6.3)}$ runs analogously.

LEMMA 6.3. *Let* $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}(3.3)(6.1)}$. *Then*:

(i) *If* $\boldsymbol{G}$ *satisfies* (6.3) *then* $\boldsymbol{G}$ *is a near-semilattice and* $p_2(\boldsymbol{G}) = 1$.

(ii) *If* $\boldsymbol{G}$ *satisfies* (6.4) *then*

(a) $\boldsymbol{G}$ *satisfies* $x(xy)^2 = xy$.

(b) $\boldsymbol{G}$ *is a near-semilattice and* $p_2(\boldsymbol{G}) = 1$.

Proof. (i) By assumption $xy = (xy^2)(yx^2)$. So $(xy)(xy^2) = (yx^2)(xy^2)^2$. Hence (Lemma 6.2(i)) $xy^2 = y(yx)^2 = (yx^2)(xy^2)^2$. Then $(xy^2)(yx^2) = ((yx^2)(xy^2))(yx^2)$. By (6.3), $(xy^2)(yx^2) = (yx^2)(xy^2)^2$. Hence $(xy^2)(yx^2) = xy^2$. By (6.1), $xy^2 = xy$ and consequently $\boldsymbol{G}$ is a near-semilattice.

(ii) (a) We have $x(xy)^2 = (yx^2)(xy) = (yx^2)((xy^2)(yx^2))$. So $x(xy)^2 = (xy^2)(yx^2)^2 = ((yx^2)y)(yx^2)^2 = y(yx^2)^3 = y(yx^2)^2 = (y(yx^2))(yx^2) = (xy^2)(yx^2) = xy$.

(b) By Lemma 4.1 of [4].

PROPOSITION 6.4. *Let* $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}(3.3)}$. *Then* $p_2(\boldsymbol{G}) = 3$ *if and only if* $\boldsymbol{G}$ *is not a near-semilattice and belongs to the variety*

$$\mathcal{G}_3^3: \quad xy = yx,\ xy = (xy^2)(yx^2) = (xy^2)x,\ xy^2 = xy^3.$$

Proof. This follows from Lemmas 6.1–6.3 and formula (1).

NONCOMMUTATIVE CASE

7. The term operation $(xy)x$**.** Now assume that $\boldsymbol{G}$ is a groupoid from the class $\mathcal{G}_{\mathrm{I\check{C}}}$ (so the term operations xy and yx are both essentially binary and distinct). If $p_2(\boldsymbol{G}) \leq 3$ then at least one of the following identities holds in $\boldsymbol{G}$ (up to duality):

$$(xy)x = x, \tag{7.1}$$

$$(xy)x = y, \tag{7.2}$$

$$(xy)x = xy, \tag{7.3}$$

$$(xy)x = yx, \tag{7.4}$$

$$(xy)x = (yx)y. \tag{7.5}$$

8. Groupoids with $(xy)x = x$

LEMMA 8.1. *Let* $\boldsymbol{G}$ *be an idempotent groupoid satisfying* (7.1) ($\boldsymbol{G} \in \mathcal{G}_{\mathrm{I}(7.1)}$). *Then the identity* $x(xy) = xy$ *holds in* $\boldsymbol{G}$.

Proof. By assumption, $(xy)x = x$. Hence $xy = ((xy)x)(xy) = x(xy)$.

Assume that $\boldsymbol{G}$ satisfies (7.1). If $p_2(\boldsymbol{G}) \le 3$, then at least one of the following identities holds in $\boldsymbol{G}$:

$$x(yx) = x, \tag{8.1}$$

$$x(yx) = y, \tag{8.2}$$

$$x(yx) = xy, \tag{8.3}$$

$$x(yx) = yx, \tag{8.4}$$

$$x(yx) = y(xy). \tag{8.5}$$

LEMMA 8.2. *Let* $\boldsymbol{G} \in \mathcal{G}_{\mathrm{I}(7.1)}$. *Then*:

(i) *If* $\boldsymbol{G}$ *satisfies* (8.1) *and* $p_2(\boldsymbol{G}) \le 3$ *then* $\boldsymbol{G} \in \mathcal{G}_5^2$ *and consequently* $p_2(\boldsymbol{G}) \le 2$.

(ii) *If* $p_2(\boldsymbol{G}) = 3$ *then* (8.1) *does not hold in* $\boldsymbol{G}$.

(iii) (8.2), (8.3), (8.5) *do not hold in* $\boldsymbol{G}$.

(iv) *If* $\boldsymbol{G}$ *satisfies* (8.4) *then* $xy^2 = y$ *holds in* $\boldsymbol{G}$.

(v) *If* $\boldsymbol{G}$ *satisfies* (8.4) *and* $p_2(\boldsymbol{G}) \le 3$ *then* $\boldsymbol{G} \in \mathcal{G}_2^2 \cup \mathcal{G}_3^2$ *and consequently* $p_2(\boldsymbol{G}) \le 2$.

(vi) *If* $p_2(\boldsymbol{G}) = 3$ *then* (8.4) *does not hold in* $\boldsymbol{G}$.

Proof. (i) Assume that (8.3) holds in $\boldsymbol{G}$. Then $xy = (xy)(y(xy)) = xy^2$. So by Lemma 8.1 we have $A_2^{(2)}(\boldsymbol{G}) = \{x, y, xy, yx, (xy)(yx), (yx)(xy)\}$. Now assume that $p_2(\boldsymbol{G}) \le 3$. So $(xy)(yx) \in \{x, y, xy, yx, (yx)(xy)\}$. Suppose that $(xy)(yx) = x$. Then $\boldsymbol{G} \in \mathcal{G}_5^2$. If $(xy)(yx) = y$, then $y = (xy^2)(y(xy)) = (xy^2)y = xy^2 = xy$, a contradiction. Assume that $(xy)(yx) = xy$. Putting yx for x we have $((yx)y)(y(yx)) = (yx)y$. Hence $y(^2yx) = y$. Thus $yx = y$, a contradiction. Now assume that $(xy)(yx) = yx$. Then $(x(yx))((yx)x) = (yx)x$. Hence $x(yx) = yx$. Thus $x = yx$, a contradiction. Finally suppose that $(xy)(yx)$ is symmetric. Then, using Lemma 8.1, we have $y = (yx)y = (y(yx))((yx)y) = ((yx)y)(y(yx)) = y(yx) = yx$, a contradiction.

(ii) By (i).

(iii) Assume that (8.2) holds in $\boldsymbol{G}$. Then $xy = x(x(yx))$. Hence, by Lemma 8.1 and (8.2), $xy = x(yx) = y$, a contradiction. Assume that (8.3) holds in $\boldsymbol{G}$. Then putting xy for y in (8.3) we get $x = xy$, a contradiction. Now assume that (8.5) holds in $\boldsymbol{G}$. Putting xy for y in (8.5) and using (7.1) and Lemma 8.1 we get $x = xy$, a contradiction.

(iv) (8.4) gives $y = (y(xy))y = (xy)y$.

