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NONLINEAR HEAT EQUATION WITH A FRACTIONAL LAPLACIAN IN A DISK

BY

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Abstract. For the nonlinear heat equation with a fractional Laplacian

$$u_t + (-\Delta)^{\alpha/2} u = u^2, \quad 1 < \alpha \le 2,$$

the first initial-boundary value problem in a disk is considered. Small initial conditions, homogeneous boundary conditions, and periodicity conditions in the angular coordinate are imposed. Existence and uniqueness of a global-in-time solution is proved, and the solution is constructed in the form of a series of eigenfunctions of the Laplace operator in the disk. First-order long-time asymptotics of the solution is obtained.

1. Introduction. The nonlinear heat equation

$$u_t - \Delta u + u^p = 0, \quad x \in \mathbb{R}^N,\ t > 0, \tag{1.1}$$

and the asymptotic behavior of its solutions were the subject of many papers (see, e.g., [4–9, 11, 12, 15] and the references there). The authors considered primarily initial-value problems imposing some restrictions on the initial data and discussing the asymptotic behavior of solutions in terms of the parameters N, p, and the exponent of decay of initial data. The existence of a global-in-time solution of the Cauchy problem for (1.1) with initial data from $L_{1,\mathrm{loc}}(\mathbb{R}^N)$ was proved in [4] and for the corresponding mixed problem in the bounded domain Ω in [15]. Using the approach of [6–8], i.e., the rescaling technique and the maximum principle, L. Herraiz [9] examined the first initial-boundary value problem for (1.1) in the domain $\mathbb{R}^N \setminus \Omega$, where Ω is bounded. For nonnegative initial data with a power decay at infinity he calculated the first-order long-time asymptotics of the classical solution.

In [26] C. E. Wayne examined the Cauchy problem for (1.1) with a sufficiently smooth nonlinear term $F(u)$ and constructed finite-dimensional invariant manifolds for it. He showed that these manifolds controlled the long-time behavior of solutions and could be used for calculating the higher-order long-time asymptotics. As an example he considered the power non-

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linearity u^4 and obtained the second-order asymptotics of the solution. For parabolic partial differential equations on bounded domains the use of invariant manifolds usually permits establishing the lowest-order long-time asymptotics (see [24] and the references therein). The linear operator of the corresponding equation has a point spectrum, which gives a natural separation of the phase space of the linear problem into stable, unstable, and central subspaces. In this case the asymptotics is characterized by the exponential decay in time rather than the power-law decay that one encounters in initial-value problems (see [26]).

The aim of the present paper is to study the long-time behavior of solutions of the fractional Laplacian version of (1.1) with the particular quadratic nonlinearity,

$$u_t + (-\Delta)^{\alpha/2} u = u^2, \qquad 1 < \alpha \leq 2. \tag{1.2}$$

The case $\alpha = 2$ corresponds to the standard (Gaussian) diffusion while $0 < \alpha < 2$ corresponds to the anomalous one (see [3]). The importance of examining fractional derivative nonlinear dissipative equations was emphasised in [1–3, 13, 14]. Nonlocal Burgers-type equations (similar to (1.2), but with the nonlinearity containing the gradient of u^p) appeared as model equations simplifying the multidimensional Navier–Stokes system with modified dissipativity [1], describing hereditary effects for nonlinear acoustic waves [16], and modeling interfacial growth mechanisms which include trapping surface effects [13]. A variety of physically motivated linear fractal differential equations with applications to hydrodynamics, statistical physics, and molecular biology can be found in [15]. We would like to point out that a rigorous investigation of the long-time behavior of solutions of the Cauchy problems for fractal Burgers-type equations has been conducted in [3], where the first two terms of the asymptotics have been found.

Below we study the first initial-boundary value problem for the equation (1.2) in a circular domain and obtain the higher-order long-time asymptotics of its solution. First, we construct a global-in-time mild solution by means of eigenfunction expansions and perturbation theory, and then we calculate the first-order long-time asymptotics of this solution. The Laplace operator in a disk has a point spectrum, therefore it is natural to expect the exponential decay of the solution in time. The sign of u^2 does not matter for the proof of the existence of global-in-time solutions since we only consider small initial data.

Note that we do not use any of the methods of [2–9, 11, 12, 24, 26]. The basic ideas of our approach stem from the monograph [14], where Cauchy problems for nonlocal evolution equations of the first order in time were considered (fractional derivative terms describing dispersive and dissipative effects appear there in connection with equations governing wave propagation). In the papers [19–23] this method was developed further and adapted not only for solving higher-order in time nonlinear dissipative equations, but also for studying initial-boundary value problems. The latter aspect is more important for us in our present investigation. In [20] the first initial-boundary value problem with small initial data was considered for the spatially 1-D Boussinesq equation on an interval. Its solution was constructed in the form of a Fourier series, whose coefficients in their turn were represented by series in a small parameter present in the initial conditions. The first-order long-time asymptotics was calculated, which showed exponentially damped time oscillations and space evolution. In [21] the second mixed problem with small initial data was studied for the same equation on an interval. Its solution was constructed, and the second-order long-time asymptotics was obtained. The main term came from the linear problem, but the second term was essentially nonlinear and contained Airy functions of a negative argument. In a certain case a blow up of the solution took place. In [22] the radially symmetric problem for the damped Boussinesq equation in a disk was examined. Its global-in-time solution was constructed in the form of a Fourier–Bessel series, and the first-order long-time asymptotics was calculated. In [23] the general spatially 2-D case in a disk was studied, and the long-time asymptotics was computed.

In the radially symmetric problem in a disk considered in [22], we encountered the "loss of smoothness effect", i.e., raising the smoothness of the initial data does not lead to the increase of the regularity of the solution. This is a consequence of the combined influence of the geometry and the nonlinearity. However, in the problem in question, as well as in [23], it does not take place. The regularity of the solution can be somewhat improved by means of imposing more periodicity conditions in the angle (in spite of the poor convergence of the eigenfunction expansion series). It is reflected by the presence of convolutions in the "angular indices" in the sums representing the eigenfunction expansion coefficients.

The presence of the fractional Laplacian, the nonlinearity, and the circular geometry in the problem in question lead to the appearance of the critical power α_{cr} which determines the decay of the residual term of the long-time asymptotics. We must also point out that this asymptotics is nonlinear since the coefficient in its main term is represented by a series of nonlinear approximations.

2. Notation and function spaces. Let $\Omega = \{(r,\theta) : |r| < 1, \theta \in [-\pi,\pi]\}$. Our main tool in studying the first initial-boundary value problem for the equation (1.2) in Ω will be expansions in series of eigenfunctions of the Laplace operator in the disk. For a function $f(r,\theta) \in L_{2,r}(\Omega)$ ($L_2(\Omega)$ with weight r) the corresponding expansion is

$$f(r,\theta) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \widehat{f}_{mn} \chi_{mn}(r,\theta), \tag{2.1}$$

where the $\chi_{mn}(r,\theta)$ are the eigenfunctions of the Laplace operator in Ω, i.e., nontrivial solutions of the problem

$$\begin{gathered} \Delta\chi = -\Lambda\chi, \quad (r,\theta) \in \Omega, \\ \chi|_{\partial\Omega} = 0, \quad \chi(r,\theta+2\pi) = \chi(r,\theta), \quad |\chi(0,\theta)| < \infty. \end{gathered} \tag{2.2}$$

These eigenfunctions and the corresponding eigenvalues are

$$\chi_{mn}(r,\theta) = J_m(\lambda_{mn} r) e^{im\theta}, \quad \Lambda_{mn} = \lambda_{mn}^2, \quad m \in \mathbb{Z},\ n \in \mathbb{N}, \tag{2.3}$$

where $J_m(z)$ are Bessel functions of index m, λ_{mn} are their positive zeros numbered in increasing order, and $n = 1, 2, \ldots$ is the number of the zero.

