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ON PEAKS IN CARRYING SIMPLICES

BY

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Dedicated to my teacher, Professor Andrzej Krzywicki, on the occasion of his retirement

Abstract. A necessary and sufficient condition is given for the carrying simplex of a dissipative totally competitive system of three ordinary differential equations to have a peak singularity at an axial equilibrium. For systems of Lotka–Volterra type that result translates into a simple condition on the coefficients.

1. Introduction. An n-dimensional system of C^1 ordinary differential equations (ODEs)

$$(\mathrm{S}_n) \qquad \dot{x}^{[i]} = x^{[i]} f^{[i]}(x),$$

where $f = (f^{[1]}, \ldots, f^{[n]}) : K \to \mathbb{R}^n$, $K := \{x = (x^{[1]}, \ldots, x^{[n]}) \in \mathbb{R}^n : x^{[i]} \geq 0 \text{ for } i = 1, \ldots, n\}$ is called *totally competitive* if

$$\frac{\partial f^{[i]}}{\partial x^{[j]}}(x) < 0$$

for all $x \in K$, $i, j = 1, \ldots, n$. We write $F = (F^{[1]}, \ldots, F^{[n]})$ with $F^{[i]}(x) = x^{[i]} f^{[i]}(x)$. The symbol $DF(x)$ denotes the Jacobian matrix of the vector field F at $x \in K$, $DF(x) = [(\partial F^{[i]}/\partial x^{[j]})(x)]_{i,j=1}^n$. Let K° stand for the interior of K in $\mathbb{R}^n$, $K^\circ = \{x \in K : x^{[i]} > 0 \text{ for all } i = 1, \ldots, n\}$.

We say that system (S_n) is *dissipative* if there is a compact invariant set $\Gamma \subset K$ attracting all bounded subsets of K. A compact invariant set $A \subset K$ is a *repeller* if $\alpha(B) \subset A$ for some neighborhood B of A in K. (For the definitions of concepts from the theory of dynamical systems see Hale [5].)

M. W. Hirsch proved in [6] the following result.

THEOREM 1.1. *Assume that* (S_n) *is a dissipative n-dimensional totally competitive system of ODEs having* $\{0\}$ *as a repeller. Then there exists a compact invariant set* Σ *with the following properties*:

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(i) *Σ is homeomorphic via radial projection to the standard $(n-1)$-dimensional probability simplex $\{x \in K : \sum_{i=1}^{n} x^{[i]} = 1\}$.*

(ii) *For each $v \in V$ with positive components the restriction $P_v|_\Sigma$ of the orthogonal projection P_v along v is a Lipeomorphism onto its image.*

(iii) *For each $x \in K \setminus \{0\}$, $\omega(x) \subset \Sigma$.*

We call Σ the *carrying simplex* for (S_n) (after M. L. Zeeman [13]).

In the aforementioned paper Hirsch asked about smoothness of Σ. The problem for $\Sigma \cap K^\circ$ was considered by P. Brunovský [3], I. Tereščák [12] and M. Benaïm [2]. The present author in [8], [9] gave criteria for (the whole of) Σ to be a neatly embedded C^1 manifold-with-corners, whereas in [10] he presented (for $n = 3$) an example of a carrying simplex which is not of class C^1 on a part of its boundary. On the other hand, for $n = 2$ the carrying simplex is always a C^1 manifold-with-boundary, diffeomorphic to the real interval $[0, 1]$ (Mierczyński [11]).

In this paper we investigate the situation (for the system (S_3)) when at some point the carrying simplex has a peak, which means that the tangent cone of Σ is a halfline.

2. Preliminaries.

We specialize to $n = 3$. For the remainder of the paper the standing assumption is:

(S_3) *is a dissipative three-dimensional totally competitive system of ODEs having $\{0\}$ as a repeller.*

The symbol $V := \{v = (v^{[1]}, v^{[2]}, v^{[3]})\}$ stands for the vector space of all three-vectors, with the Euclidean norm $\|\cdot\|$. For $I \subset \{1, 2, 3\}$ we write $V^I := \{v \in V : v^{[i]} = 0 \text{ for } i \in I\}$, and $K^I := \{x \in K : x^{[i]} = 0 \text{ for } i \in I\}$. We also use a dual notation: V_I means $V^{\{1,2,3\} \setminus I}$. Denote the ith vector of the standard basis of V by e_i. Further, $\Sigma^I := \Sigma \cap K^I$, $\Sigma_I := \Sigma \cap K_I$, $\Sigma^\circ := \Sigma \cap K^\circ$.

