

*MULTIPLE SOLUTIONS FOR NONLINEAR DISCONTINUOUS
ELLIPTIC PROBLEMS NEAR RESONANCE*

BY

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Abstract. We consider a quasilinear elliptic eigenvalue problem with a discontinuous right hand side. To be able to have an existence theory, we pass to a multivalued problem (elliptic inclusion). Using a variational approach based on the critical point theory for locally Lipschitz functions, we show that we have at least three nontrivial solutions when $\lambda \rightarrow \lambda_1$ from the left, λ_1 being the principal eigenvalue of the p -Laplacian with the Dirichlet boundary conditions.

1. Introduction. In a recent paper (see Kourogenis–Papageorgiou [10]), we examined quasilinear elliptic problems at resonance with discontinuous right hand side and we proved the existence of a nontrivial solution. In the present paper we examine quasilinear elliptic problems near resonance with discontinuities. Semilinear problems near resonance with a continuous right hand side were studied by Mawhin–Schmitt [12], [13], Chiappinelli–De Figueiredo [4] and Chiappinelli–Mawhin–Nugari [5]. In [13] the equation under consideration is an ordinary differential equation (i.e. $N = 1$) and the authors employ a sign condition to establish the existence of three nontrivial solutions. In [12] an analogous abstract result with the sign condition replaced by a Landesman–Lazer type hypothesis can be found. The authors obtain two solutions, one negative and the other positive. In [4], a similar multiplicity result is shown under the hypothesis of linear growth as $x \rightarrow \infty$. In all three papers the approximation of the first eigenvalue is from the left. In [5] the parameter λ is to the right of the first eigenvalue. Again the authors prove the existence of two solutions, one of them positive. All the aforementioned papers use bifurcation theory.

1991 *Mathematics Subject Classification*: Primary 35J20.

Key words and phrases: multiple solutions, discontinuous function, elliptic inclusion, first eigenvalue, p -Laplacian, Rayleigh quotient, nonsmooth Palais–Smale condition, coercive functional, Clarke subdifferential, critical point, generalized directional derivative.

The first named author supported by the General Secretariat of Research and Technology of Greece.

Recently there appeared the interesting works of Ambrosetti–Garcia Azorero–Peral [1] and Ramos–Sanchez [15]. The authors of [1] study (Section 4) the existence of positive solutions for the eigenvalue problem $-\Delta_p x = \lambda f(x)$, $x|_\Gamma = 0$, where $f(x) \simeq x^{p-1}$ near 0 and infinity, and they prove a bifurcation result, both from zero and from infinity. Their approach is based on degree-theoretic arguments. The work of Ramos–Sanchez [15] is closer to ours. They study the semilinear version (i.e. $p = 2$) of our problem with the right hand side function $f(z, x)$ continuous in both variables. In Section 2 of [15], they examine the case when λ approaches λ_1 (the first eigenvalue of $(-\Delta, H_0^1(Z))$) from the left. In Theorem 2.6, they prove the existence of three nontrivial solutions (as we do here in Theorem 7). As we already said, in their problem $f(z, x)$ is jointly continuous, they assume that F is bounded below on $Z \times \mathbb{R}_+$, they have a hypothesis similar to our hypothesis $H(f)$ (iii), but they impose the asymptotic condition on f instead of F (the potential function corresponding to f) as we do, and they also have hypothesis $H(f)$ (iv). Their approach is different from ours and uses the theory of elliptic variational inequalities. It should be mentioned that in Section 4 of [15] they also study the case when λ approaches λ_1 from the right.

Our approach here is variational and is based on the critical point theory for nonsmooth locally Lipschitz functionals, as developed by Chang [3]. For the convenience of the reader, in the next section we outline the basic aspects of this theory.