(v) Assume that $p_2(\boldsymbol{G}) \le 3$. If $(xy)(yx) = x$ then $\boldsymbol{G} \in \mathcal{G}_2^2$. Assume that $(xy)(yx) = y$. Then putting yx for x we have $((yx)y)(y(yx)) = y$. Hence $y(y(yx)) = y$ and by Lemma 8.1, $xy = y$, a contradiction. Now suppose that $(xy)(yx) = xy$. Hence $((xy)(yx))(yx) = (xy)(yx) = xy$. By (iv) we obtain $yx = xy$, which means that $\boldsymbol{G}$ is commutative. Hence using (7.1) and (8.4) we deduce that $\mathrm{card}(G) = 1$, a contradiction. If $(xy)(yx) = yx$ then $\boldsymbol{G} \in \mathcal{G}_3^2$.

Now assume that $(xy)(yx) = (yx)(xy)$. Putting xy for x and using (iv) we get $((xy)y)(y(xy)) = (y(xy))((xy)y)$ and so $y(y(xy)) = (y(xy))y$. Therefore $y(xy) = (xy)y$ and so $xy = y$, a contradiction.

(vi) By (v).

As a consequence of Lemma 8.2 we have the following proposition:

PROPOSITION 8.3. *Let* $\boldsymbol{G} \in \mathcal{G}_{\mathrm{I}(7.1)}$. *Then one of the following conditions holds* (*recall that* $\mathrm{card}(G) > 1$):

(i) $p_2(\boldsymbol{G}) = 2$ *and* $\boldsymbol{G} \in \mathcal{G}_2^2 \cup \mathcal{G}_3^2 \cup \mathcal{G}_5^2$ *or*
(ii) $p_2(\boldsymbol{G}) \geq 4$.

9. Groupoids with $(xy)x = y$. In this section we deal with groupoids $\boldsymbol{G}$ satisfying (7.2) and such that $p_2(\boldsymbol{G}) = 3$. At least one of the following identities holds in $\boldsymbol{G}$:

$$xy^2 = x, \tag{9.1}$$

$$xy^2 = y, \tag{9.2}$$

$$xy^2 = xy, \tag{9.3}$$

$$xy^2 = yx, \tag{9.4}$$

$$xy^2 = yx^2. \tag{9.5}$$

We start with the following obvious, but useful lemma (cf. Lemma 5.2 of [7]):

LEMMA 9.1. *Let* $\boldsymbol{G} \in \mathcal{G}_{\mathrm{I}}$. *Then*:

(i) *The following conditions are equivalent*:
   (a) $\boldsymbol{G}$ *satisfies the identity* (7.2): $(xy)x = y$.
   (b) $\boldsymbol{G}$ *satisfies the identity* (7.2)$'$: $x(yx) = y$.

(ii) *If* $\boldsymbol{G}$ *satisfies* (7.2) *or* (7.2)$'$ *then* $\boldsymbol{G}$ *is a quasigroup.*

LEMMA 9.2. *Let* $\boldsymbol{G} \in \mathcal{G}_{\mathrm{I}(7.2)}$. *Then*:

(i) *If* (9.1) *holds in* $\boldsymbol{G}$, *then* $\boldsymbol{G}$ *is a Steiner quasigroup and* $p_2(\boldsymbol{G}) = 1$ (cf. Lemma 3.1 of [8]).

(ii) (9.2) *does not hold in* $\boldsymbol{G}$.

Proof. (i) (9.1) yields $y((xy)y) = yx$. By Lemma 9.1(i) we get $xy = yx$. So $\boldsymbol{G}$ is commutative.

(ii) (9.2) implies $(xy)y = yy$. By Lemma 9.1(ii), $xy = y$, a contradiction.

PROPOSITION 9.3. *Let* $\boldsymbol{G} \in \mathcal{G}_{\mathrm{I}(7.2)}$. *Then one of the following conditions holds*:

(i) $p_2(\boldsymbol{G}) = 1$ *and* $\boldsymbol{G}$ *is a Steiner quasigroup or*
(ii) $p_2(\boldsymbol{G}) = 2$ *and* $\boldsymbol{G} \in \mathcal{G}_7^2$ *or*
(iii) $p_2(\boldsymbol{G}) \geq 4$.

P r o o f. (i) If xy^2 is not essentially binary then $\boldsymbol{G}$ is a Steiner quasigroup (by Lemma 9.2).

(ii) Suppose that (9.3) holds in $\boldsymbol{G}$. As $\boldsymbol{G}$ is a quasigroup (Lemma 9.1) it follows that $xy = x$, a contradiction.

Assume that (9.4) holds in $\boldsymbol{G}$. Then $y = ((xy)y)(xy) = (yx)(xy)$. Therefore $y(yx) = ((yx)(xy))(yx) = xy$. So $p_2(\boldsymbol{G}) \leq 2$, more exactly $\boldsymbol{G}$ is a member of $\mathcal{G}_7^2$.

(iii) Now assume that $\boldsymbol{G}$ satisfies (9.5). By Lemmas 9.1 and 9.2 we have $y(yx) \notin \{x, y, yx\}$. Suppose that $y(yx) = xy$. Then $(xy)(yx) = (y(yx))(yx)$. Hence by (9.5) we get $(xy)(yx) = ((yx)y)y$. By (7.2), $(xy)(yx) = xy$. Hence $((xy)(yx))(xy) = xy$. Thus $yx = xy = y(yx)$. By Lemma 9.1(ii) we get $yx = y$, a contradiction with the assumption that $\boldsymbol{G}$ is proper. Assume that $y(yx) = xy^2 = yx^2$. Then $y = (yx)(y(yx)) = (yx)((yx)x) = (x(yx))(yx) = y(yx)$, a contradiction. Therefore $y(yx) \notin \{x, y, xy, yx, (yx)y\}$ and $p_2(\boldsymbol{G}) \geq 4$.

PROPOSITION 9.4. *Let $\boldsymbol{G}$ be an idempotent groupoid such that $p_2(\boldsymbol{G}) = 3$. Then the term operations $(xy)x$ and $x(yx)$ are both essentially binary.*

P r o o f. By Propositions 8.3, 9.3 and their dual versions.

10. Groupoids with $(xy)x = xy$. Assume that $\boldsymbol{G}$ satisfies (7.3). If $p_2(\boldsymbol{G}) \leq 3$ then at least one of (8.1)–(8.5) holds in $\boldsymbol{G}$. Note that (8.5) is a dual case of (7.5).

LEMMA 10.1. *If $\boldsymbol{G}$ satisfies* (8.3) *then the identity $xy^2 = (xy)(yx)$ holds in $\boldsymbol{G}$.*

P r o o f. By (8.3) we have $(xy)(yx) = (xy)(y(xy)) = (xy)y$.

LEMMA 10.2. *Let $\boldsymbol{G} \in \mathcal{G}_{\mathrm{I}(7.3)}$. Then*:

(i) *If $p_2(\boldsymbol{G}) = 3$ then $x(yx)$ is essentially binary.*
(ii) *If $\boldsymbol{G}$ satisfies* (8.4) *then $xy^2 = xy$ in $\boldsymbol{G}$.*

P r o o f. (i) By Proposition 9.4.