The system of functions $\{\chi_{mn}(r,\theta)\}_{m\in\mathbb{Z},\, n\in\mathbb{N}}$ is orthogonal and complete in $L_{2,r}(\Omega)$ (see [18, 25]). Denoting the scalar product in $L_{2,r}(\Omega)$ by $(\cdot,\cdot)_{r,0}$ and the corresponding norm by $\|\cdot\|_{r,0}$ we can write

$$(\chi_{mn}, \chi_{kl})_{r,0} = \delta_{mk}\delta_{nl}\|\chi_{mn}\|_{r,0}^2,$$

where δ_{ij} is the Kronecker symbol. By Parseval's identity we have

$$\|f\|_{r,0}^2 = \sum_{m,n} |\widehat{f}_{mn}|^2 \|\chi_{mn}\|_{r,0}^2, \quad \text{where} \quad \widehat{f}_{mn} = (f, \chi_{mn})_{r,0} / \|\chi_{mn}\|_{r,0}^2.$$

Denoting by $\|\cdot\|_r$ the norm in the weighted space $L_{2,r}(0,1)$ we obtain

$$\|\chi_{mn}\|_{r,0}^2 = 2\pi \|J_m(\lambda_{mn} r)\|_r^2 = 2\pi \int_0^1 r J_m^2(\lambda_{mn} r)\, dr = \pi J_{m+1}^2(\lambda_{mn}).$$

The following estimates are valid for large positive λ (see [18]):

$$C_1/\lambda \le \|J_m(\lambda r)\|_r^2 \le C_2/\lambda. \tag{2.4}$$

We shall also need the asymptotics of λ_{mn} as $n \to \infty$. For bounded m we have the following asymptotic formula uniform in m (McMahon's expansion, see [10]):

$$\lambda_{mn} = \mu_{mn} + O\left(\frac{1}{\mu_{mn}}\right), \quad \mu_{mn} = \left(m + 2n - \frac{1}{2}\right)\frac{\pi}{2}, \quad n \to \infty. \tag{2.5}$$

On the basis of the weighted Lebesgue space $L_{2,r}(\Omega)$, we can introduce the weighted Sobolev spaces $H_r^s(\Omega) \equiv W_{2,r}^s(\Omega)$ with the norm defined by the formula

$$\|f\|_{r,s}^2 = \sum_{m,n} \lambda_{mn}^{2s} |\widehat{f}_{mn}|^2 \|\chi_{mn}\|_{r,0}^2,$$

where $\lambda_{mn} > 0$ for all $m \in \mathbb{Z}$, $n \in \mathbb{N}$.

We shall also use the Banach space $C^k([0,\infty), H^s_r(\Omega))$ equipped with the norm

$$\|u\|_{C^k} = \sum_{j=0}^{k} \sup_{t\in[0,\infty)} \|\partial_t^j u(t)\|_{s,r}.$$

3. Main results. We consider the first initial-boundary value problem for the equation (1.2) in the unit disk. Using polar coordinates (r,θ) we can pose this problem as follows:

$$(3.1)\qquad \begin{aligned} &u_t + (-\Delta)^{\alpha/2}u = u^2, && (r,\theta)\in\Omega,\ t>0,\\ &u(r,\theta,0) = \varepsilon^2\varphi(r,\theta), && (r,\theta)\in\Omega,\\ &u|_{\partial\Omega} = 0, && t>0,\\ &|u(0,\theta,t)| < \infty, &&\\ &\text{periodicity conditions in } \theta \text{ with period } 2\pi, && \end{aligned}$$

where $1<\alpha\le 2$, $\varepsilon=\mathrm{const}>0$; $\varphi(r,\theta)$ is a real-valued function, $\Delta = (1/r)\partial_r(r\partial_r) + (1/r^2)\partial_\theta^2$.

Set $A=(-\Delta)^{\alpha/2}$, $1<\alpha\le 2$, where Δ is defined on sufficiently smooth functions satisfying the conditions (2.2).

Definition. We call $u(t)$ a *mild solution* of the problem (3.1) if it satisfies the integral equation

$$(3.2)\qquad u(t) = \varepsilon^2\exp(-tA)\varphi + \int_0^t \exp(-(t-\tau)A)u^2(\tau)\,d\tau, \qquad t>0,$$

in some Banach space E.

Define $\Omega_\delta^{(1)} = \{(r,\theta) : 0\le r<\delta,\ \theta\in[-\pi,\pi]\}$ for sufficiently small $\delta>0$ and $\Omega_\delta^{(2)} = \overline{\Omega}\setminus\Omega_\delta^{(1)}$. Note that $\Omega_\delta^{(2)}$ is a closed domain. Now we formulate some assumptions for a sufficiently smooth function $f(r,\theta)$, $(r,\theta)\in\Omega$.

Assumptions A.

$$\begin{gathered} \partial_\theta^k f(r,-\pi) = \partial_\theta^k f(r,-\pi), \qquad k=0,1;\\ f(1,\theta)=0, \qquad \partial_\theta^2 f(0,\theta) = \partial_\theta^2 f(1,\theta) = 0;\\ V_0^1(\sqrt{r}\,\partial_r f(r,\theta)) = V_1(\theta)\in L_1(-\pi,\pi),\\ \lim_{r\to0^+}\sqrt{r}\,\partial_r f(r,\theta) = F_1(\theta)\in L_1(-\pi,\pi),\\ V_0^1(\sqrt{r}\,\partial_r\partial_\theta^2 f(r,\theta)) = V_{1,2}(\theta)\in L_1(-\pi,\pi),\\ \lim_{r\to0^+}\sqrt{r}\,\partial_r\partial_\theta^2 f(r,\theta) = F_{1,2}(\theta)\in L_1(-\pi,\pi). \end{gathered}$$

Theorem 1. *If $1<\alpha\le 2$ and $\varphi(r,\theta)$ satisfies Assumptions* A, *then there is $\varepsilon_0>0$ such that for $\varepsilon\in[0,\varepsilon_0]$ there exists a unique mild solution* *of the problem* (2.1) *in the space* $C^0([0,\infty), H_r^s(\Omega))$, $s<\alpha-1/2$. *It can be represented as*

$$u(r,\theta,t)=\sum_{n=1}^{\infty}\widehat{u}_{0n}(t)J_0(\lambda_{0n}r) + \sum_{m,n=1}^{\infty}J_m(\lambda_{mn}r)[\widehat{u}_{mn}(t)e^{im\theta}+\overline{\widehat{u}_{mn}(t)}e^{-im\theta}], \tag{3.3}$$

where the bar denotes complex conjugation and the coefficients $\widehat{u}_{mn}(t)$ *are defined below. Moreover*, $u(r,\theta,t)$ *is continuous and bounded in* $\Omega_\delta^{(2)}\times[0,\infty)$ *and can be represented there as*

$$u(r,\theta,t)=\sum_{N=0}^{\infty}\varepsilon^{N+1}u^{(N)}(r,\theta,t), \tag{3.4}$$

where the functions $u^{(N)}(r,\theta,t)$ *are defined in the proof* (see (5.9)) *and the series converges absolutely and uniformly with respect to* $(r,\theta)\in\Omega_\delta^{(2)}$, $t\in[0,\infty)$, *and* $\varepsilon\in[0,\varepsilon_0]$.

Remark 3.1. It is easy to construct an example of a function $\varphi(r,\theta)$ satisfying the hypotheses of Theorem 1 by separation of variables. Indeed, we have

$$\begin{gathered}\varphi(r,\theta)=R(r)\Theta(\theta),\quad\text{where}\quad R(0)=R(1)=0,\\ V_0^1(\sqrt{r}R'(r))=c_1<\infty,\qquad \lim_{r\to0^+}\sqrt{r}R'(r)=c_2<\infty;\\ \Theta^{(k)}(-\pi)=\Theta^{(k)}(\pi);\quad k=0,1;\quad \Theta''(\theta)\in L_1(-\pi,\pi).\end{gathered}$$

Remark 3.2. Representation (3.3) is a series of regular perturbations with respect to the initial data and can be used as an asymptotic series in the domain $\Omega_\delta^{(2)}\times[0,\infty)$.

Now we sketch briefly the proof of Theorem 1. Seeking the solution of (3.1) in the form of an expansion in eigenfunctions of the Laplace operator in the disk,

$$u(r,\theta,t)=\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\widehat{u}_{mn}(t)\chi_{mn}(r,\theta),$$

and calculating the series expansion coefficients of the nonlinearity $(u^2)^{\wedge}_{mn}(t)$ we substitute the corresponding expansions into (3.1) and obtain an initial-value problem for $\widehat{u}_{mn}(t)$. Integrating it with respect to t we deduce a nonlinear integral equation for $\widehat{u}_{mn}(t)$. To solve this equation we use perturbation theory. Representing the series expansion coefficients as formal series in ε,

$$\widehat{u}_{mn}(t)=\sum_{N=0}^{\infty}\varepsilon^{N+1}\widehat{v}_{mn}^{(N)}(t), \tag{3.5}$$

we substitute them into the integral equation and obtain

$$\widehat{v}_{mn}^{(0)}(t)=\varepsilon\widehat{\varphi}_{mn}\exp(-\lambda_{mn}^{\alpha}t), \tag{3.6}$$

where the $\widehat{\varphi}_{mn}$ are the coefficients of the expansion of the initial function $\varphi(r,\theta)$, and $\widehat{v}_{mn}^{(N)}(t)$, $N\ge 1$, are the nonlinear approximations defined by the recurrence formulas

$$\begin{aligned}\widehat{v}_{mn}^{(N)}(t)=\int_0^t &\exp[-\lambda_{mn}^{\alpha}(t-\tau)]\\ &\times\Big\{\sum_{\substack{p,l\ge 0;\, q,s\ge 1\\ p+l=m}} a_{mnpqls}\sum_{j=1}^{N}\widehat{v}_{pq}^{(j-1)}(\tau)\widehat{v}_{ls}^{(N-j)}(\tau)\\ &+\sum_{\substack{p,q,l,s\ge 1\\ l-p=m}} a_{mnpqls}\sum_{j=1}^{N}\overline{\widehat{v}_{pq}^{(j-1)}(\tau)}\widehat{v}_{ls}^{(N-j)}(\tau)\\ &+\sum_{\substack{p,q,l,s\ge 1\\ p-l=m}} a_{mnpqls}\sum_{j=1}^{N}\widehat{v}_{pq}^{(j-1)}(\tau)\overline{\widehat{v}_{ls}^{(N-j)}(\tau)}\Big\}\,d\tau.\end{aligned} \tag{3.7}$$