2. Preliminaries. Chang’s critical point theory for locally Lipschitz functionals is based on the subdifferential theory of Clarke [6] for such functionals. In the previous paper [10], in Section 2, we presented the basic definitions and facts from these theories that are needed in our analysis. The notation introduced there will also be used in this paper. So if X is a Banach space and $f : X \rightarrow \mathbb{R}$ is locally Lipschitz we define the *generalized directional derivative*

$$f^0(x; h) = \overline{\lim}_{\substack{x' \rightarrow x \\ \lambda \downarrow 0}} \frac{f(x' + \lambda h) - f(x')}{\lambda}$$

and the *generalized subdifferential*

$$\partial f(x) = \{x^* \in X^* : (x^*, h) \leq f^0(x; h) \text{ for all } h \in X\}.$$

A point $x \in X$ is a *critical point* of f if $0 \in \partial f(x)$. We say that f satisfies the *nonsmooth Palais–Smale condition* (nonsmooth PS-condition) if any sequence $\{x_n\}_{n \geq 1}$ along which $\{f(x_n)\}_{n \geq 1}$ is bounded and $m(x_n) = \inf\{\|x^*\| : x^* \in \partial f(x_n)\} \rightarrow 0$ as $n \rightarrow \infty$, has a strongly convergent sub-

sequence. This notion generalizes the classical one for C^1 -functionals (see Rabinowitz [14]).

Consider the nonnegative p -Laplacian ($2 \leq p < \infty$) differential operator

$$-\Delta_p x = -\operatorname{div}(\|Dx\|^{p-2} Dx)$$

with Dirichlet boundary conditions; we use the notation $(-\Delta_p, W_0^{1,p}(Z))$. The *first (principal) eigenvalue* λ_1 of this operator is the least real number λ for which the nonlinear elliptic problem

$$(1) \quad \begin{cases} -\operatorname{div}(\|Dx(z)\|^{p-2} Dx(z)) = \lambda|x(z)|^{p-2}x(z) & \text{a.e. on } Z, \\ x|_\Gamma = 0, \end{cases}$$

has a nontrivial solution. The first eigenvalue λ_1 is positive, isolated and simple (i.e. the associated eigenfunctions are constant multiples of each other).

Furthermore, we have the following variational characterization of $\lambda_1 > 0$ (Rayleigh quotient):

$$\lambda_1 = \min\{\|Dx\|_p^p / \|x\|_p^p : x \in W_0^{1,p}(Z)\}.$$

This minimum is realized at the normalized eigenfunction u_1 . Note that if u_1 minimizes the Rayleigh quotient, then so does $|u_1|$ and so we infer that the first eigenfunction u_1 does not change sign on Z . In fact, we can show that $u_1 \neq 0$ a.e. on Z and so we may assume that $u_1(z) > 0$ a.e. on Z . Also, using nonlinear elliptic regularity theory (see Tolksdorf [16]), we can show that $u_1 \in C^{1,\alpha}$ for some $\alpha > 0$. For details we refer to Lindqvist [11]. The Lyusternik–Schnirelmann theory gives, in addition to λ_1 , a whole strictly increasing sequence $\{\lambda_n\}_{n \geq 1}$ of positive numbers for which there exist nontrivial solutions of the eigenvalue problem (1). In other words, the spectrum $\sigma(-\Delta_p)$ of $(-\Delta_p, W_0^{1,p}(Z))$ contains at least these points. However, in general, nothing is known about the possible existence of other points in $\sigma(-\Delta_p) \subseteq [\lambda_1, \infty) \subseteq \mathbb{R}_+$. Nevertheless, we can define

$$\mu = \inf\{\lambda > 0 : \lambda \text{ is an eigenvalue of } (-\Delta_p, W_0^{1,p}(Z)), \lambda \neq \lambda_1\}.$$

Because $\lambda_1 > 0$ is isolated, we have $\mu > \lambda_1$ and if V is a topological complement of $X = \langle u_1 \rangle = \mathbb{R}u_1$, then

$$\mu_V = \inf\{\|Dv\|_p^p / \|v\|_p^p : v \in V\} > \lambda_1, \quad \mu = \sup_V \mu_V.$$

The following theorem is due to Chang [6] and extends to a nonsmooth setting the well-known Mountain Pass Theorem due to Ambrosetti–Rabinowitz [2].