(ii) By (7.3) and (8.4) we have $(xy)y = (y(xy))y = y(xy) = xy$.

LEMMA 10.3. *Let $\boldsymbol{G} \in \mathcal{G}_{\mathrm{I}(7.3)(8.3)}$. Then*:

(i) *If $p_2(\boldsymbol{G}) \leq 2$ then $p_2(\boldsymbol{G}) = 1$ and $\boldsymbol{G}$ is a near-semilattice or $p_2(\boldsymbol{G}) = 2$ and $\boldsymbol{G} \in \mathcal{G}_1^2 \cup \mathcal{G}_4^{2d}$.*

(ii) *If $p_2(\boldsymbol{G}) = 3$ then*:

(a) *The term operation xy^2 is symmetric and $x(xy) = xy$ in $\boldsymbol{G}$.*
(b) *The clone $\{x, y, x \circ y\}$, where $x \circ y = xy^2$, is a proper subclone of $\{x, y, xy, yx, xy^2\}$.*

(iii) *If xy^2 is symmetric and ${}^2xy = xy$ in $\boldsymbol{G}$ then $p_2(\boldsymbol{G}) \leq 3$.*

Proof. (i) Since $p_2(\boldsymbol{G}) \le 3$, at least one of (9.1)–(9.5) holds. Assume that (9.1) holds. By Lemma 10.1, $(xy)(yx) = x$. Then $x(yx) = ((xy)(yx))(yx)$. Hence $x(yx) = xy$. Moreover $x(xy) = ((xy)(yx))(xy)$. So $x(xy) = x$. Thus $\boldsymbol{G} \in \mathcal{G}_1^2$. Suppose that $\boldsymbol{G}$ satisfies (9.2). Then $((yx)y)y = y$. Therefore $yx = y$, a contradiction. Now suppose that (9.3) holds. Assume that $x(xy) = x$. By Proposition 9.4 the term operation $x(yx)$ is essentially binary. If $x(yx) = xy$ then $\boldsymbol{G} \in \mathcal{G}_4^{2\mathrm{d}}$, and $p_2(\boldsymbol{G}) \le 2$. If $x(yx) = yx$ then $x(x(yx)) = x(yx)$ and $x = xy$, a contradiction. If $x(yx) = y(xy)$ then putting xy for y we obtain $x((xy)x) = (xy)(x(xy))$. Hence $x = (xy)x = xy$, a contradiction. Suppose that $\boldsymbol{G}$ satisfies (9.4). Then $((xy)y)(xy) = (yx)(xy)$. By (7.3) and Lemma 10.1 we have $xy^2 = xy$. Hence $yx = xy$. So $\boldsymbol{G}$ is a near-semilattice and $p_2(\boldsymbol{G}) = 1$.

(ii) (a) By (i) the term operation xy^2 is symmetric in $\boldsymbol{G}$. As $p_2(\boldsymbol{G}) = 3$, at least one of the following identities holds:

(10.1) $$ {}^2xy = x, $$
(10.2) $$ {}^2xy = y, $$
(10.3) $$ {}^2xy = xy, $$
(10.4) $$ {}^2xy = yx, $$
(10.5) $$ {}^2xy = {}^2yx. $$

Suppose that (10.1) holds. Then $x = (x(xy))(xy) = ((xy)x)x$. Hence $x = (xy)x = xy$, a contradiction. Suppose that (10.2) holds. Then $(x(xy))x = yx$. Hence $y = x(xy) = yx$, a contradiction. Suppose that (10.4) holds. Then $(x(xy))x = (yx)x$. Hence $(yx)x = x(xy) = yx$, so $\boldsymbol{G}$ is commutative, a contradiction. Suppose that $x(xy)$ is symmetric. As $p_2(\boldsymbol{G}) = 3$ we have $x(xy) = (xy)y$. Hence $xy = (xy)x = ((xy)x)x = (x(xy))(xy) = ((xy)y)(xy) = (xy)y$. Hence $\boldsymbol{G}$ is commutative, a contradiction. Therefore $\boldsymbol{G}$ satisfies $x(xy) = xy$.

(b) Obvious (by (7.3), (8.3) and Lemma 10.1).

(iii) $A^{(2)}(\boldsymbol{G}) = \{x, y, xy, yx, (xy)y\}$. Indeed, observe that $A_3^{(2)}(\boldsymbol{G}) = A_2^{(2)}(\boldsymbol{G}) = \{x, y, xy, yx, (xy)y\}$. For example $((xy)y)y = (y(xy))(xy) = (yx)(xy) = yx^2 = xy^2$.

LEMMA 10.4. *Let $\boldsymbol{G} \in \mathcal{G}_{\mathrm{I}(7.3)(8.4)}$. Then*:

(i) *If $p_2(\boldsymbol{G}) \le 2$ then $p_2(\boldsymbol{G}) = 1$ and $\boldsymbol{G}$ is a near-semilattice.*

(ii) *If $p_2(\boldsymbol{G}) = 3$ then*:

(a) *The term operation $(xy)(yx)$ is symmetric and $x(xy) = xy$, $x((xy)(yx)) = ((xy)(yx))x = (xy)(yx)$ in $\boldsymbol{G}$.*

(b) *The clone $\{x, y, x \circ y\}$, where $x \circ y = (xy)(yx)$, is a proper subclone of $\{x, y, xy, yx, (xy)(yx)\}$.*

(iii) *If* $(xy)(yx)$ *is symmetric and* $x(xy) = xy$, $(xy)(yx) = x((xy)(yx)) = ((xy)(yx))x$ *in* $\boldsymbol{G}$ *then* $p_2(\boldsymbol{G}) \le 3$.

Proof. (i) Assume that $p_2(\boldsymbol{G}) \le 3$. Then at least one of (10.1)–(10.5) holds in $\boldsymbol{G}$. Suppose that $x(xy) = x$. Then by (7.3) and (8.4) we have $(xy)x = y(xy)$. Putting xy for y we get $(x(xy))x = (xy)((x(xy))$. Hence $x = (xy)x = xy$. Thus $\operatorname{card}(G) = 1$, a contradiction. Now assume that $x(xy) = y$. Hence $(x(xy))x = yx$. By (7.3) we get $x(xy) = yx$. Hence $y = yx$, a contradiction. Now suppose that $x(xy) = yx$. Lemma 10.2 shows that $yx = (yx)(xy) = (xy)((xy)(yx)) = xy$. Thus $\boldsymbol{G}$ is a near-semilattice. Assume $x(xy) = y(yx)$. Hence, by (7.3) and (8.4) we get $y(yx) = y((yx)y) = (yx)y$. By (7.3), $y(yx) = yx$. So $\boldsymbol{G}$ is commutative and hence a near-semilattice. So if $p_2(\boldsymbol{G}) \le 3$ then $x(xy) = xy$ in $\boldsymbol{G}$.