For a closed set $A \subset \mathbb{R}^3$ and $x \in A$, $\mathcal{C}_x(A)$ denotes the *tangent cone* of A at x, $\mathcal{C}_x(A) := \{\alpha v : \alpha \ge 0$, there is a sequence $\{x_k\} \subset A \setminus \{x\}$, $x_k \to x$ as $k \to \infty$, such that $(x_k - x)/\|x_k - x\| \to v\}$. The cone $\mathcal{C}_x(A)$ is closed, and if x is not isolated in A then $\mathcal{C}_x(A) \ne \emptyset$.

For further reference we restate Hirsch's result:

THEOREM 2.1. *There exists a compact invariant set Σ with the following properties*:

(a) *Σ is homeomorphic via radial projection to the standard two-dimensional probability simplex $\{x \in K : \sum_{i=1}^{3} x^{[i]} = 1\}$.*

(b) *For each $v \in V$ with positive components the mapping $P_v|_\Sigma$ is a Lipeomorphism onto its image.*

(c) *Let* $v \in V^i$ *with* $v^{[j]} > 0$ *for both* $j \neq i$. *Then the mapping* $P_v|_{\Sigma \cap K^i}$ *is a Lipeomorphism onto its image.*

(d) *For each* $x \in K \setminus \{0\}$, $\omega(x) \subset \Sigma$.

An equilibrium is called *axial* if only one of its coordinates is positive. By Theorem 2.1(a) there are precisely three axial equilibria $y_i \in K_i$, and $\Sigma_i = \{y_i\}$.

We say Σ has a *peak singularity* at $y \in \Sigma$ if there is a nonzero vector $p \in V$ such that $\mathcal{C}_y(\Sigma) = \{\alpha p : \alpha \geq 0\}$.

PROPOSITION 2.2. *If* Σ *has a peak singularity at* y *then* y *is an axial equilibrium.*

Proof. Suppose first that $y \in \Sigma^\circ$, that is, all three coordinates of y are positive. Denote by P the orthogonal projection along $v = (1, 1, 1)$ on $S := \{x \in \mathbb{R}^3 : x^{[1]} + x^{[2]} + x^{[3]} = 0\}$. Theorem 2.1(b) states that $P|_\Sigma$ is a Lipeomorphism (hence a homeomorphism) onto its image. Put L to be a Lipschitz constant of the inverse $(P|_\Sigma)^{-1}$. The projection P takes the set $\Sigma \cap K^\circ$ onto the interior of $P\Sigma$ in S. Consequently, the tangent cone $\mathcal{C}_{Py}(P\Sigma)$ is the (two-dimensional) tangent space of S at Py, that is, $\mathcal{C}_{Py}(P\Sigma) = \{v \in V : v^{[1]} + v^{[2]} + v^{[3]} = 0\}$.

Take a unit vector $r \in \mathcal{C}_{Py}(P\Sigma)$. There is a sequence $\{x_k\} \subset \Sigma^\circ \setminus \{y\}$ such that $\lim_{k\to\infty} x_k = y$ and $\lim_{k\to\infty} (Px_k - Py)/\|Px_k - Py\| = r$. By choosing a subsequence if necessary, we can assume $\lim_{k\to\infty} (x_k - y)/\|x_k - y\| = q$. As the derivative of P at y is equal to P, one has $Pq = \beta r$. We claim that $\beta \neq 0$. Indeed,

$$\|Pq\| = \lim_{k\to\infty} \left\| P \frac{x_k - y}{\|x_k - y\|} \right\| = \lim_{k\to\infty} \frac{\|Px_k - Py\|}{\|x_k - y\|} \geq \frac{1}{L}.$$

We have thus proved that P takes $\mathcal{C}_y(\Sigma)$ onto $\mathcal{C}_{Py}(P\Sigma)$. Therefore $\mathcal{C}_y(\Sigma)$ contains two noncollinear vectors, so Σ cannot have a peak singularity at y.