THEOREM 1. *If X is a reflexive Banach space, $R : X \rightarrow \mathbb{R}$ is a locally Lipschitz functional which satisfies the PS-condition and $x_0, x_1 \in X$, $\beta \in \mathbb{R}$ and $\varrho > 0$ are such that*

$$(i) \quad \|x_0 - x_1\| > \varrho,$$

- (iv) $\lim_{|t| \rightarrow \infty} \int_Z F(z, tu_1(z)) dz = \infty$;
- (v) $\lim_{x \rightarrow 0} pF(z, x)/|x|^p < -\lambda_1$ and $\lim_{x \rightarrow 0} F(z, x)/|x|^p > -\infty$ uniformly for almost all $z \in Z$.

REMARK. Hypotheses $H(f)$ are more general than the corresponding ones used by Kourogenis–Papageorgiou [10]. Indeed, while $H(f)$ (i) is common in both papers, $H(f)$ (ii) is weaker than the one of [10], where it is assumed that $|f(z, x)| \leq a(z)$ for almost all $z \in Z$ and all $x \in \mathbb{R}$, with $a \in L^\infty(Z)$. Moreover, $H(f)$ (iii) is weaker than the corresponding one in [10], since it does not imply that $\lim_{x \rightarrow \pm\infty} f(z, x)$ exist and are finite for almost all $z \in Z$. Finally, $H(f)$ (v) is common in both papers and is needed in order to apply Theorem 1.

We introduce the energy functional $R_\lambda : W_0^{1,p}(Z) \rightarrow \mathbb{R}$ defined by

$$R_\lambda(x) = \frac{1}{p} \|Dx\|_p^p - \frac{\lambda}{p} \|x\|_p^p - \int_Z F(z, x(z)) dz.$$

Since $J : W_0^{1,p}(Z) \rightarrow \mathbb{R}$ defined by $J(x) = \int_Z F(z, x(z)) dz$ is locally Lipschitz (see Chang [3]) and

$$x \rightarrow \frac{1}{p} \|Dx\|_p^p \rightarrow \frac{\lambda}{p} \|x\|_p^p$$

are continuous convex functions on $W_0^{1,p}(Z)$, hence locally Lipschitz on $W_0^{1,p}(Z)$, it follows that $R_\lambda(\cdot)$ is locally Lipschitz.

PROPOSITION 2. *If hypotheses $H(f)$ hold and $\lambda < \lambda_1$, then $R_\lambda(\cdot)$ is coercive.*

PROOF. Suppose not. Then we can find $\{x_n\}_{n \geq 1} \subseteq W_0^{1,p}(Z)$ such that $\|x_n\|_{1,p} \rightarrow \infty$ as $n \rightarrow \infty$, and $R_\lambda(x_n) \leq M$ for all $n \geq 1$. We have

$$\frac{1}{p} \|Dx_n\|_p^p - \frac{\lambda}{p} \|x_n\|_p^p - \int_Z F(z, x(z)) dz \leq M.$$

By $H(f)$ (iii), given $\varepsilon > 0$ we can find $M_1 = M_1(\varepsilon) > 0$ such that $F(z, x) \leq (\varepsilon/p)|x|^p$ for almost all $z \in Z$ and all $|x| > M_1$. Also, for almost all $z \in Z$ and all $|x| \leq M_1$ we have $|F(z, x)| \leq a_1(z)$ with $a_1 \in L^\infty(Z)$ (see hypothesis $H(f)$ (ii)). So finally, we can say that for almost all $z \in Z$ and all $x \in \mathbb{R}$ we have $F(z, x) \leq a_1(z) + (\varepsilon/p)|x|^p$. Hence

$$\frac{1}{p} \|Dx_n\|_p^p - \frac{\lambda}{p} \|x_n\|_p^p - \|a_1\|_1 - \frac{\varepsilon}{p} \|x_n\|_p^p \leq M,$$

that is,

$$\frac{1}{p} \|Dx_n\|_p^p \leq M + \|a_1\|_1 + \frac{\lambda + \varepsilon}{p} \|x_n\|_p^p.$$