Now at least one of the following identities holds:

(10.6) $$(xy)(yx) = x,$$

(10.7) $$(xy)(yx) = y,$$

(10.8) $$(xy)(yx) = xy,$$

(10.9) $$(xy)(yx) = yx,$$

(10.10) $$(xy)(yx) = {}^2yx.$$

Suppose that (10.6) holds. Then $((xy)(yx))(xy) = x(xy)$. Hence, by (7.3), $x = (xy)(yx) = xy$, a contradiction. Assume that (10.7) holds. So $((xy)(yx))(xy) = y(xy) = xy$ and by (7.3), $y = (xy)(yx) = xy$, a contradiction. Now assume (10.8). Hence $(yx)((xy)(yx)) = (yx)(xy) = yx$. By (8.4) we get $xy = yx$, so $\boldsymbol{G}$ is a near-semilattice. Now assume (10.9). Then $((xy)(yx))(xy) = (yx)(xy) = xy$. By (7.3), $yx = (xy)(yx) = xy$. So $\boldsymbol{G}$ is a near-semilattice again. Thus if $p_2(\boldsymbol{G}) \le 3$ then either $\boldsymbol{G}$ is a near-semilattice and $p_2(\boldsymbol{G}) = 1$, or $(xy)(yx)$ is symmetric and $p_2(\boldsymbol{G}) = 3$.

(ii) (a) Assume that $p_2(\boldsymbol{G}) = 3$. By what was proved above, $x(xy) = xy$ and $(xy)(yx)$ is symmetric in $\boldsymbol{G}$. Now, $x((xy)(yx)) \in \{x, y, xy, yx\}$ or $x((xy)(yx)) = y((yx)(xy)) = (xy)(yx)$. Suppose that $x((xy)(yx)) = x$. Then $x = x((x(xy))((xy)x))$. Hence $x = x((xy)(xy)) = x(xy) = xy$, a contradiction. Now, suppose that $x((xy)(yx)) = y$. Then $y = (xy)(((xy)y)(y(xy)))$. Hence $y = (xy)(((xy)y)(xy)) = (xy)((xy)y) = (xy)y$, a contradiction. Assume $x((xy)(yx)) = xy$. Then $((xy)(yx))(x((xy)(yx))) = ((xy)(yx))(xy)$. Hence, by (8.4) and (7.3), we have $xy = x((xy)(yx)) = (xy)(yx)$, so $\boldsymbol{G}$ is commutative, a contradiction. Assume that $x((xy)(yx)) = yx$. Then $((xy)(yx))(x((xy)(yx))) = ((xy)(yx))(yx) = ((yx)(xy))(yx)$. Hence $yx = (yx)(xy)$, a contradiction. Therefore we have $x((xy)(yx)) = y((yx)(xy)) = (xy)(yx)$.

(b) Obvious.

(iii) By formula (1).

11. Groupoids with $(xy)x = yx$. Assume that $\boldsymbol{G}$ satisfies (7.4). If $p_2(\boldsymbol{G}) \leq 3$ then at least one of (8.1)–(8.5) holds in $\boldsymbol{G}$. Note that (8.5) is a dual case of (7.5).

LEMMA 11.1. *Assume that $\boldsymbol{G}$ satisfies* (7.4). *Then*:

(i) *The term operation $x(yx)$ is essentially binary.*
(ii) *If* (8.3) *holds in $\boldsymbol{G}$ then $\boldsymbol{G}$ is a near-semilattice.*
(iii) *If $\boldsymbol{G}$ satisfies* (8.4) *then ${}^2yx = (xy)(yx)$ in $\boldsymbol{G}$.*

Proof. (i) Assume that $x(yx) = x$. Then $xy = (xy)(y(xy))$. So $xy = (xy)y$. Hence $xy = ((xy)y)(xy) = y(xy) = y$, a contradiction.

Now suppose that $x(yx) = y$. By Lemma 9.1(ii), $\boldsymbol{G}$ is a quasigroup. Hence, (7.4) yields $xy = y$, a contradiction.

(ii) Suppose that (8.3) is satisfied in $\boldsymbol{G}$. Putting xy for x in (8.3) we get $(xy)(y(xy)) = (xy)y$. Hence $(xy)(yx) = (xy)y$. Then $((xy)(yx))(xy) = ((xy)y)(xy)$. Hence $(yx)(xy) = y(xy) = yx$. Therefore $yx = (yx)(yx) = ((yx)(xy))(yx) = (xy)(yx) = xy$. So $\boldsymbol{G}$ is commutative.

(iii) By (7.4) we have $((yx)y)(yx) = y(yx)$. Hence $(xy)(yx) = y(yx)$.

LEMMA 11.2. *Assume that $\boldsymbol{G}$ satisfies* (7.4) *and* (8.4). *Then*:

(i) *If $p_2(\boldsymbol{G}) \leq 2$ then either $p_2(\boldsymbol{G}) = 1$ and $\boldsymbol{G}$ is a near-semilattice*, *or $p_2(\boldsymbol{G}) = 2$ and $\boldsymbol{G} \in \mathcal{G}_1^{2\mathrm{d}} \cup \mathcal{G}_4^2 \cup \mathcal{G}_6^{2\mathrm{d}}$.*
(ii) *If $p_2(\boldsymbol{G}) = 3$ then*:
  (a) *$xy^2 = xy$ and the term operation ${}^2xy$ is symmetric in $\boldsymbol{G}$.*
  (b) *The clone $\{x, y, x \circ y\}$, where $x \circ y = {}^2xy$, is a proper subclone of the clone $\{x, y, xy, yx, {}^2xy\}$.*
(iii) *If $xy^2 = xy$ and ${}^2xy$ is symmetric in $\boldsymbol{G}$ then $p_2(\boldsymbol{G}) \leq 3$.*

Proof. (i) and (ii). Since $p_2(\boldsymbol{G}) \leq 3$, at least one of (9.1)–(9.5) holds in $\boldsymbol{G}$. Suppose that (9.1) holds. Then $(x(xy))(xy) = x$. By Lemma 11.1(iii) we get $((yx)(xy))(xy) = x$. So $yx = x$, a contradiction.

Now suppose that (9.2) holds. We have ${}^2yx \in \{x, y, xy, yx, {}^2xy\}$. If $y(yx) = x$ then evidently $\boldsymbol{G} \in \mathcal{G}_1^{2\mathrm{d}}$. Assume that $y(yx) = y$. Then (by Lemma 11.1(iii)), $y = y(yx) = (xy)(yx)$. Hence, by (7.4), $y = (xy)((xy)x) = xy$, a contradiction. Now assume that $y(yx) = xy$. Then $(y(yx))(yx) = (xy)(yx)$. Hence $yx = (xy)(yx) = y(yx) = xy$, a contradiction. Now suppose that $y(yx) = yx$. Then $y(yx) = (xy)(yx) = yx$ and consequently $\boldsymbol{G} \in \mathcal{G}_4^2$. Suppose that $y(yx) = x(xy)$. Then by (8.4) we have $x(x(yx)) = x(yx) = yx$. Hence, as $x(xy)$ is commutative, we get $(yx)((yx)x) = yx$. Therefore $yx = (yx)(yx^2) = (yx)x = x$, a contradiction.