Using time estimates of $\widehat{v}_{mn}^{(N)}(t)$, $N\ge 0$, which show the decay in λ_{mn}, we prove that the formally constructed function (3.3) really represents a mild solution of (3.1) from the required function space. Since in $\Omega_{\delta}^{(2)}\times[0,\infty)$ the series (3.3) with $\widehat{u}_{mn}(t)$ defined by (3.5) converges absolutely and uniformly we can change the order of summation and obtain (3.4).

We prove the uniqueness in the following way. Assuming that there exist two solutions of the problem in question, $u^{(1)}$ and $u^{(2)}$, we set $w=u^{(1)}-u^{(2)}$ and expand w in a series of $\chi_{mn}(r,\theta)$. Thus we obtain an integral equation for the series expansion coefficients $\widehat{w}_{mn}(t)$. From this equation we deduce a local-in-time estimate of $\|w(t)\|_{r,0}$ which contains a contradiction. Extending this estimate to all $t\ge 0$ we establish the global-in-time uniqueness.

THEOREM 2. *Under the hypotheses of Theorem* 1, *there exists a constant* C *such that for all* $t>0$,

$$\|u-\widetilde{u}\|_{r,s}\le C\begin{cases}\exp(-\lambda_{11}^{\alpha}t), & 1<\alpha\le\alpha_{\mathrm{cr}},\\ \exp(-2\lambda_{01}^{\alpha}t), & \alpha_{\mathrm{cr}}\le\alpha\le 2,\end{cases} \tag{3.8}$$

where $\widetilde{u}(r,\theta,t)=B_{\varepsilon}J_0(\lambda_{01}r)\exp(-\lambda_{01}^{\alpha}t)$, λ_{01} *and* λ_{11} *are the first positive zeros of the Bessel functions* $J_0(z)$ *and* $J_1(z)$ *respectively*, $\alpha_{\mathrm{cr}}=$

$\ln 2/\ln(\lambda_{11}/\lambda_{01}) \simeq 1.52$, *and the coefficient* B_ε *will be defined in the proof* (see (6.2), (6.3)).

4. Auxiliary results. Let $f(x,\omega)$ be defined on $[0,1]\times[a,b]$, $-\infty < a,b < \infty$. We denote by $V_0^1(f(x,\omega))$ the total variation of $f(x,\omega)$ in $x\in[0,1]$. Consider the integral

$$I_m(\lambda,\omega)=\int_0^1 xf(x,\omega)J_m(\lambda x)\,dx,\qquad m\ge 0,\ \lambda>0,\ \omega\in[a,b].$$

The following lemma can be found in [23], and it is the extension of the proposition given in [25, p. 595] to the case when the integral in question depends on two parameters, λ and ω. Since the proof is not long we present it below for the reader's convenience.

LEMMA 1. *Suppose that for each fixed* $\omega\in[a,b]$ *the function* $\sqrt{x}f(x,\omega)$ *has a bounded total variation in* $x\in[0,1]$. *Moreover, assume that this variation is absolutely integrable in* $\omega\in[a,b]$, *i.e.*, $V_0^1(\sqrt{x}f(x,\omega)) = V_f(\omega)\in L_1(a,b)$, *and*

$$\lim_{x\to 0^+}\sqrt{x}f(x,\omega)=F(\omega)\in L_1(a,b).$$

Then for all $m\ge 0$, $\lambda>0$, *and* $\omega\in[a,b]$,

$$|I_m(\lambda,\omega)|\le\frac{C_\omega}{\lambda^{3/2}},$$

where $C_\omega\in L_1(a,b)$ *and is independent of* m *and* λ.

P r o o f. It follows from the asymptotics of Bessel functions as $x\to\infty$ that for any $z\in(0,\infty)$,

$$\Big|\int_0^z\sqrt{x}J_m(x)\,dx\Big|\le c<\infty,$$

where c is independent of m and z.

Set $\sqrt{x}f(x,\omega)=\widetilde{f}_\omega(x)$. We can represent $\widetilde{f}_\omega(x)$ as

$$\widetilde{f}_\omega(x)=f_\omega^{(1)}(x)-f_\omega^{(2)}(x),$$

where $f_\omega^{(1)}(x)=V_0^x(\widetilde{f}_\omega(x))$ is the total variation of $\widetilde{f}_\omega(x)$ in $[0,x]$, $x\in[0,1]$, and $f_\omega^{(2)}(x)=V_0^x(\widetilde{f}_\omega(x))-\widetilde{f}_\omega(x)$. The functions $f_\omega^{(1)}(x)$ and $f_\omega^{(2)}(x)$ are nondecreasing in x for each fixed $\omega\in[a,b]$. We have

$$\begin{aligned}
&f_\omega^{(1)}(0)=0,\qquad f_\omega^{(1)}(1)=V_0^1(\widetilde{f}_\omega(x))=V_f(\omega)\in L_1(a,b),\\
&f_\omega^{(2)}(0)=-\widetilde{f}_\omega(0)=-F(\omega)\in L_1(a,b),\\
&f_\omega^{(2)}(1)=V_0^1(\widetilde{f}_\omega(x))-\widetilde{f}_\omega(1)=V_f(\omega)-\widetilde{f}_\omega(1).
\end{aligned}$$

Note that $|\widetilde{f}_\omega(1)| \le |\widetilde{f}_\omega(0)| + V_0^1(\widetilde{f}_\omega(x)) = |F(\omega)| + V_f(\omega)$. Applying the second mean value theorem for integrals we deduce that

$$\begin{aligned}\Big|\int_0^1 \sqrt{x} f_\omega^{(1)}(x) J_m(\lambda x)\,dx\Big| &\le |f_\omega^{(1)}(0)|\Big|\int_0^\xi \sqrt{x} J_m(\lambda x)\,dx\Big| \\ &\quad + |f_\omega^{(1)}(1)|\Big|\int_\xi^1 \sqrt{x} J_m(\lambda x)\,dx\Big| \\ &\le C V_f(\omega)\lambda^{-3/2}.\end{aligned}$$

Estimating the integral $\int_0^1 \sqrt{x} f_\omega^{(2)}(\lambda x) J_m(x)\,dx$ in an analogous manner and combining the results we obtain the required estimate. ■

The next lemma permits one to increase the decay of $I_m(\lambda,\omega)$ in λ.

LEMMA 2. *Let $f(x,\omega)$ have a partial derivative $\partial_x f(x,\omega)$ in $[0,1]$ and let $f(0,\omega) = f(1,\omega) = 0$ (in case $m = 0$ only $f(1,\omega) = 0$). Assume that for any fixed $\omega \in [a,b]$, $\sqrt{x}\,\partial_x f(x,\omega)$ has a bounded total variation in $x \in [0,1]$ which is absolutely integrable in $\omega \in [a,b]$, i.e., $V_0^1(\sqrt{x}\partial_x f(x,\omega)) = V_{f,1}(\omega) \in L_1(a,b)$, and*

$$\lim_{x\to 0^+} \sqrt{x}\,\partial_x f(x,\omega) = F_1(\omega) \in L_1(a,b).$$

Then for $m \ge 0$, $\lambda > 0$ and $\omega \in [a,b]$,

$$|I_m(\lambda,\omega)| \le \frac{C_\omega(m+1)}{\lambda^{5/2}},$$

where $C_\omega \in L_1(a,b)$ and is independent of m and λ.