Since $\|x_n\|_{1,p} \rightarrow \infty$, we have $\|Dx_n\|_p \rightarrow \infty$ and so from the last inequality it follows that $\|x_n\|_p \rightarrow \infty$. Let $y_n = x_n/\|x_n\|_p$, $n \geq 1$. Dividing by $\|x_n\|_p^p$ we obtain

$$(4) \quad \frac{1}{p} \|Dy_n\|_p^p \leq \frac{M}{\|x_n\|_p^p} + \frac{\|a_1\|_1}{\|x_n\|_p^p} + \frac{1}{p}(\lambda + \varepsilon),$$

hence $\{y_n\}_{n \geq 1} \subseteq W_0^{1,p}(Z)$ is bounded (by Poincaré's inequality). Thus by passing to a subsequence if necessary, we may assume that $y_n \rightharpoonup y$ in $W_0^{1,p}(Z)$, $y_n \rightarrow y$ in $L^p(Z)$, $y_n(z) \rightarrow y(z)$ a.e. on Z as $n \rightarrow \infty$ and $|y_n(z)| \leq h_1(z)$ a.e. on Z with $h_1 \in L^p(Z)$.

Passing to the limit in (4) and using the fact that $\|Dy\|_p \leq \underline{\lim} \|Dy_n\|_p$ (weak lower semicontinuity of the norm functional), we obtain

$$\frac{1}{p} \|Dy\|_p^p \leq \frac{\lambda + \varepsilon}{p}.$$

Let $\varepsilon > 0$ be such that $\lambda + \varepsilon < \lambda_1$ (recall that by hypothesis $\lambda < \lambda_1$). Also, since $\|y_n\|_p = 1$, $n \geq 1$, we have $\|y\|_p = 1$, i.e. $y \neq 0$. So we can write

$$\|Dy\|_p^p < \lambda_1 \|y\|_p^p,$$

which contradicts the variational characterization of λ_1 (Rayleigh quotient, see Section 2). This proves the coercivity of $R_\lambda(\cdot)$. ■

Let $X = \langle u_1 \rangle = \mathbb{R}u_1$ and V a topological complement. Then $W_0^{1,p}(Z) = X \oplus V$.

PROPOSITION 3. *If hypotheses H(f) hold, then there exists $\beta < 0$ such that $R_\lambda(v) \geq \beta$ for all $v \in V$ and all $0 < \lambda < \lambda_1$.*

Proof. From Section 2 we know that there exists $\mu > \lambda_1$ such that for all $v \in V$ we have

$$\|Dv\|_p^p \geq \mu \|v\|_p^p.$$

Also, since $0 < \lambda < \lambda_1$, we have

$$R_\lambda(v) \geq \frac{1}{p} \|Dv\|_p^p - \frac{\lambda_1}{p} \|v\|_p^p - \int_Z F(z, v(z)) dz.$$

From the proof of Proposition 2 we know that given $\varepsilon > 0$ we can find $a_1 \in L^\infty(Z)$ (depending on $\varepsilon > 0$) such that for almost all $z \in Z$ and all $x \in \mathbb{R}$ we have

$$F(z, x) \leq a_1(z) + \frac{\varepsilon}{p} |x|^p.$$

So we can write

$$R_\lambda(v) \geq \frac{1}{p} \|Dv\|_p^p - \frac{\lambda_1}{p} \|v\|_p^p - \|a_1\|_1 - \frac{\varepsilon}{p} \|v\|_p^p \geq \frac{1}{p} \left(1 - \frac{\lambda_1 + \varepsilon}{\mu} \right) \|Dv\|_p^p - \|a_1\|_1.$$

Choose $\varepsilon > 0$ so that $\lambda_1 + \varepsilon < \mu$. From the last inequality it follows that $R_\lambda(\cdot)$ is coercive on V , uniformly in $0 < \lambda < \lambda_1$. Thus we can find $\beta < 0$ such that $R_\lambda(v) \geq \beta$ for all $v \in V$ and all $0 < \lambda < \lambda_1$. ■

PROPOSITION 4. *If hypotheses H(f) hold, then there exists $\hat{t} > 0$ such that for every $|t| > \hat{t}$ there exists $\delta_t > 0$ such that $R_\lambda(\pm tu_1) < \beta$ for all $\lambda \in (\lambda_1 - \delta, \lambda_1)$.*