Now assume that (9.3) holds and consider the same cases as above. If $y(yx) = x$, then $(yx)((yx)x) = y$ and $yx = x$, a contradiction. Suppose that $y(yx) = y$. Then $y(y(xy)) = y$. By (8.4) we get $xy = y$, a contradiction.

Now suppose that $y(yx) = xy$. Lemma 11.1(iii) gives $xy = (xy)(yx)$. Hence $xy = ((xy)(yx))(xy)$. Thus by (7.4), $xy = (yx)(xy) = yx$. So $\boldsymbol{G}$ is a near-semilattice. Suppose that $y(yx) = yx$. Then $\boldsymbol{G} \in \mathcal{G}_5^{2\mathrm{d}}$. Evidently if $y(yx) = x(xy)$ and $y(yx)$ is essentially binary and different from xy then $p_2(\boldsymbol{G}) = 3$.

Assume that (9.4) holds. Then $((xy)y)(xy) = (yx)(xy)$. By (7.4) and (8.4), $((xy)y)(xy) = y(xy) = xy$. Hence $xy = ((yx)(xy))(xy) = (yx)(xy)$. So xy is commutative and consequently $\boldsymbol{G}$ is a near-semilattice.

Now assume that (9.5) holds. If $y(yx) = x$ then, by (8.4), $(y(yx))(yx) = x(yx) = yx$. Therefore, by (7.4), we get $yx = ((yx)y)y = (xy)y$, a contradiction. Now assume that $y(yx) = y$. Hence $y = y(yx) = (y(yx))(yx)$. Then, as $xy^2 = yx^2$, we have $y = ((yx)y)y) = (xy)y$, a contradiction. Suppose now that $y(yx) = xy$. Then, by Lemma 11.1(iii), we have $xy = y(yx) = (xy)(yx) = (y(yx))(yx)$. Hence, by (7.4), $xy = ((yx)y)y = (xy)y$. So $\boldsymbol{G}$ is a near-semilattice. Suppose that $y(yx) = yx$. Then $(y(yx))(yx) = yx$. Hence $yx = ((yx)y)y = (xy)y$ and $\boldsymbol{G}$ is a near-semilattice. Assume that $y(yx) = x(xy)$. Then, since $xy^2 = yx^2$ and $p_2(\boldsymbol{G}) \leq 3$, we have $(y(yx))(yx) = ((yx)x)(yx)$. By (8.4) and (7.4), $yx = x(yx) = ((yx)x)(yx)$. Then, as $xy^2 = yx^2$, $yx = ((yx)y)y = (xy)y$. So $\boldsymbol{G}$ is a near-semilattice again.

(iii) By Lemma 11.1(iii) and using formula (1).

12. Groupoids with $(xy)x = (yx)y$. In this section we deal with proper, noncommutative groupoids $\boldsymbol{G}$ such that $(xy)x$ is symmetric in $\boldsymbol{G}$ and $(xy)x \notin \{xy, yx\}$. As in the whole paper, we assume that $p_2(\boldsymbol{G}) \leq 3$ so at least one of (8.1)–(8.5) holds in $\boldsymbol{G}$.

LEMMA 12.1. *Assume that $\boldsymbol{G} \in \mathcal{G}_{\mathrm{I}(7.5)}$ and $(xy)x \notin \{xy, yx\}$. Then*:

(i) *The term operation $x(yx)$ is essentially binary.*

(ii) (8.3) *does not hold in $\boldsymbol{G}$.*

Proof. (i) Assume that (8.1) holds in $\boldsymbol{G}$. Then $x = (x(yx))x$. Hence—by (7.5) and using the dual version of Lemma 8.1—we get $x = ((yx)x)(yx) = yx$, a contradiction. Suppose that (8.2) holds in $\boldsymbol{G}$. So, by Lemma 9.1(ii), $\boldsymbol{G}$ is a quasigroup. Then $(x(yx))x = yx$, $((yx)x)(yx) = yx$, $(yx)x = yx$, $yx = y$, a contradiction.

(ii) Assume that $\boldsymbol{G}$ satisfies (7.5) and (8.3). Consider identities (9.1)–(9.5). Assume that (9.1) holds. Then $((xy)y)(xy) = x(xy)$ and $x(xy) = (y(xy))y$. By (8.3) and (7.5), $x(xy) = (yx)y = (xy)x = y(yx)$. Hence $((xy)x)(xy) = (x(xy))(xy)$. By (9.1), $((xy)x)(xy) = x$. By (7.5), $(x(xy))x = x$. Hence $x = (((x(xy))x)x$. By (9.1), $x = x(xy)$, a contradiction. Now suppose that (9.2) holds. By (8.3), $((xy)y)(xy) = y(xy) = yx$. By (7.5), $yx = ((xy)y)(xy) = (y(xy))y$. By (8.3), $yx = (yx)y$ and consequently yx is commutative. By (9.2), $y = (xy)y = (yx)y = yx$, a contradiction. Assume that

(9.3) holds. Then, by (7.5), $xy = ((xy)y)(xy) = (y(xy))y$. By (8.3), $xy = (yx)y$, a contradiction. Suppose that (9.4) holds. By Lemma 10.1 we get $xy = (yx)x = (yx)(xy) = ((xy)y)(xy)$. By (7.5) and (8.3), $xy = (y(xy))y = (yx)y$, a contradiction. Now, assume that (9.5) holds, $xy^2 \notin \{x, y, xy, yx\}$ and $p_2(\boldsymbol{G}) = 3$. Then $(xy)y = (yx)y$. Hence $y((xy)y) = y((yx)y)$. Therefore, by (8.3), $yx = y(xy) = y(yx)$. Then $yx = (yx)(yx) = (yx)(y(yx)) = (yx)y$, a contradiction.

LEMMA 12.2. *Let $\boldsymbol{G} \in \mathcal{G}_{\mathrm{I}(7.5)(8.4)}$. Then*:

(i) *If $p_2(\boldsymbol{G}) = 3$ then*:

(a) *$\boldsymbol{G}$ satisfies the following identities*:

$$xy^2 = xy, \tag{12.1}$$

$$(xy)(yx) = (yx)(xy), \tag{12.2}$$

$$(xy)x = (yx)y = (xy)(yx) = (yx)(xy). \tag{12.3}$$

(b) *$\boldsymbol{G}$ satisfies one of the following identities*:

$$^2xy = xy, \tag{12.4}$$

$$^2xy = {}^2yx. \tag{12.5}$$

(c) *The clone $\{x, y, x\circ y\}$, where $x\circ y = (xy)(yx)$, is a proper subclone of $\{x, y, xy, yx, (xy)x\}$. Moreover $(G, \circ)$ is a near-semilattice.*

(ii) *If $\boldsymbol{G}$ satisfies* (7.5), (8.4) *and* (12.3) *then $p_2(\boldsymbol{G}) \le 3$.*