Suppose now that only one of the coordinates of y is zero, say $y \in \Sigma^3 \setminus (\Sigma_1 \cup \Sigma_2)$. Thm. 1 in Mierczyński [11] yields that Σ^3 is a C^1 one-dimensional manifold-with-boundary containing y in its (manifold) interior $\Sigma^3 \setminus (\Sigma_1 \cup \Sigma_2)$. Hence $\mathcal{C}_y(\Sigma^3) \subset \mathcal{C}_y(\Sigma^3)$ is a one-dimensional vector space. Therefore Σ does not have a peak singularity at y. ∎

For an axial equilibrium y_i the Jacobian matrix $DF(y_i)$ leaves both the two-dimensional vector subspaces V^j, $j \neq i$, as well as their one-dimensional intersection V_i, invariant. As $V_i = \operatorname{span} e_i$, e_i is an eigenvector of $DF(y_i)$. Adapting the terminology from Mierczyński [9] we will call the eigenvalue of $DF(y_i)$ corresponding to e_i the *internal eigenvalue* at y_i. By the *external eigenvalue* at y_i in K^j, $j \neq i$, we mean the (unique) eigenvalue of the quotient linear mapping $(DF(y_i)|_{V^j})/V_i$. An eigenvector for $DF(y_i)$ belonging to an external eigenvalue is called an *external eigenvector* (such an eigenvector need not exist, see Lemma 3.1(iii)).

We are now ready to formulate our principal result.

MAIN THEOREM. *The carrying simplex Σ has a peak singularity at an axial equilibrium y_i if and only if the internal eigenvalue at y_i is larger than or equal to the maximum external eigenvalue at y_i. In that case*, $\mathcal{C}_{y_i}(\Sigma) = \{-\alpha e_i : \alpha \ge 0\}$.

3. Proof of the Main Theorem. To streamline the argument and limit the number of indices we assume in the present section that the axial equilibrium under consideration is $y = y_1$. Similarly, we write $e = e_1$.

LEMMA 3.1. (i) *If the internal eigenvalue at y is larger than the external eigenvalue at y in K^3 [resp. in K^2] then there is an external eigenvector in V^3 [resp. in V^2] of the form $(1, a_2, 0)$ with $a_2 > 0$ [resp. of the form $(1, 0, a_3)$ with $a_3 > 0$].*

(ii) *If the internal eigenvalue at y is smaller than the external eigenvalue at y in K^3 [resp. in K^2] then there is an external eigenvector in V^3 [resp. in V^2] of the form $(1, -b_2, 0)$ with $b_2 > 0$ [resp. of the form $(1, 0, -b_3)$ with $b_3 > 0$].*

(iii) *If the internal eigenvalue at y is equal to the external eigenvalue at y in K^3 [resp. in K^2] then there are no external eigenvectors in V^3 [resp. in V^2].*

Proof. It suffices to observe that the matrix of the restriction of $DF(y)$ to K^j, $j = 2, 3$, has the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ with $a < 0$ and $b < 0$, and compute eigenvectors (compare Lemma 2 in [11]). ■

3.1. *Necessity.* Suppose by way of contradiction that the internal eigenvalue at y is smaller than the external eigenvalue at y in, say, K^3. By Mierczyński [11], $\Sigma^3 = \Sigma \cap K^3$ is a C^1 one-dimensional manifold-with-boundary. By the theory of invariant manifolds (see e.g. Hirsch, Pugh and Shub [7]) any locally invariant C^1 one-dimensional submanifold passing through y is tangent either to e or to another eigenvector of $DF(y)|_{V^3}$ (not collinear with e); moreover, in the former case the submanifold is locally unique. As K_1 is an invariant one-dimensional submanifold tangent at y to e, Σ^3 cannot be locally equal to it, since otherwise the radial projection of Σ would not be injective (Theorem 2.1(a)). Consequently, Σ^3 is tangent at y to the vector $(1, -b_2, 0)$ with nonzero second component (by Lemma 3.1(ii)). Hence $\mathcal{C}_y(\Sigma) \supset \mathcal{C}_y(\Sigma^3)$ contains $(1, -b_2, 0)$. On the other hand, any vector in $\mathcal{C}_y(\Sigma^3) \subset \mathcal{C}_y(\Sigma)$ has zero second component (and $\mathcal{C}_y(\Sigma^3) \ne \{0\}$). Therefore $\mathcal{C}_y(\Sigma)$ contains two noncollinear vectors, so Σ does not have a peak singularity at y.