Proof. We have

$$\begin{aligned} R_\lambda(tu_1) &= \frac{|t|^p}{p} \|Du_1\|_p^p - \frac{\lambda|t|^p}{p} \|u_1\|_p^p - \int_Z F(z, tu_1(z)) dz \\ &= \frac{|t|^p}{p} \|Du_1\|_p^p - \frac{\lambda|t|^p}{\lambda_1 p} \|Du_1\|_p^p - \int_Z F(z, tu_1(z)) dz. \end{aligned}$$

By H(f)(iv), we can find $\hat{t} > 0$ so large that for every $|t| > \hat{t}$,

$$-(\beta + 1) < \int_Z F(z, \pm tu_1(z)) dz.$$

Let $\delta = \delta_t = \lambda_1 p / (t^p \|Du_1\|_p^p) > 0$. Then for $\lambda_1 - \delta < \lambda < \lambda_1$ we have

$$R_\lambda(\pm tu_1) < \frac{t^p \delta}{\lambda_1 p} \|Du_1\|_p^p + \beta - 1 = \beta. \quad \blacksquare$$

The next proposition shows that $R_\lambda(\cdot)$ satisfies a sort of nonsmooth Palais–Smale condition over closed and convex subsets of $W_0^{1,p}(Z)$.

PROPOSITION 5. *If hypotheses H(f) hold, $K \subseteq W_0^{1,p}(Z)$ is a nonempty, closed and convex set and $\{x_n\}_{n \geq 1} \subseteq K$ and $\varepsilon_n > 0$, $\varepsilon_n \downarrow 0$ satisfy $|R_\lambda(x_n)| \leq M$, $0 \leq R_\lambda^0(x_n; y - x_n) + \varepsilon_n \|y - x_n\|$ for all $y \in K$, then $\{x_n\}_{n \geq 1} \subseteq W_0^{1,p}(Z)$ has a strongly convergent subsequence.*

Proof. Since by hypothesis $\{R_\lambda(x_n)\}_{n \geq 1}$ is bounded and because $R_\lambda(\cdot)$ is coercive (see Proposition 2) we infer that $\{x_n\}_{n \geq 1} \subseteq W_0^{1,p}(Z)$ is bounded. So by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$ and $x_n \rightarrow x$ in $L^p(Z)$ as $n \rightarrow \infty$. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets of the pair $(W_0^{1,p}(Z), W^{-1,q}(Z))$. We know that $R_\lambda^0(x_n; x - x_n) = \sup\{\langle x^*, x - x_n \rangle : x^* \in \partial R_\lambda(x_n)\}$, $n \geq 1$. Because $\partial R_\lambda(x_n) \subseteq W^{-1,q}(Z)$ is weakly compact, we can find $x_n^* \in \partial R_\lambda(x_n)$ such that $R_\lambda^0(x_n; x - x_n) = \langle x_n^*, x - x_n \rangle$, $n \geq 1$. We have

$$x_n^* = A(x_n) - \lambda k(x_n) - v_n, \quad n \geq 1, \quad k(x) = |x|^{p-2}x,$$

where $A : W_0^{1,p}(Z) \rightarrow W^{-1,q}(Z)$ is defined by

$$\langle A(x), y \rangle = \int_Z \|Dx(z)\|^{p-2} (Dx(z), Dy(z))_{\mathbb{R}^N} dz$$

for all $y \in W_0^{1,p}(Z)$ and $v_n \in \partial J(x_n)$ with $J(x_n) = \int_Z F(z, x_n(z)) dz$. It is easy to see that A is monotone, demicontinuous, hence maximal monotone (see Hu–Papageorgiou [8]) and $f_1(z, x_n(z)) \leq v_n(z) \leq f_2(z, x_n(z))$ a.e. on Z for all $n \geq 1$ (see [10]). Then $\{v_n\}_{n \geq 1} \subseteq L^q(Z)$ is bounded and

$$\begin{aligned} 0 &\leq \langle x_n^*, x - x_n \rangle + \varepsilon_n \|x - x_n\| \\ &= \langle A(x_n), x - x_n \rangle - \int_Z v_n(z)(x - x_n)(z) dz \\ &\quad - \lambda(k(x_n), x - x_n)_{pq} + \varepsilon_n \|x - x_n\|, \end{aligned}$$

hence $\overline{\lim} \langle A(x_n), x_n - x \rangle \leq 0$ since

$$\int_Z v_n(z)(x - x_n)(z) dz \xrightarrow{n \rightarrow \infty} 0, \quad (k(x_n), x - x_n)_{pq} \xrightarrow{n \rightarrow \infty} 0, \quad \varepsilon_n \|x - x_n\| \xrightarrow{n \rightarrow \infty} 0.$$