Proof. We shall use the notations $f_\omega(x) = f(x,\omega)$ and $f'_\omega(x) = \partial_x f(x,\omega)$. Changing the variable $\xi = \lambda x$ and integrating by parts we obtain

$$\begin{aligned} I_m(\lambda,\omega) &= \frac{1}{\lambda^2}\int_0^\lambda \xi f_\omega(\xi/\lambda) J_m(\xi)\,d\xi \\ &= -\frac{1}{\lambda^2}\int_0^\lambda \Big[\frac{1}{\lambda} f'_\omega(\xi/\lambda)\xi - m f_\omega(\xi/\lambda)\Big] J_{m+1}(\xi)\,d\xi \\ &= -\frac{1}{\lambda}\int_0^1 [f'_\omega(x)x - m f_\omega(x)] J_{m+1}(\lambda x)\,dx.\end{aligned}$$

In order to justify these calculations we note that the hypotheses imply that there exists a constant $M_\omega \in L_1(a,b)$ such that $|\sqrt{x} f'_\omega(x)| \le M_\omega$ for

$x \in [0,1]$. Therefore, $f'_\omega(x)$ is absolutely integrable in $x \in [0,1]$. Expanding $f_\omega(x)$ around $x_0 = 0$ and using the boundary condition $f_\omega(0) = 0$ we get, for $x \in (0,1]$,

$$f_\omega(x) = f'_\omega(\vartheta_1 x)x, \qquad 0 < \vartheta_1 < 1.$$

Substituting this expression into the integrand we obtain

$$I_m(\lambda,\omega) = -\frac{1}{\lambda}\int_0^1 [f'_\omega(x) - mf'_\omega(\vartheta_1 x)]xJ_{m+1}(\lambda x)\,dx.$$

Applying Lemma 1 we deduce the required estimate. When $m = 0$ we do not need to expand $f_\omega(x)$ around $x_0 = 0$ and, therefore, we do not need the condition $f_\omega(0) = 0$. ■

REMARK 4.1. The fact that Assumptions A are valid for the initial function means that $\varphi(r,\theta)$ satisfies the hypotheses of Lemma 2 for $m = 0$ and its second derivative $\partial_\theta^2\varphi(r,\theta)$ satisfies the hypotheses of Lemma 2 in the general case.

In the sequel we shall calculate the eigenfunction expansion coefficients of u^2 by means of multiplying two series, i.e.,

$$\begin{aligned}(u^2)^\wedge_{mn}(t) &= \frac{1}{\|\chi_{mn}\|^2_{r,0}}\Big(\sum_{p,q}\widehat{u}_{pq}(t)\chi_{pq}\cdot\sum_{l,s}\widehat{u}_{ls}(t)\chi_{ls},\,\chi_{mn}\Big)_{r,0}\\ &= \frac{1}{\|\chi_{mn}\|^2_{r,0}}\sum_{p,q,l,s}(\chi_{pq}\chi_{ls},\chi_{mn})_{r,0}\widehat{u}_{pq}(t)\widehat{u}_{ls}(t).\end{aligned}$$

Therefore, we shall need estimates of the coefficients

$$a_{mnpqls} = \frac{g_{mnpqls}}{\|J_m(\lambda_{mn}r)\|^2_r}, \qquad g_{mnpqls} = \int_0^1 rJ_m(\lambda_{mn}r)J_p(\lambda_{pq}r)J_l(\lambda_{ls}r)\,dr \tag{4.1}$$

for integers $m,p,l \ge 0$ and $n,q,s \ge 1$.

LEMMA 3. *The following inequality holds*:

$$|a_{mnpqls}| \le C\frac{\sqrt{\lambda_{mn}}}{\sqrt{\lambda_{pq}\lambda_{ls}}}, \tag{4.2}$$

where the constant is independent of m,n,p,q,l,s.

Proof. Using (2.4) and the estimate [18, 25]

$$|J_\nu(z)| \le C/\sqrt{z}, \qquad \nu \ge 0,\ z > 0, \tag{4.3}$$

for each of the Bessel functions in the integrand we obtain (4.2). ■

LEMMA 4. *For a function $f(r,\theta)$ satisfying Assumptions* A *the following estimate holds for integers $m \ge 0$, $n \ge 1$:*

$$|\widehat{f}_{mn}| \le \frac{C}{\lambda_{mn}^{3/2}(m+1)}. \tag{4.4}$$

Proof. By Lemma 2 and (2.4), for $m = 0$ we have

$$|\widehat{f}_{0n}| \le \frac{1}{2\pi\|J_0(\lambda_{0n}r)\|_r^2}\int_{-\pi}^{\pi} d\theta\, \Big|\int_0^1 rJ_0(\lambda_{0n}r)f(r,\theta)\,dr\Big| \le \frac{1}{\lambda_{0n}^{3/2}}\int_{-\pi}^{\pi} C_\theta\,d\theta \le \frac{C}{\lambda_{0n}^{3/2}},$$

where $C_\theta \in L_1(-\pi,\pi)$. To justify the use of the iterated integral above we note that it follows from the hypotheses of the theorem that there exists $N_\theta \in L_1(-\pi,\pi)$ such that $|\partial_r f(r,\theta)| \le N_\theta/\sqrt{r}$, $r \in (0,1)$. Therefore, since $f(1,\theta) = 0$, we have

$$|f(r,\theta)| \le \int_r^1 |\partial_\xi f(\xi,\theta)|\,d\xi \le cN_\theta$$

uniformly in $r \in [0,1]$.

For $m \ge 1$ we can integrate two times in θ using the periodicity conditions $\partial_\theta^k f(r,-\pi) = \partial_\theta^k f(r,-\pi)$, $k = 0,1$, to get

$$\widehat{f}_{mn} = \frac{1}{\|J_m(\lambda_{mn}r)\|_r^2}\int_0^1 rJ_m(\lambda_{mn}r)\wp_m(r)\,dr,$$

$$\wp_m(r) = -\frac{1}{2\pi m^2}\int_{-\pi}^{\pi} e^{-im\theta}\partial_\theta^2 f(r,\theta)\,d\theta.$$

Changing the order of integration and applying Lemma 2 and (2.4) we deduce that

$$|\widehat{f}_{mn}| \le \frac{1}{2\pi\|J_m(\lambda_{mn}r)\|_r^2 m^2}\int_{-\pi}^{\pi} d\theta\,\Big|\int_0^1 rJ_m(\lambda_{mn}r)\partial_\theta^2 f(r,\theta)\,dr\Big| \le \frac{1}{\lambda_{mn}^{3/2}(m+1)}\int_{-\pi}^{\pi} C_\theta\,d\theta \le \frac{C}{\lambda_{mn}^{3/2}(m+1)}.$$

The inequality (4.4) is established. ■

LEMMA 5. *The following estimates are valid for the functions* (3.6), (3.7) *with $m \ge 0$, $n \ge 1$, $N \ge 0$, $t > 0$:*

$$|\widehat{v}_{mn}^{(N)}(t)| \le c^N(N+1)^{-2}\lambda_{mn}^{-(\alpha-1/2)}(m+1)^{-1}\exp(-\lambda_{01}^{\alpha}t). \tag{4.5}$$

Proof. First, we notice that the estimates (4.4) hold for the coefficients $\widehat{\varphi}_{mn}$. Next, we use induction on N. For $N=0$ and sufficiently small ε we have, from (3.6),

$$|\widehat{v}^{(0)}_{mn}(t)| \le \varepsilon |\widehat{\varphi}_{mn}| \exp(-\lambda^{\alpha}_{mn} t) \le \lambda_{mn}^{-3/2}(m+1)^{-1}\exp(-\lambda^{\alpha}_{01}t).$$

Assuming that (4.5) holds for all $\widehat{v}^{(k)}_{mn}(t)$, $0\le k\le N-1$, we shall prove that it is valid for $k=N$. We shall estimate a typical term on the right-hand side of (3.7) using the inequality [14, p. 181]

$$j^{-2}(N+1-j)^{-2} \le 2^2(N+1)^{-2}[j^{-2}+(N+1-j)^{-2}], \qquad 1\le j\le N.$$

Denoting this term by $\Im^{(N)}_{mn}(t)$, we have, by (2.4),

$$\begin{aligned}
|\Im^{(N)}_{mn}(t)| &\le c\lambda_{mn}\int_0^t \exp[-\lambda^{\alpha}_{mn}(t-\tau)] \\
&\qquad \times \sum_{\substack{p,l\ge 0;\, q,s\ge 1\\ p+l=m}} |a_{mnpqls}| \sum_{j=1}^{N} |\widehat{v}^{(j-1)}_{pq}(\tau)|\cdot|\widehat{v}^{(N-j)}_{ls}(\tau)|\,d\tau \\
&\le c\lambda^{1/2}_{mn} S_{mn}(t)\varGamma_m \text{Ç}^{(N)},
\end{aligned}$$

where

$$\begin{aligned}
S_{mn}(t) &= \exp(-\lambda^{\alpha}_{mn}t)\int_0^t \exp[(\lambda^{\alpha}_{mn}-2\lambda^{\alpha}_{01})\tau]\,d\tau,\\
\varGamma_m &= \sum_{\substack{p,l\ge 0;\, q,s\ge 1\\ p+l=m}} \frac{1}{\lambda^{\alpha}_{pq}}\cdot\frac{1}{\lambda^{\alpha}_{ls}}\cdot\frac{1}{(p+1)(l+1)} > 0,\\
\text{Ç}^{(N)} &= \sum_{j=1}^{N} c^{j-1}c^{N-j} j^{-2}(N+1-j)^{-2} \le c^{N-1}(N+1)^{-2}.
\end{aligned}$$

Now we prove that

$$S_{mn}(t) \le C\frac{\exp(-\lambda^{\alpha}_{01}t)}{\lambda^{\alpha}_{mn}}. \tag{4.6}$$

For this purpose we consider several subcases.