3.2. *Sufficiency.* Put $\mathcal{C} := \mathcal{C}_y(\Sigma)$. We write A for the linear operator $DF(y)$. In the standard basis, A has the matrix

$$\begin{bmatrix} d_{11} & d_{12} & d_{13} \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$$

with

$$d_{11} = y^{[1]}\frac{\partial f^{[1]}}{\partial x^{[1]}}(y) < 0, \quad d_{12} = y^{[1]}\frac{\partial f^{[1]}}{\partial x^{[2]}}(y) < 0, \quad d_{13} = y^{[1]}\frac{\partial f^{[1]}}{\partial x^{[3]}}(y) < 0,$$
$$d_{22} = f^{[2]}(y), \quad d_{33} = f^{[3]}(y).$$

We will prove sufficiency by carefully analyzing the action of the group $\{e^{tA}\}_{t\in\mathbb{R}}$ on the tangent cone $\mathcal{C}$.

As $y \in \Sigma$ is an equilibrium and Σ is invariant, each of the linear operators e^{tA} leaves $\mathcal{C}$ invariant.

Put $\mathcal{C}^{\mathrm{N}} := \mathcal{C} \cap \mathbb{S}$, where $\mathbb{S} := \{v \in V : \|v\| = 1\}$ is the unit sphere in V. For $t \in \mathbb{R}$ define the mapping $\psi_t : \mathbb{S} \to \mathbb{S}$ as

$$\psi_t v := \frac{e^{tA}v}{\|e^{tA}v\|}.$$

The family $\psi = \{\psi_t\}_{t\in\mathbb{R}}$ is the solution flow of the system of ODEs

$$\dot{v} = Av - \langle Av, v\rangle v,$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product on $\mathbb{R}^3$. The flow ψ leaves $\mathcal{C}^{\mathrm{N}}$ invariant.

LEMMA 3.2. *For any nonzero $v \in \mathcal{C}$ we have $v^{[1]} < 0$, $v^{[2]} \geq 0$, $v^{[3]} \geq 0$.*

Proof. The last two inequalities follow by the definition of $\mathcal{C}$ and the fact that $(x-y)^{[2]} \geq 0$ and $(x-y)^{[3]} \geq 0$ for any $x \in K$. Suppose first that there is $v \in \mathcal{C}$ with $v^{[1]} > 0$. As a consequence of the definition of $\mathcal{C}$ there is a point $x \in \Sigma$ such that $x^{[1]} > y^{[1]}$, $x^{[2]} \geq y^{[2]}$ and $x^{[3]} \geq y^{[3]}$. If $x^{[2]} > y^{[2]}$ and $x^{[3]} > y^{[3]}$ then the restriction $P_{x-y}|_{\Sigma}$ of the orthogonal projection along $x-y$ is not injective, which contradicts Theorem 2.1(b). If $x^{[2]} > y^{[2]}$ and $x^{[3]} = y^{[3]}$ then $x \in \Sigma^3 \subset K^3$ and $P_{x-y}|_{\Sigma^3}$ is not injective, which is in contradiction with Theorem 2.1(c). The case $x^{[2]} = y^{[2]}$ and $x^{[3]} > y^{[3]}$ is treated in an analogous way. If $x^{[2]} = y^{[2]}$ and $x^{[3]} = y^{[3]}$ then $y \in \Sigma_1$ and the radial projection of Σ is not injective, contrary to Theorem 2.1(a).

Suppose now that there is a nonzero $v \in \mathcal{C}$ with $v^{[1]} = 0$. Then at least one of the remaining components of v is positive. We have

$$\frac{d}{dt}(e^{tA}v)^{[1]}\bigg|_{t=0} = (Av)^{[1]} = d_{12}v^{[2]} + d_{13}v^{[3]} < 0,$$

from which it follows that $(e^{-tA}v)^{[1]} > 0$ for $t > 0$ sufficiently close to 0. As $e^{-tA}v \in \mathcal{C}$ for all $t \in \mathbb{R}$, this is in contradiction to the above paragraph. ∎

Denote by λ_1 the internal eigenvalue at v, and by λ_2 [resp. λ_3] the external eigenvalue at v in K^2 [resp. in K^3]. The symbol w_j, $j = 2, 3$, stands for the unit external eigenvector corresponding to λ_j (provided it exists) having positive first component.