But A , being maximal monotone, is generalized pseudomonotone (see for example [8], Remark III.6.3, p. 365). So we have

$$\langle A(x_n), x_n \rangle \xrightarrow{n \rightarrow \infty} \langle A(x), x \rangle, \quad \text{hence} \quad \|Dx_n\|_p \xrightarrow{n \rightarrow \infty} \|Dx\|_p.$$

Recall that $Dx_n \xrightarrow{w} Dx$ in $L^p(Z, \mathbb{R}^N)$ and that $L^p(Z, \mathbb{R}^N)$ is uniformly convex. So it has the Kadec–Klee property (see [8], Definition I.1.72, p. 28). Thus $Dx_n \rightarrow Dx$ in $L^p(Z, \mathbb{R}^N)$, from which we conclude that $x_n \rightarrow x$ in $W_0^{1,p}(Z)$ as $n \rightarrow \infty$. ■

Because $\partial R_\lambda(x) \subseteq W^{-1,q}(Z)$ is weakly compact, we can find $x^* \in \partial R_\lambda(x)$ such that $\|x^*\| = m(x) = \inf\{\|y^*\| : y^* \in \partial R_\lambda(x)\}$. Then the same proof as for Proposition 5 gives us the following result:

PROPOSITION 6. *If hypotheses H(f) hold, then $R_\lambda(\cdot)$ satisfies the non-smooth (PS)-condition.*

4. Multiplicity theorem. In this section we state and prove the main result of this paper, which shows that problem (3) has at least three non-trivial solutions for all $\lambda \in (\lambda_1 - \delta, \lambda_1)$ with $\delta > 0$ small enough (i.e. for equations near resonance).

THEOREM 7. *If hypotheses H(f) hold, then there exists $\delta > 0$ such that for all $\lambda \in (\lambda_1 - \delta, \lambda_1)$ problem (3) has at least three nontrivial solutions.*

Proof. From Proposition 3 of [10], we know that we can find $\beta_1, \beta_2 > 0$ such that for all $0 < \lambda \leq \lambda_1$ and all $x \in W_0^{1,p}(Z)$, we have $R_\lambda(x) \geq \beta_1 \|x\|_{1,p}^p - \beta_2 \|x\|_{1,p}^\theta$ with $\theta > p$. So we can find $\varrho > 0$ small enough so that $\inf\{R_\lambda(x) : \|x\|_{1,p} = \varrho\} > 0$ for all $\lambda \in (0, \lambda_1]$. Let $\beta \in \mathbb{R}$ be as in Proposition 3 and let $\hat{t} > 0$ be as in Proposition 4. Choose $t_1 > \max\{\hat{t}, \varrho\}$ and let $\delta = \delta_{t_1}$ be as in Proposition 4. Then $t_1 u_1 \in \overline{B}_\varrho(0)$ and $R_\lambda(t_1 u_1) < \beta < 0$. So we can apply Theorem 1 and obtain $y_0 \neq 0$ such that $R_\lambda(y_0) > 0 > \beta$ and $0 \in \partial R_\lambda(y_0)$.

Let $U^\pm = \{x \in W_0^{1,p}(Z) : x = \pm tu_1 + v, t > 0, v \in V\}$. We show that $R_\lambda(\cdot)$ attains its infimum on both subsets U^+ and U^- . To this end let $m_+ = \inf[R_\lambda(x) : x \in U^+] = \inf[R_\lambda(x) : x \in \overline{U^+}]$ (since $R_\lambda(\cdot)$ is locally Lipschitz). Let

$$\overline{R}_\lambda(x) = \begin{cases} R_\lambda(x) & \text{if } x \in \overline{U^+}, \\ +\infty & \text{otherwise.} \end{cases}$$