(i) If $m=0$, $n=1$, then

$$\begin{aligned}
S_{01}(t) &= \exp(-\lambda^{\alpha}_{01}t)\int_0^t \exp(-\lambda^{\alpha}_{01}\tau)\,d\tau = \exp(-\lambda^{\alpha}_{01}t)\frac{1-\exp(-\lambda^{\alpha}_{01}t)}{\lambda^{\alpha}_{01}}\\
&\le \frac{\exp(-\lambda^{\alpha}_{01}t)}{\lambda^{\alpha}_{01}}.
\end{aligned}$$

(ii) If $m = 0$, $n \geq 2$, then $\lambda_{0n} \geq \lambda_{02} > 2\lambda_{01}$ since $\lambda_{02} \simeq 5.52$ and $\lambda_{01} \simeq 2.42$ (see [10]). Therefore, for $1 < \alpha \leq 2$,

$$\lambda_{0n}^{\alpha} - 2\lambda_{01}^{\alpha} \geq \lambda_{02}^{\alpha} - 2\lambda_{01}^{\alpha} \geq \lambda_{01}^{\alpha}\left[\left(\frac{\lambda_{02}}{\lambda_{01}}\right)^{\alpha} - 2\right] > 0,$$

and for $n \geq 2$ we have

$$\begin{aligned} S_{0n}(t) &= \exp(-\lambda_{0n}^{\alpha}t)\int_0^t \exp[(\lambda_{0n}^{\alpha} - 2\lambda_{01}^{\alpha})\tau]\,d\tau \\ &= \exp(-\lambda_{0n}^{\alpha}t)\frac{\exp[(\lambda_{0n}^{\alpha} - 2\lambda_{01}^{\alpha})t] - 1}{\lambda_{0n}^{\alpha} - 2\lambda_{01}^{\alpha}} \\ &\leq \frac{\exp(-2\lambda_{01}^{\alpha}t)}{\lambda_{0n}^{\alpha}[1 - 2(\lambda_{01}/\lambda_{0n})^{\alpha}]} \leq C\frac{\exp(-\lambda_{01}^{\alpha}t)}{\lambda_{0n}^{\alpha}}. \end{aligned}$$

(iii) If $m = 1$, $n = 1$, then $\lambda_{11} \simeq 3.83$, and $\lambda_{11}^{\alpha} - 2\lambda_{01}^{\alpha} < 0$ if $1 < \alpha < \alpha_{\mathrm{cr}}$, where $\alpha_{\mathrm{cr}} = \ln 2/\ln(\lambda_{11}/\lambda_{01}) \simeq 1.53$, $\lambda_{11}^{\alpha} - 2\lambda_{01}^{\alpha} = 0$ for $\alpha = \alpha_{\mathrm{cr}}$; and $\lambda_{11}^{\alpha} - 2\lambda_{01}^{\alpha} > 0$ for $\alpha_{\mathrm{cr}} < \alpha \leq 2$.

Therefore, if $1 < \alpha < \alpha_{\mathrm{cr}}$, we can write

$$\begin{aligned} S_{11}(t) &= \exp(-\lambda_{11}^{\alpha}t)\int_0^t \exp[-(2\lambda_{01}^{\alpha} - \lambda_{11}^{\alpha})\tau]\,d\tau \\ &= \exp(-\lambda_{11}^{\alpha}t)\frac{1 - \exp[-(2\lambda_{01}^{\alpha} - \lambda_{11}^{\alpha})t]}{2\lambda_{01}^{\alpha} - \lambda_{11}^{\alpha}} \\ &\leq C\frac{\exp(-\lambda_{11}^{\alpha}t)}{\lambda_{11}^{\alpha}[2(\lambda_{01}/\lambda_{11})^{\alpha} - 1]} \leq C\frac{\exp(-\lambda_{01}^{\alpha}t)}{\lambda_{11}^{\alpha}}. \end{aligned}$$

If $\alpha = \alpha_{\mathrm{cr}}$, then

$$S_{11}(t) = \exp(-\lambda_{11}^{\alpha}t)\int_0^t d\tau = t\exp(-\lambda_{11}^{\alpha}t) = t\exp(-2\lambda_{01}^{\alpha}t) \leq C\frac{\exp(-\lambda_{01}^{\alpha}t)}{\lambda_{11}^{\alpha}}.$$

If $\alpha_{\mathrm{cr}} < \alpha \leq 2$, we can repeat the considerations of item (ii) to get

$$S_{11}(t) = \exp(-\lambda_{11}^{\alpha}t)\int_0^t \exp[(\lambda_{11}^{\alpha} - 2\lambda_{01}^{\alpha})\tau]\,d\tau \leq C\frac{\exp(-2\lambda_{01}^{\alpha}t)}{\lambda_{11}^{\alpha}} \leq C\frac{\exp(-\lambda_{01}^{\alpha}t)}{\lambda_{11}^{\alpha}}.$$

(iv) If $m \geq 1$, $n \geq 2$, then $\lambda_{mn}^{\alpha} - 2\lambda_{01}^{\alpha} \geq \lambda_{12}^{\alpha} - 2\lambda_{01}^{\alpha} > 0$ since $\lambda_{12} \simeq 7.02$. Therefore, the same arguments as in (ii) lead to

$$\begin{aligned} S_{mn}(t) &= \exp(-\lambda_{mn}^{\alpha}t)\frac{\exp[(\lambda_{mn}^{\alpha} - 2\lambda_{01}^{\alpha})t] - 1}{\lambda_{mn}^{\alpha} - 2\lambda_{01}^{\alpha}} \\ &\leq C\frac{\exp(-2\lambda_{01}^{\alpha}t)}{\lambda_{mn}^{\alpha}} \leq C\frac{\exp(-\lambda_{01}^{\alpha}t)}{\lambda_{mn}^{\alpha}}. \end{aligned}$$

The estimate (4.6) is established.

Next, we examine $\mathit{\Gamma}_m$ and prove that for $m \geq 0$,

$$\mathit{\Gamma}_m \leq \frac{C}{m+1}. \tag{4.7}$$

If $m = 0$, then $p = l = 0$ and

$$\mathit{\Gamma}_0 = \sum_{q,s=1}^{\infty} \frac{1}{\lambda_{0q}^{\alpha}} \cdot \frac{1}{\lambda_{0s}^{\alpha}} < \infty$$

since $\alpha > 1$. Assuming that $m \geq 1$ we can estimate $\mathit{\Gamma}_m$ as follows:

$$\mathit{\Gamma}_m \leq \frac{1}{m+2} \sum_{q,l,s=1}^{\infty} \frac{1}{\lambda_{m-l,q}^{\alpha}} \cdot \frac{1}{\lambda_{ls}^{\alpha}} \left(\frac{1}{m-l+1} + \frac{1}{l+1} \right) \leq \frac{C}{m+1}.$$

The convergence of the triple sum follows from the asymptotics (2.5) and the comparison with the corresponding triple integral. Combining (4.6) and (4.7) we establish (4.5) by induction. ∎

Corollary. *The following inequalities hold for* $N \geq 0$, $m = 0$, $n \geq 2$:

$$|\widehat{v}_{mn}^{(N)}(t)| \leq c^N (N+1)^{-2} \lambda_{mn}^{-(\alpha-1/2)} (m+1)^{-1} \exp(-2\lambda_{01}^{\alpha} t). \tag{4.8}$$

Moreover, for $m \geq 1$, $n \geq 1$,

$$|\widehat{v}_{mn}^{(N)}(t)| \leq c^N (N+1)^{-2} \lambda_{mn}^{-(\alpha-1/2)} (m+1)^{-1} \exp(-\lambda_{11}^{\alpha} t). \tag{4.9}$$