Proof. (i) (a) $p_2(\boldsymbol{G}) = 3$ so one of (9.1)–(9.5) holds in $\boldsymbol{G}$. Suppose that (9.1) holds. Then, by (8.4), we have $xy = x((yx)x) = (yx)x = y$, a contradiction. Now suppose that (9.2) holds. Then, by (8.4), $x = (yx)x = (x(yx))x$. By (7.5), $x = ((yx)x)(yx) = x(yx)$. By (8.4), $x = yx$, a contradiction. Assume that (9.4) is satisfied. Then $yx = (yx)(yx) = (x(yx))(yx)$. By (9.4), $yx = (yx)x = y$, a contradiction. Now assume that (9.5) holds and $xy^2 \notin \{x, y, xy, yx\}$. Then (recall that $p_2(\boldsymbol{G}) = 3$)

$$(xy)x = (yx)y = (xy)y = (yx)x. \tag{12.6}$$

Consider identities (10.1)–(10.5). Suppose that (10.1) holds. Then $x = x(xy) = (x(xy))(xy)$. Hence $x = ((xy)x)x = ((yx)x)x$. Thus $x = (x(yx))x = (yx)x$, a contradiction. Suppose that (10.2) holds. Hence $(xy)(x(xy)) = (xy)y$. By (8.4), $(xy)y = (xy)(x(xy)) = x(xy) = y$, a contradiction. Assume that (10.3) holds. By (8.4) and (9.5) we have $xy = (xy)(xy) = (x(xy))(xy) = ((xy)x)x$. By (12.6) we obtain $xy = ((xy)x)x = ((yx)x)x$. By (9.5), $xy = ((yx)x)x = (x(yx))(yx) = yx$, a contradiction. Suppose that (10.4) holds. Hence—putting yx for y—we have $x(x(yx)) = (yx)x$. So by (8.4), $(yx)x = x(x(yx)) = x(yx) = yx$. By (9.5), $\boldsymbol{G}$ is commutative, a contradiction. Now suppose that (10.5) holds and ${}^2xy$ is essentially binary. Thus

$$x(xy) = y(yx) = (xy)x = (yx)y = (xy)y = (yx)x. \tag{12.7}$$

Putting yx for y in (10.5) we obtain $x(x(yx)) = (yx)((yx)x)$. By (8.4) and (12.7) we get $yx = x(x(yx)) = (yx)((yx)x) = (yx)(y(yx)) = y(yx)$. So $\boldsymbol{G}$ is commutative, a contradiction. Thus (12.1) holds in $\boldsymbol{G}$. So in the sequel we can assume that (7.5), (8.4) and (12.1) are satisfied in $\boldsymbol{G}$.

As $p_2(\boldsymbol{G}) \le 3$ at least one of identities (10.6)–(10.9) and (12.2) holds in $\boldsymbol{G}$. Assume that (10.6) holds. Then $(x(yx))((yx)x) = x$. Hence $yx = x$, a contradiction. Suppose that (10.7) holds. Then $((xy)y)(y(xy)) = y$. By (12.1), $(xy)(y(xy)) = y$. Hence $xy = y$, a contradiction. Now suppose that (10.8) holds. Then $((xy)(yx))(xy) = xy$. By (7.5) we obtain $xy = ((yx)(xy))(yx) = yx$, a contradiction. Assume that (10.9) holds. Then, by (7.5), $xy = (yx)(xy) = ((xy)(yx))(xy) = ((yx)(xy))(yx) = yx$, a contradiction. So $(xy)(yx)$ is symmetric (i.e. (12.2) holds). The assumption $p_2(\boldsymbol{G}) \le 3$ yields

$$(xy)x = (yx)y = (xy)(yx) = (yx)(xy).$$

(b) At least one of (10.1)–(10.5) holds in $\boldsymbol{G}$. Suppose that (10.1) holds. Then $x = x(x(yx))$. By (8.4) we have $x = x(yx) = yx$, a contradiction. Assume that (10.2) holds. By (12.1), $y = (xy)((xy)y) = xy$, a contradiction. Assume now that (10.4) holds. Then by (12.1) we have $yx = (yx)x = (x(xy))x$. By (12.3), $yx = ((xy)x)(xy) = ((xy)(yx))(xy) = ((yx)(xy))(yx) = xy$, a contradiction.

(c) Obvious.

(ii) By formula (1).

LEMMA 12.3. *Let* $\boldsymbol{G} \in \mathcal{G}_{\mathrm{I}(7.5)(8.5)}$. *If* $p_2(\boldsymbol{G}) = 3$ *then*:

(i) *The following identities hold*:

$$(xy)x = (yx)y = x(yx) = y(xy). \tag{12.8}$$

(ii) $\boldsymbol{G}$ *satisfies exactly one of the following identities*:

$$xy^2 = x, \tag{12.9}$$

$$xy^2 = yx^2. \tag{12.10}$$

Proof. (i) By the assumption $p_2(\boldsymbol{G}) = 3$.

(ii) Consider identities (9.1)–(9.5). Suppose that (9.2) holds or equivalently $(yx)x = x$. Putting xy for y and using (12.8) we obtain $x = ((xy)x)x = (x(yx))x = ((yx)x)(yx)$. By (9.2) we get $x = x(yx)$, a contradiction. Assume that (9.3) holds. Hence—using (9.3) and (12.8)—we have $x = (xy)(xy) = ((xy)y)(xy) = (y(xy))y$. So $x = ((yx)y)y = (yx)y$, a contradiction. Suppose now that (9.4) holds. By (12.8) we have $x(xy) = x((yx)x) = (yx)(x(yx))$.

Hence $x(xy) = ((yx)x)(yx) = (xy)(yx)$. So we have

$$x(xy) = (xy)(yx). \tag{12.11}$$

Consider identities (10.1)–(10.5). Suppose that (10.1) holds. By (12.11) and (9.4) we have $y = (yx)(xy) = ((xy)y)(xy)$. By (12.8) we get $y = (xy)(y(xy)) = (xy)((xy)x)$. Then by (10.1), $y = xy$, a contradiction. The term operation ${}^2xy$ is dual to xy^2. So ${}^2xy \neq y$, i.e. (10.2) does not hold in $\boldsymbol{G}$. Assume that (10.3) holds. By (10.3) we get $xy = (xy)(xy) = (xy)(x(xy))$. Hence—using (12.8)—we have $xy = x((xy)x) = x(x(yx))$. Thus $xy = x(yx)$, a contradiction. Now suppose that (10.4) holds. Then by (12.11) we have $yx = (xy)(yx)$. So $yx = ((xy)(yx))(yx)$. By (9.4), (12.11) and (10.4) we have $yx = (yx)(xy) = y(yx) = xy$, a contradiction. Now assume that (10.5) holds. So evidently the term operation ${}^2xy$ is essentially binary. As $p_2(\boldsymbol{G}) = 3$ we have

$$(xy)x = (yx)y = x(yx) = y(xy) = x(xy) = y(yx). \tag{12.12}$$

Putting yx for y in (9.4) we get $(x(yx))(yx) = (yx)x = xy$. Hence by (12.12) we obtain $xy = (x(yx))(yx) = (y(yx))(yx) = (yx)y$. So xy is commutative, a contradiction.