Evidently, $\overline{R}_\lambda(\cdot)$ is a lower semicontinuous function on the Banach space $W_0^{1,p}(Z)$ which is bounded below (see Proposition 2). By Ekeland's variational principle (strong form, see for example De Figueiredo [7] or Hu-Papageorgiou [8]), we can find $\{x_n\}_{n \geq 1} \subseteq U^+$ such that $R_\lambda(x_n) \downarrow m_+$ as $n \rightarrow \infty$ and

$$\overline{R}_\lambda(x_n) \leq \overline{R}_\lambda(y) + \varepsilon_n \|y - x_n\| \quad \text{for all } y \in W_0^{1,p}(Z),$$

hence

$$R_\lambda(x_n) \leq R_\lambda(y) + \varepsilon_n \|y - x_n\| \quad \text{for all } y \in \overline{U^+}.$$

Because $\overline{U^+}$ is convex, for every $t \in (0, 1)$ and every $w \in \overline{U^+}$, we have $y_n = (1-t)x_n + tw \in \overline{U^+}$ for all $n \geq 1$. So we have

$$-\varepsilon_n \|w - x_n\| \leq \frac{R_\lambda(x_n + t(w - x_n)) - R_\lambda(x_n)}{t}$$

and therefore

$$0 \leq R_\lambda^0(x_n; w - x_n) + \varepsilon_n \|w - x_n\| \quad \text{for all } w \in \overline{U^+}.$$

Proposition 5 says that by passing to a subsequence if necessary, we may assume that $x_n \rightarrow y_1$ as $n \rightarrow \infty$ in $W_0^{1,p}(Z)$. If $y_1 \in \partial U^+ = V$, then because of Proposition 3, we have $\lim R_\lambda(x_n) = R_\lambda(y_1) = m_+ > \beta$. On the other hand, Proposition 4 implies that there exists $\delta > 0$ such that $m_+ < \beta$ for $\lambda \in (\lambda_1 - \delta, \lambda_1)$. So we have a contradiction from which it follows that $y_1 \in U^+$, $y_1 \neq 0$. Thus y_1 is a local minimum of $R_\lambda(\cdot)$ and so $0 \in \partial R_\lambda(y_1)$ (see Section 2). Working similarly on the other open set U^- , we obtain $y_2 \in U^-$, $y_1 \neq y_2 \neq y_0$ such that $0 \in \partial R_\lambda(y_2)$ for all $\lambda \in (\lambda_1 - \delta, \lambda_1)$.

Finally, let $y = y_k$, $k \in \{1, 2, 0\}$. Since $0 \in \partial R_\lambda(y)$ (where $\lambda_1 - \delta_1 < \lambda < \lambda_1$, $\delta_1 = \delta_{t_1}$), we have

$$A(y) - \lambda |y|^{p-2} y = v \quad \text{in } W_0^{1,p}(Z)$$

with $f_1(z, y(z)) \leq v(z) \leq f_2(z, y(z))$ a.e. on Z . Thus for $\theta \in C_0^\infty(Z)$ we have

$$\langle A(y), \theta \rangle - \lambda \int_Z |y(z)|^{p-2} y(z) \theta(z) dz = \int_Z v(z) \theta(z) dz,$$

hence

$$\int_Z \|Dy(z)\|^{p-2} (Dy(z), D\theta(z))_{\mathbb{R}^N} dz = \int_Z (v(z) + \lambda |y(z)|^{p-2} y(z)) \theta(z) dz$$

and consequently

$$\langle -\operatorname{div}(\|Dy\|^{p-2}Dy), \theta \rangle = \int_Z (v(z) + \lambda|y(z)|^{p-2}y(z))\theta(z) dz$$

(by Green's formula; see for example Kenmochi [9]). Since $C_0^\infty(Z)$ is dense in $W_0^{1,p}(Z)$, we conclude that $y \in W_0^{1,p}(Z)$ solves problem (3).

This proves that y_1, y_2, y_0 are three distinct nontrivial solutions of (3). ■

REMARK. It will be very interesting to know if a similar multiplicity theorem is also valid for the resonant problem.

Acknowledgments. The authors wish to express their gratitude to a very knowledgeable referee for his corrections, remarks, constructive criticism and for furnishing additional relevant references.

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Received 16 September 1998;
revised 28 January 1999