Proof. Again we use induction on N. For $N = 0$ we have

$$|\widehat{v}_{0n}^{(0)}(t)| \leq \varepsilon |\widehat{\varphi}_{0n}| \exp(-\lambda_{0n}^{\alpha} t) \leq \lambda_{0n}^{-3/2} \exp(-\lambda_{02}^{\alpha} t) \leq \lambda_{0n}^{-3/2} \exp(-2\lambda_{01}^{\alpha} t).$$

Assuming that (4.8) is valid for all $\widehat{v}_{0n}^{(k)}(t)$, $0 \leq k \leq N-1$, we estimate $\widehat{v}_{0n}^{(N)}(t)$. Since for $m = 0$ the condition $p + l = m$ yields $p = 0$, $l = 0$ we should use the inequalities (4.5) to estimate the term

$$\sum_{j=1}^{N} |\widehat{v}_{0q}^{(j-1)}(\tau)| \cdot |\widehat{v}_{0s}^{(N-j)}(\tau)|$$

and, therefore, obtain for $\widehat{v}_{0n}^{(N)}(t)$ the same estimate as in item (ii) of the proof of Lemma 5. Similar considerations are used to establish (4.9). ∎

5. Proof of Theorem 1

5.1. *Construction of solutions.* We seek solutions of (3.1) in the form

$$u(r,\theta,t) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \widehat{u}_{mn}(t) \chi_{mn}(r,\theta) \tag{5.1}$$

$$\text{with} \quad \widehat{u}_{mn}(t) = \frac{(u, \chi_{mn})_{r,0}(t)}{\|\chi_{mn}\|_{r,0}^2}.$$

Since for integer $m \geq 0$, $J_{-m}(z) = (-1)^m J_m(z)$ and $\lambda_{-m,n} = \lambda_{m,n}$, $n \geq 1$ (see [18, 25]), we can deduce that

$$\widehat{u}_{-m,n}(t) = (-1)^m \overline{\widehat{u}_{-m,n}(t)}, \qquad m \geq 0,\ n \geq 1.$$

Therefore, we can rewrite (5.1) as

$$\begin{aligned} u(r,\theta,t) &= \sum_{n=1}^{\infty} \widehat{u}_{0n}(t) J_0(\lambda_{0n} r) \\ &\quad + \sum_{m,n=1}^{\infty} J_m(\lambda_{mn} r)[\widehat{u}_{mn}(t) e^{im\theta} + \overline{\widehat{u}_{mn}(t)} e^{-im\theta}] \\ &= \sum_{m,n}^{\infty}{}^{*} \widehat{u}_{mn}(t) \chi_{mn}(r,\theta). \end{aligned} \tag{5.2}$$

The right-hand side of (5.2) will be used as the notation for the sum on the left-hand side.

First, we expand u^2 in a series of the type (5.1), then we substitute it into (2.1) to obtain

$$\begin{cases} \widehat{u}'_{mn}(t) + \lambda_{mn}^{\alpha} \widehat{u}_{mn}(t) = (u^2)^{\wedge}_{mn}(t), & t > 0, \\ \widehat{u}_{mn}(0) = \varepsilon^2 \widehat{\varphi}_{mn}, \quad m \in \mathbb{Z},\ n \in \mathbb{N}, \end{cases} \tag{5.3}$$

where $\widehat{\varphi}_{mn}$ are the coefficients of the corresponding expansion of $\varphi(r,\theta)$, i.e.,

$$\varphi(r,\theta) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \widehat{\varphi}_{mn} \chi_{mn}(r,\theta), \qquad \widehat{\varphi}_{mn} = \frac{(\varphi, \chi_{mn})_{r,0}}{\|\chi_{mn}\|^2_{r,0}}.$$

Note that the estimates (4.4) are valid for $\widehat{\varphi}_{mn}$.

Next, we obtain the series expansion coefficients of u^2 by multiplying two series. We have

$$\begin{aligned} &(u^2)^{\wedge}_{mn}(t) \\ &\quad = \frac{1}{\|\chi_{mn}\|^2_{r,0}} \int_0^1 dr\, r J_m(\lambda_{mn} r) \int_{-\pi}^{\pi} d\theta\, e^{-im\theta} \\ &\qquad \times \Big\{ \sum_{q=1}^{\infty} \widehat{u}_{0q}(t) J_0(\lambda_{0q} r) + \sum_{q,p=1}^{\infty} J_p(\lambda_{pq} r)[\widehat{u}_{pq}(t) e^{ip\theta} + \overline{\widehat{u}_{pq}(t)} e^{-ip\theta}] \Big\} \\ &\qquad \times \Big\{ \sum_{s=1}^{\infty} \widehat{u}_{0s}(t) J_0(\lambda_{0s} r) + \sum_{s,l=1}^{\infty} J_l(\lambda_{ls} r)[\widehat{u}_{ls}(t) e^{il\theta} + \overline{\widehat{u}_{ls}(t)} e^{-il\theta}] \Big\}. \end{aligned}$$

Calculating the integrals in θ we deduce that for $m \geq 0$, $n \geq 1$,

$$(5.4)\quad (u^2)^\wedge_{mn}(t) = \sum_{\substack{p,l\geq 0;\, q,s\geq 1\\ p+l=m}} a_{mnpqls}\widehat{u}_{pq}(t)\widehat{u}_{ls}(t) + \sum_{\substack{p,q,l,s\geq 1\\ l-p=m}} a_{mnpqls}\overline{\widehat{u}_{pq}(t)}\widehat{u}_{ls}(t) + \sum_{\substack{p,q,l,s\geq 1\\ p-l=m}} a_{mnpqls}\widehat{u}_{pq}(t)\overline{\widehat{u}_{ls}(t)},$$

where the coeffcients a_{mnpqls} are defined by (4.1).

Setting $\widehat{\Phi}_{mn} = \varepsilon\widehat{\varphi}_{mn}$ (it is convenient to keep ε in these coefficients in order to simplify some estimates) we integrate (5.3) in t to get

$$(5.5)\quad \widehat{u}_{mn}(t) = \varepsilon\widehat{\Phi}_{mn}\exp(-\lambda^\alpha_{mn}t) + \int_0^t \exp[-\lambda^\alpha_{mn}(t-\tau)](u^2)^\wedge_{mn}(\tau)\,d\tau.$$

For solving this nonlinear integral equation we use perturbation theory. Representing $\widehat{u}_{mn}(t)$ as a formal series in ε,

$$(5.6)\quad \widehat{u}_{mn}(t) = \sum_{N=0}^{\infty} \varepsilon^{N+1}\widehat{v}^{(N)}_{mn}(t),$$

we substitute (5.6) into (5.5), compare the coefficients of equal powers of ε and get the following recurrence formulas for $m\geq 0$, $n\geq 1$, $N\geq 0$, $t>0$:

$$(5.7)\quad \widehat{v}^{(0)}_{mn}(t) = \widehat{\Phi}_{mn}\exp(-\lambda^\alpha_{mn}t),$$

$$\widehat{v}^{(N)}_{mn}(t) = \int_0^t \exp[-\lambda^\alpha_{mn}(t-\tau)] \times\Big\{ \sum_{\substack{p,l\geq 0;\, q,s\geq 1\\ p+l=m}} a_{mnpqls}\sum_{j=1}^{N}\widehat{v}^{(j-1)}_{pq}(\tau)\widehat{v}^{(N-j)}_{ls}(\tau) + \sum_{\substack{p,q,l,s\geq 1\\ l-p=m}} a_{mnpqls}\sum_{j=1}^{N}\overline{\widehat{v}^{(j-1)}_{pq}(\tau)}\widehat{v}^{(N-j)}_{ls}(\tau) + \sum_{\substack{p,q,l,s\geq 1\\ p-l=m}} a_{mnpqls}\sum_{j=1}^{N}\widehat{v}^{(j-1)}_{pq}(\tau)\overline{\widehat{v}^{(N-j)}_{ls}(\tau)}\Big\}\,d\tau,\qquad N\geq 1.$$

In order to prove that the formally constructed function (5.2), (5.6), (5.7) is really a mild solution of (2.1) in the space $C^0([0,\infty), H^s_r(\Omega))$, $s<\alpha-1/2$, we should examine the convergence of the series

$$(5.8)\quad u(r,\theta,t) = {\sum_{m,n}}^{*}\Big[\sum_{N=0}^{\infty}\varepsilon^{N+1}\widehat{v}^{(N)}_{mn}(t)\Big]\chi_{mn}(r,\theta),$$

where the $\widehat{v}_{mn}^{(N)}(t)$ have the estimates (4.5), (4.8), (4.9). Making use of (5.6) and choosing $\varepsilon \le \varepsilon_0 < 1/c$ we can establish analogous estimates for $\widehat{u}_{mn}^{(N)}(t)$ (with $c^N(N+1)^{-2}$ replaced by c).