LEMMA 12.4. *Let* $\boldsymbol{G} \in \mathcal{G}_{\mathrm{I}(7.5)(8.5)}$. *Then*:

(i) *If* $p_2(\boldsymbol{G}) = 3$ *then* $\boldsymbol{G} \in \mathcal{G}_{10}^3 \cup \mathcal{G}_{11}^3$ *where* $\mathcal{G}_{10}^3 = \mathcal{G}_{\mathrm{I}(12.8)(12.9)}$, $\mathcal{G}_{11}^3 = \mathcal{G}_{\mathrm{I}(12.13)}$ *and*

$$(xy)x = (yx)y = x(yx) = {}^2xy = (xy)(yx) = xy^2 = xy^3. \tag{12.13}$$

(ii) *If* $\boldsymbol{G} \in \mathcal{G}_{10}^3 \cup \mathcal{G}_{11}^3$ *then* $p_2(\boldsymbol{G}) \leq 3$.

(iii) *If* $\boldsymbol{G} \in \mathcal{G}_{10}^3$ *then the clone* $\{x, y, x \circ y\}$, *where* $x \circ y = (xy)x$, *is minimal*, *and* $\boldsymbol{G}$ *is polynomially equivalent to an affine space over* GF(5).

(iv) *If* $\boldsymbol{G} \in \mathcal{G}_{11}^3$ *then the clone in* (iii) *is a proper subclone of* $\{x, y, xy, yx, (xy)x\}$.

P r o o f. (i) By Lemma 12.3 it is enough to prove (12.13). Identity (12.8) was proved, so we must prove that if $\boldsymbol{G} \in \mathcal{G}_{\mathrm{I}(12.8)(12.10)}$ and $p_2(\boldsymbol{G}) = 3$ then

$${}^2xy = {}^2yx, \tag{12.14}$$

$$(xy)(yx) = (yx)(xy), \tag{12.15}$$

$$xy^2 = xy^3. \tag{12.16}$$

From $p_2(\boldsymbol{G}) = 3$ we infer that

$$(xy)x = (yx)y = x(yx) = y(xy) = xy^2 = yx^2. \tag{12.17}$$

As in the proof of Lemma 12.3 consider identities (10.1)–(10.5). Suppose that (10.1) holds. Putting xy for x in this identity we obtain $(xy)((xy)y) = xy$. Hence—using (12.17)—we have $xy = (xy)((xy)y) = (xy)(y(xy))$. Using (12.17) again we obtain $xy = (xy)(y(xy)) = y((xy)y) = y(y(xy)) = y$, a contradiction. Now suppose that (10.2) holds. Putting yx for y in (10.2) and using (12.17) we get $yx = x((xy)x) = (xy)(x(xy)) = (xy)y$. Thus $\boldsymbol{G}$ is commutative, a contradiction. Assume that (10.3) holds. So $xy = (xy)(xy) = (x(xy))(xy)$. By (12.17) we have $xy = ((xy)x)x = ((yx)x)x = ((yx)x)(yx)$. Hence $xy = ((yx)y)(yx) = (y(yx))(yx)$. By (10.3) we obtain $xy = (yx)(yx) = yx$. So $\boldsymbol{G}$ is commutative, a contradiction. Suppose that (10.4) holds. Then $xy = y(yx) = (yx)((yx)y)$. By (12.17) we have $xy = (yx)((yx)x) = x(yx)$, a contradiction. By the assumption $p_2(\boldsymbol{G}) = 3$ we find that (10.5) holds. More exactly,

$$(xy)x = (yx)y = x(yx) = y(xy) = xy^2 = yx^2 = {}^2xy = {}^2yx. \tag{12.18}$$

Now consider identities (10.6)–(10.9) and (12.2). Suppose that (9.2) holds in $\boldsymbol{G}$. Putting xy for y we have $(x(xy))((xy)x) = x$. By (12.18) the term operation $(x(xy))((xy)x)$ is commutative. So we have a contradiction. Now suppose that (10.7) holds. Putting xy for x we get $((xy)y)(y(xy)) = y$, a contradiction. Assume that (10.8) is satisfied. Then $xy = ((xy)(yx))(xy)$. By (7.5) we have $xy = ((yx)(xy))(yx) = yx$, a contradiction. Now assume that (10.9) holds. Then $((xy)(yx))(yx) = yx$. By (12.10) we have $yx = ((yx)(xy))(xy) = xy$, a contradiction. Thus $(xy)(yx)$ is commutative and consequently

$$(xy)x = (yx)y = x(yx) = xy^2 = {}^2xy = (xy)(yx). \tag{12.19}$$

Now consider the identities

$$xy^3 = x, \tag{12.20}$$

$$xy^3 = y, \tag{12.21}$$

$$xy^3 = xy, \tag{12.22}$$

$$xy^3 = yx, \tag{12.23}$$

$$xy^3 = yx^3. \tag{12.24}$$

Assume that (12.20) is satisfied. Then $x = ((x(xy))(xy))(xy)$. By (12.19) we obtain $x = (((xy)x)x)(xy) = (((yx)x)x)(xy)$. By (12.20) we have $x = x(xy)$, a contradiction. Suppose that (12.21) holds. Then—using (12.19)—we get $y = ((xy)y)y = (y(yx))y = ((yx)y)(yx)$. Hence $y = (y(yx))(yx)$. Therefore $y(yx) = ((y(yx))(yx))(yx) = yx$, a contradiction. Now assume that (12.22) holds. By (12.10) we have $xy = (y(xy))(xy)$. Hence $xy = (xy)(xy) = ((y(xy))(xy))(xy) = y(xy)$, a contradiction. Suppose that (12.23) holds. By (12.19), $yx = ((yx)y)y = (y(yx))(yx)$. Hence $yx = ((y(yx))(yx))(yx) = (yx)y$, a contradiction. Thus $\boldsymbol{G}$ satisfies (12.24) and from the assumption $p_2(\boldsymbol{G}) = 3$ we get

$$(xy)x = (yx)y = x(yx) = xy^2 = {}^2xy = (xy)(yx) = xy^3. \tag{12.25}$$

(ii) Assume that $\boldsymbol{G} \in \mathcal{G}^3_{10}$. Observe that

$$x(xy) = yx, \tag{12.26}$$

$$(xy)(yx) = y. \tag{12.27}$$

Indeed, putting yx for x in (12.9) we obtain $yx = ((yx)y)y$. By (12.8) we have $yx = (y(xy))y = ((xy)y)(xy) = x(xy)$. So we have (12.26). To prove (12.27) observe that $y = (y(yx))(yx)$. By (12.26) we have $y = (xy)(yx)$. By (1) we have $p_2(\boldsymbol{G}) \leq 3$.

Assume that $\boldsymbol{G} \in \mathcal{G}^3_{11}$. By (1)—using (12.25)—we have $p_2(\boldsymbol{G}) \leq 3$.

(iii) Assume that $\boldsymbol{G} \in \mathcal{G}^3_{10}$. Consider the binary operation $x \circ y = (xy)x$. Evidently this operation is commutative. It is easy to prove that $(x \circ y) \circ y = xy$. So $\boldsymbol{G}$ is polynomially equivalent to an affine space over GF(5) and its clone is minimal.