Using (2.4) and (2.5) we deduce that the series

$$\sum_{m,n} \lambda_{mn}^{2s} |\widehat{u}_{mn}(t)|^2 \|J_m(\lambda_{mn} r)\|_r^2$$

representing $\|u\|_{r,s}^2$ converges uniformly with respect to $t \ge 0$ for $s < \alpha - 1/2$. To this end we apply Fubini–Tonelli's theorem to establish the convergence of the iterated series $\sum_m \sum_n$ by means of the comparison with the integral

$$\int_{A_1}^{\infty} \frac{1}{m+1} \int_{B_1}^{\infty} (m + 2n - 1/2)^{2s-2\alpha}\, dn$$

with sufficiently large $A_1, B_1 > 0$. Thus, $u \in C^0([0,\infty), H_r^s(\Omega))$ with $s < \alpha - 1/2$.

We note that for $(r,\theta) \in \Omega_\delta^{(2)}$ and $t \ge 0$ the series (5.2) converges absolutely and uniformly. Indeed, using the estimate (4.3) we get

$$\Big| \sum_{m,n}^{*} \widehat{u}_{mn}(t) \chi_{mn}(r,\theta) \Big| \le \frac{C}{\sqrt{r}} \bigg[\sum_{n=1}^{\infty} \frac{|\widehat{u}_{0n}(t)|}{\sqrt{\lambda_{0n}}} + \sum_{m,n=1}^{\infty} \frac{|\widehat{u}_{mn}(t)|}{\sqrt{\lambda_{mn}}} \bigg].$$

Therefore, for $\varepsilon \le \varepsilon_0 < 1/c$ (where c comes from the estimates (4.5)) we can interchange the order of summation in (5.8) and obtain

$$\begin{aligned} u(r,\theta,t) &= \sum_{N=0}^{\infty} \varepsilon^{N+1} u^{(N)}(r,\theta,t), \\ u^{(N)}(r,\theta,t) &= \sum_{m,n}^{*} \widehat{v}_{mn}^{(N)}(t) \chi_{mn}(r,\theta). \end{aligned} \tag{5.9}$$

From the absolute and uniform (in $r, \theta, t, \varepsilon$) convergence of this series it follows that $u(r,\theta,t)$ is continuous and bounded in $\Omega_\delta^{(2)} \times [0,\infty)$.

5.2. *Uniqueness of solutions.* Assume that there exist two mild solutions $u^{(1)}$ and $u^{(2)}$ of the problem (2.1) in $C^0([0,\infty), H_r^s(\Omega))$, $s < \alpha - 1/2$. Then each of them can be expanded in a series of the type (5.2), and the estimates (4.5) are valid for the corresponding coefficients. Setting $w = u^{(1)} - u^{(2)}$ we expand w in a series of the same type and obtain

$$w(r,\theta,t) = \sum_{m,n}^{*} \widehat{w}_{mn}(t) \chi(r,\theta),$$

where

$$(5.10)\quad \widehat{w}_{mn}(t) = \int_0^t \exp[-\lambda_{mn}^\alpha (t-\tau)]\{[(u^{(1)})^2]^\wedge_{mn}(\tau) - [(u^{(2)})^2]^\wedge_{mn}(\tau)\}\, d\tau.$$

A typical term in the difference in the integrand can be represented as

$$H_{mn}(t) = \sum_{p,q,l,s} a_{mnpqls}[\widehat{u}^{(1)}_{pq}(t)\widehat{w}_{ls}(t) + \widehat{u}^{(2)}_{ls}(t)\widehat{w}_{pq}(t)].$$

For brevity we omit here the convolutions in p and l.

Using the Cauchy–Schwarz inequality and (2.4), (4.2) we can write, for $k = 1, 2$,

$$\begin{aligned}
\Big|\sum_{p,q,l,s} & a_{mnpqls}\widehat{u}^{(k)}_{pq}(t)\widehat{w}_{ls}(t)\Big| \\
&\le C\sqrt{\lambda_{mn}} \sum_{p,q,l,s} \frac{|\widehat{u}^{(k)}_{pq}(t)|}{\sqrt{\lambda_{pq}}} \cdot \frac{|\widehat{w}_{ls}(t)|}{\sqrt{\lambda_{ls}}} \\
&\le C\sqrt{\lambda_{mn}} \Big(\sum_{p,q} \frac{|\widehat{u}^{(k)}_{pq}(t)|^2}{\sqrt{\lambda_{pq}}}\Big)^{1/2} \Big(\sum_{l,s} \frac{|\widehat{w}_{ls}(t)|^2}{\sqrt{\lambda_{ls}}}\Big)^{1/2} \\
&\le C\sqrt{\lambda_{mn}} \Big(\sum_{p,q} |\widehat{u}^{(k)}_{pq}(t)|^2 \|J_p(\lambda_{pq}r)\|_r^2\Big)^{1/2} \\
&\quad \times \Big(\sum_{l,s} |\widehat{w}_{ls}(t)|^2 \|J_p(\lambda_{pq}r)\|_r^2\Big)^{1/2} \\
&\le C\sqrt{\lambda_{mn}} \|u^{(k)}(t)\|_{r,0} \|w(t)\|_{r,0}.
\end{aligned}$$

Since $\|u^{(k)}(t)\|_{r,0} < \infty$ uniformly in $t \ge 0$ we deduce from (5.10) that

$$|\widehat{w}_{mn}(t)|^2 \le C\lambda_{mn} \Big(\int_0^t \exp[-\lambda_{mn}^\alpha (t-\tau)] \|w(\tau)\|_{r,0}\, d\tau\Big)^2.$$

Multiplying this inequality by $\|\chi_{mn}\|_{r,0}^2$ and summing the result over m, n, we find that for some $h > 0$ and $t \in [0, h]$,

$$\|w(t)\|_{r,0}^2 \le CQ(t)(\sup_{t\in[0,h]} \|w(t)\|_{r,0}^2),$$

where

$$\begin{aligned}
Q(t) &= \sum_{m,n} \lambda_{mn} \|\chi_{mn}\|_{r,0}^2 \Big(\int_0^t \exp[-\lambda_{mn}^\alpha (t-\tau)]\, d\tau\Big)^2 \\
&= \sum_{m,n} \lambda_{mn} \|\chi_{mn}\|_{r,0}^2 \Big(\frac{1 - \exp(-\lambda_{mn}^\alpha t)}{\lambda_{mn}^\alpha}\Big)^2.
\end{aligned}$$

Note that $Q(t)$ is a nondecreasing continuous function on $[0,h]$ and $Q(0)=0$. Therefore,

$$(\sup_{t\in[0,T_1]}\|w(t)\|_{r,0})^2 \le CQ(t)(\sup_{t\in[0,h]}\|w(t)\|_{r,0})^2 \le C(h)(\sup_{t\in[0,h]}\|w(t)\|_{r,0})^2,$$

where $C(h)=CQ(h)$. We can make the constant $C(h)$ less than one by an appropriate choice of h. This contradiction yields the uniqueness for $t\in[0,h]$.

Next, we continue this process for the intervals $[T_1,T_2],[T_2,T_3],\dots$ $\dots,[T_k,T_{k+1}],\dots$ with $T_k=kh$ and $k\to\infty$. Since

$$\int_{T_k}^{t}\exp[-\lambda_{mn}^{\alpha}(t-\tau)]\,d\tau=\frac{1-\exp[-\lambda_{mn}^{\alpha}(t-T_k)]}{\lambda_{mn}^{\alpha}}$$

we deduce that for $t\in[T_k,T_{k+1}]$,

$$(\sup_{t\in[T_k,T_{k+1}]}\|w(t)\|_{r,0})^2 \le CQ(t-T_k)(\sup_{t\in[T_k,T_{k+1}]}\|w(t)\|_{r,0})^2.$$

Setting $t=T_k+\eta$, $\eta\in[0,h]$, so that $Q(t-T_k)=Q(\eta)$, and observing that the condition $CQ(\eta)\le CQ(h)<1$ has already been satisfied we establish the uniqueness of solutions for all $t\ge 0$. ∎

6. Proof of Theorem 2: long-time asymptotics. Recalling (5.2) we can represent the solution as

$$u(r,\theta,t)=\widehat{u}_{01}(t)J_0(\lambda_{01}r)+R_1(r,t)+R_2(r,\theta,t), \tag{6.1}$$

where

$$R_1(r,t)=\sum_{n=2}^{\infty}\widehat{u}_{0n}(t)J_0(\lambda_{0n}r),$$
$$R_2(r,\theta,t)=\sum_{m,n=1}^{\infty}J_m(\lambda_{mn}r)[\widehat{u}_{mn}(t)e^{im\theta}+\overline{\widehat{u}_{mn}(t)}e^{-im\theta}].$$

First, we obtain a subtle asymptotic estimate of $\widehat{u}_{01}(t)$ and then estimate the residual terms $R_{1,2}$. By (5.6) we have

$$\widehat{u}_{01}(t)=\sum_{N=0}^{\infty}\varepsilon^{N+1}\widehat{v}_{01}^{(N)}(t).$$

Adding and subtracting the integrals from t to ∞ in the integral representations of $\widehat{v}_{01}^{(N)}(t)$, $N\ge 1$, we can write

$$\widehat{v}_{01}^{(0)}(t) = \exp(-\lambda_{01}^{\alpha} t) B_{\varepsilon}^{(0)}, \quad \widehat{v}_{01}^{(N)}(t) = \exp(-\lambda_{01}^{\alpha} t)[B_{\varepsilon}^{(N)} + R_{01}^{(N)}(t)],$$

$$B_{\varepsilon}^{(0)} = \varepsilon \widehat{\varphi}_{01}, \quad B_{\varepsilon}^{(N)} = \int_0^{\infty} \exp(\lambda_{01}^{\alpha} \tau) G_{01}^{(N)}(\widehat{v}(\tau))\, d\tau,$$

$$R_{01}^{(N)}(t) = \int_t^{\infty} \exp(\lambda_{01}^{\alpha} \tau) G_{01}^{(N)}(\widehat{v}(\tau))\, d\tau,$$

$$\begin{aligned} G_{01}^{(N)}(\widehat{v}(t)) = & \sum_{q,s=1}^{\infty} a_{010q0s} \sum_{j=1}^{\infty} \widehat{v}_{0q}^{(j-1)}(\tau) \widehat{v}_{0s}^{(N-j)}(\tau) \\ & + \sum_{q,l,s=1}^{\infty} a_{01lqls} \sum_{j=1}^{\infty} \overline{\widehat{v}_{lq}^{(j-1)}} \widehat{v}_{ls}^{(N-j)}(\tau) \\ & + \sum_{q,l,s=1}^{\infty} a_{01lqls} \sum_{j=1}^{\infty} \widehat{v}_{lq}^{(j-1)}(\tau) \overline{\widehat{v}_{ls}^{(N-j)}(\tau)}, \quad N \geq 1, \end{aligned} \tag{6.2}$$

where $\widehat{v}_{mn}^{(s)}(t),\ 0 \leq s \leq N-1$, are defined by (5.7).

Next, we estimate $R_B^{(N)}(t)$. By (4.2), (4.5), (4.8), (4.9) we have

$$\begin{aligned} |R_{01}^{(N)}(t)| & \\ & \leq c \int_t^{\infty} \exp(\lambda_{01}^{\alpha} \tau) \\ & \quad \times \Big[\sum_{q,s=1}^{\infty} \frac{1}{\lambda_{0q}^{\alpha}} \cdot \frac{1}{\lambda_{0s}^{\alpha}} \exp(-2\lambda_{01}^{\alpha} \tau) + \sum_{l,q,s=1}^{\infty} \frac{1}{\lambda_{lq}^{\alpha}} \cdot \frac{1}{\lambda_{ls}^{\alpha}} \exp(-2\lambda_{11}^{\alpha} \tau) \Big] d\tau \\ & \leq c \exp(-\lambda_{01}^{\alpha} t). \end{aligned}$$

Therefore, for $t \geq 0$ we obtain

$$|\widehat{u}_{01}(t) - B_{\varepsilon} \exp(-\lambda_{01}^{\alpha} t)| \leq C \exp(-2\lambda_{01}^{\alpha} t),$$

where

$$B_{\varepsilon} = \sum_{N=0}^{\infty} \varepsilon^{N+1} B_{\varepsilon}^{(N)}, \tag{6.3}$$

and this series converges absolutely and uniformly with respect to $\varepsilon \in [0, \varepsilon_0]$.

Using (4.8) and (4.9) we deduce that for $t \geq 0$,

$$\|R_1(t)\|_{r,s} \leq C \exp(-2\lambda_{01}^{\alpha}), \quad \|R_2(t)\|_{r,s} \leq C \exp(-\lambda_{11}^{\alpha}). \tag{6.4}$$

Taking into account that $\lambda_{11}^{\alpha} \leq 2\lambda_{01}^{\alpha}$ for $1 < \alpha \leq \alpha_{\mathrm{cr}}$ and $\lambda_{11}^{\alpha} \geq 2\lambda_{01}^{\alpha}$ for $\alpha_{\mathrm{cr}} \leq \alpha \leq 2$ and combining (6.1), (6.3), and (6.4) we obtain (3.8). ∎

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REFERENCES

[1] C. Bardos, P. Penel, U. Frisch and P. Sulem, *Modified dissipativity for a nonlinear evolution equation arising in turbulence*, Arch. Rational Mech. Anal. 71 (1979), 237–256.

[2] P. Biler, T. Funaki and W. Woyczynski, *Fractal Burgers equations*, J. Differential Equations 148 (1998), 9–46.

[3] P. Biler, G. Karch and W. Woyczynski, *Asymptotics for multifractal conservation laws*, Studia Math. 135 (1999), 231–252.

[4] H. Brezis, *Semilinear equations in R^N without conditions at infinity*, Appl. Math. Optim. 12 (1984), 271–282.

[5] H. Brezis and A. Friedman, *Nonlinear parabolic equations involving measures as initial conditions*, J. Math. Pures Appl. 62 (1983), 73–97.

[6] M. Escobedo, O. Kavian and H. Matano, *Large time behavior of solutions of a dissipative semilinear heat equation*, Comm. Partial Differential Equations 20 (1995), 1427–1452.

[7] A. Gmira and L. Veron, *Large time behaviour of the solutions of a semilinear parabolic equation in $\mathbb{R}^N$*, J. Differential Equations 53 (1984), 258–276.

[8] L. Herraiz, *Asymptotic behavior of solutions of some semilinear parabolic problems*, Ann. Inst. Henri Poincaré Anal. Non Linéaire 16 (1999), 49–105.

[9] —, *A nonlinear parabolic problem in an exterior domain*, J. Differential Equations 142 (1998), 371–412.

[10] E. Jahnke, F. Emde and F. Lösch, *Tables of Higher Functions*, 6th ed., Teubner, Stuttgart, 1960.

[11] S. Kamin and L. A. Peletier, *Large time behaviour of solutions of the heat equation with absorption*, Ann. Scuola Norm. Sup. Pisa 12 (1985), 393–408.

[12] S. Kamin and M. Ughi, *On the behaviour as $t \to \infty$ of the solutions of the Cauchy problem for certain nonlinear parabolic equations*, J. Math. Anal. Appl. 128 (1987), 456–469.

[13] J. A. Mann, Jr. and W. Woyczynski, *Rough surfaces generated by nonlinear transport*, invited paper, Symposium on Nonlinear Diffusion, TMS International Meeting, September 1997.

[14] P. I. Naumkin and I. A. Shishmarëv, *Nonlinear Nonlocal Equations in the Theory of Waves*, Transl. Math. Monographs 133, Amer. Math. Soc., Providence, 1994.

[15] L. Oswald, *Isolated positive singularities for a nonlinear heat equation*, Houston J. Math. 14 (1988), 543–572.

[16] A. I. Saichev and G. M. Zaslavsky, *Fractional kinetic equations*: *solutions and applications*, Chaos 7 (1997), 753–764.

[17] N. Sugimoto, *"Generalized" Burgers equations and fractional calculus*, in: Nonlinear Wave Motion, A. Jeffrey (ed.), Longman Sci., Harlow, 1989, 162–179.

[18] G. Tolstov, *Fourier Series*, Dover, New York, 1962.

[19] V. V. Varlamov, *On the Cauchy problem for the damped Boussinesq equation*, Differential Integral Equations 9 (1996), 619–634.

[20] V. V. Varlamov, *On the initial-boundary value problm for the damped Boussinesq equation*, Discrete Contin. Dynam. Systems 4 (1998), 431–444.
[21] —, *Long-time asymptotics of solutions of the second initial-boundary value problem for the damped Boussinesq equation*, Abstract Appl. Anal., 2 (1998), 97–115.
[22] —, *On the damped Boussinesq equation in a circle*, Nonlinear Anal., to appear.
[23] —, *On the spatially two-dimensional Boussinesq equation in a circular domain*, ibid., submitted.
[24] M. Vishik, *Asymptotic Behaviour of Solutions of Evolutionary Equations*, Cambridge Univ. Press, Cambridge, 1992.
[25] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge Univ. Press, London, 1966.
[26] C. E. Wayne, *Invariant manifolds for parabolic partial differential equations on unbounded domains*, Arch. Rational Mech. Anal. 138 (1997), 279–306.

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