(iv) Assume that $\boldsymbol{G} \in \mathcal{G}^3_{11}$. As above $x \circ y = (xy)x$ is commutative. By (12.13) we have $(x \circ y) \circ y = x \circ y$. So $(G, \circ)$ is a near-semilattice and the clone of $\boldsymbol{G}$ is not minimal.

13. The proofs of Theorems 2.3 and 2.4

Proof of Theorem 2.3. Let $\boldsymbol{G} = (G, \cdot)$ be a groupoid such that $p_2(\boldsymbol{G}) = 3$.

Let $\boldsymbol{G} \in \mathcal{G}_{\mathrm{IC}(p_2=3)}$. Using Lemma 3.2 we infer that xy, xy^2, xy^3 are essentially binary. If $xy^2 = yx^2$, then Lemma 3.3 shows that $\boldsymbol{G} \in \mathcal{G}^3_1$. If xy^2 is essentially binary and noncommutative, then by Lemma 3.4, $\boldsymbol{G}$ satisfies either (3.1), (3.2) or (3.3). If $\boldsymbol{G}$ satisfies (3.1), then the statement follows from Proposition 4.4. If $\boldsymbol{G}$ satisfies (3.2), then Proposition 5.5 shows that $\boldsymbol{G} \in \mathcal{G}^3_2$. If $\boldsymbol{G}$ satisfies (3.3), then Proposition 6.4 yields $\boldsymbol{G} \in \mathcal{G}^3_3$.

Let $\boldsymbol{G} \in \mathcal{G}_{\mathrm{I\check{C}}(p_2=3)}$. By Proposition 9.4, $\boldsymbol{G}$ satisfies (7.3), (7.4) or (7.5). If $\boldsymbol{G}$ satisfies (7.3), then Lemma 10.2 shows that $\boldsymbol{G}$ satisfies (8.3), (8.4) or (8.5). If $\boldsymbol{G} \in \mathcal{G}_{\mathrm{I\check{C}}(p_2\leq 3)(7.3)(8.3)}$ then by Lemma 10.3, $\boldsymbol{G} \in \mathcal{G}^3_5$. If $\boldsymbol{G} \in \mathcal{G}_{\mathrm{I\check{C}}(p_2\leq 3)(7.3)(8.4)}$ then by Lemma 10.4, $\boldsymbol{G} \in \mathcal{G}^3_6$. If $\boldsymbol{G} \in \mathcal{G}_{\mathrm{I\check{C}}(p_2\leq 3)(7.3)(8.5)}$ then by the dual version of Lemma 12.2 we infer that $\boldsymbol{G} \in \mathcal{G}^{3\mathrm{d}}_8 \cup \mathcal{G}^{3\mathrm{d}}_9$. If $\boldsymbol{G} \in \mathcal{G}_{\mathrm{I\check{C}}(p_2\leq 3)(7.4)}$ then using Lemma 11.2 we see that $\boldsymbol{G} \in \mathcal{G}^3_7$. If $\boldsymbol{G} \in \mathcal{G}_{\mathrm{I\check{C}}(p_2\leq 3)(7.5)}$ then Lemmas 12.2 and 12.4 give $\boldsymbol{G} \in \mathcal{G}^3_8 \cup \mathcal{G}^3_9 \cup \mathcal{G}^3_{10} \cup \mathcal{G}^3_{11}$.

Thus we have proved that if $\boldsymbol{G} \in \mathcal{G}_{\mathrm{I}}$ and $p_2(\boldsymbol{G}) = 3$, then $\boldsymbol{G} \in \mathcal{G}_{\mathrm{I}(p_2\leq 3)} = \mathcal{G}^3_1 \cup \ldots \cup \mathcal{G}^3_{11} \cup \mathcal{G}^{3\mathrm{d}}_1 \cup \ldots \cup \mathcal{G}^{3\mathrm{d}}_{11}$.

To prove the converse we use the identities of the varieties $\mathcal{G}^3_i$ and $\mathcal{G}^{3\mathrm{d}}_i$ $(i = 1, \ldots, 11)$ and also the formula for $A^{(2)}(\boldsymbol{G})$.

Proof of Theorem 2.4. By Theorem 2.3 and Lemma 4.3, Proposition 5.5 and Lemmas 6.2, 10.3, 10.4, 11.2, 12.2, and 12.4.

14. Examples. In this section we prove that the classes described in Theorem 2.3 are all nonempty. Below we give the tables of eleven groupoids $\boldsymbol{G}_1, \ldots, \boldsymbol{G}_{11}$:

$\boldsymbol{G}_1$	0	1	2	3	4
0	0	2	3	4	4
1	2	1	3	4	4
2	3	3	2	4	4
3	4	4	4	3	4
4	4	4	4	4	4

$\boldsymbol{G}_2$	0	1	2	3	4
0	0	2	4	3	2
1	2	1	3	2	4
2	4	3	2	4	3
3	3	2	4	3	2
4	2	4	3	2	4

$\boldsymbol{G}_3$	0	1	2	3	4
0	0	2	4	2	4
1	2	1	3	3	2
2	4	3	2	3	4
3	2	3	3	3	2
4	4	2	4	2	4

$\boldsymbol{G}_4$	0	1	2	3	4
0	0	2	4	1	3
1	2	1	3	4	0
2	4	3	2	0	1
3	1	4	0	3	2
4	3	0	1	2	4

$\boldsymbol{G}_5$	0	1	2	3	4
0	0	2	2	2	2
1	3	1	3	3	3
2	2	4	2	4	4
3	4	3	4	3	4
4	4	4	4	4	4

$\boldsymbol{G}_6$	0	1	2	3	4
0	0	2	2	3	4
1	3	1	2	3	4
2	2	2	2	4	4
3	3	3	4	3	4
4	4	4	4	4	4

$\boldsymbol{G}_7$	0	1	2	3	4
0	0	2	4	3	4
1	3	1	2	4	4
2	3	2	2	4	4
3	3	2	4	3	4
4	3	2	4	4	4

$\boldsymbol{G}_8$	0	1	2	3	4
0	0	2	2	3	4
1	3	1	2	3	4
2	4	2	2	4	4
3	3	4	4	3	4
4	4	4	4	4	4

$\boldsymbol{G}_9$	0	1	2	3	4
0	0	2	4	3	4
1	3	1	2	4	4
2	4	2	2	4	4
3	3	4	4	3	4
4	4	4	4	4	4

$\boldsymbol{G}_{10}$	0	1	2	3	4
0	0	2	3	4	1
1	3	1	4	2	0
2	4	0	2	1	3
3	1	4	0	3	2
4	2	3	1	0	4

$\boldsymbol{G}_{11}$	0	1	2	3	4
0	0	2	4	4	4
1	3	1	4	4	4
2	4	4	2	4	4
3	4	4	4	3	4
4	4	4	4	4	4

We leave it to the reader to check that $\boldsymbol{G}_i \in \mathcal{G}_i^3$ for $i = 1, \ldots, 11$. The author has checked it using a program written by Marek Żabka.

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Institute of Mathematics
Silesian Technical University
44-100 Gliwice, Poland
E-mail: jagalusz@zeus.polsl.gliwice.pl

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