Suppose that $u \in \mathcal{C}^{\mathrm{N}} \setminus \operatorname{span} e$. The idea of the proof is to find a vector in $\mathcal{C}$ with first component positive, contradicting Lemma 3.2.

We consider four cases (up to relabeling).

Case I: $\lambda_1 > \lambda_2 > \lambda_3$. For the flow ψ the set $\{w_3, -w_3\}$ is a repeller, its dual attractor being $V^2 \cap \mathbb{S}$. The flow ψ restricted to $V^2 \cap \mathbb{S}$ has repeller $\{w_2, -w_2\}$ with $\{e, -e\}$ as its dual attractor (for those concepts see Conley [4] or Akin [1]).

Therefore, if $u \notin V^2$ (notice that in such a case $u^{[2]} > 0$) then $\psi_{-t}u$ converges, as $t \to \infty$, to either w_3 or $-w_3$. The latter case is impossible, since as $u^{[2]} > 0$ and $(-w_3)^{[2]} < 0$ (Lemma 3.1(i)), the image of the mapping $\mathbb{R} \ni t \mapsto \psi_t u$ would meet $V^2 \cap \mathbb{S} = \{v \in \mathbb{S} : v^{[2]} = 0\}$, which is invariant under ψ. By the closedness of the tangent cone we have $w_3 \in \mathcal{C}^{\mathrm{N}}$, which contradicts Lemma 3.2.

Similarly, if $u \in V^2$ (notice that in such a case $u^{[3]} > 0$) then $\psi_{-t}u$ converges, as $t \to \infty$, to either w_2 or $-w_2$. The latter case is impossible, since as $u^{[3]} > 0$ and $(-w_2)^{[3]} < 0$ (Lemma 3.1(i)), the image of the mapping $\mathbb{R} \ni t \mapsto \psi_t u$ would meet $V^3 \cap \mathbb{S} = \{v \in \mathbb{S} : v^{[3]} = 0\}$, which is invariant under ψ. By the closedness of the tangent cone, $w_2 \in \mathcal{C}^{\mathrm{N}}$, contradicting Lemma 3.2.

Case II: $\lambda_1 = \lambda_2 > \lambda_3$. The set $\{w_3, -w_3\}$ is a repeller, with dual attractor $V^2 \cap \mathbb{S}$. If $u \notin V^2$ the proof goes along the lines of Case I.

On $V^2 \cap \mathbb{S}$, $\{e, -e\}$ is the set of equilibria, and for $u \in V^2$ we have $\psi_t u \to e$ or $\psi_{-t}u \to e$ as $t \to \infty$ (compare the proof of Lemma 2 in [11]). By the closedness of the tangent cone, $e \in \mathcal{C}$, which is impossible.

Case III: $\lambda_1 > \lambda_2 = \lambda_3$. The flow ψ has a repeller, $\operatorname{span}\{w_2, w_3\} \cap \mathbb{S}$, consisting of fixed points. Its dual attractor is $\{e, -e\}$.

We write u as $\alpha e + \beta \widetilde{u}$, where $\widetilde{u}$ is a unit vector in $\operatorname{span}\{w_2, w_3\}$ such that $\widetilde{u}^{[2]} \geq 0$ and $\widetilde{u}^{[3]} \geq 0$ (at least one of these components must be positive). Such a $\widetilde{u}$ is unique. Writing $\widetilde{u} = \gamma w_2 + \delta w_3$ and observing that w_2 has sign pattern $(+, 0, +)$ and w_3 has sign pattern $(+, +, 0)$ (Lemma 3.1(i)) we have $\widetilde{u}^{[1]} > 0$. The vector subspace $U := \operatorname{span}\{e, \widetilde{u}\}$ is invariant under A, hence $U \cap \mathbb{S}$ is invariant under ψ. The flow ψ restricted to $U \cap \mathbb{S}$ has repeller $\{\widetilde{u}, -\widetilde{u}\}$ with dual attractor $\{e, -e\}$. Consequently, $\psi_{-t}u \to \widetilde{u}$ or $\psi_{-t}u \to -\widetilde{u}$ as $t \to \infty$ (in fact, the former is the case, but we do not need it here). By Lemma 3.2 neither $\widetilde{u}$ nor $-\widetilde{u}$ can belong to $\mathcal{C}$, a contradiction.

CASE IV: $\lambda_1 = \lambda_2 = \lambda_3$. In this case we will investigate the action of e^{tA} on V rather than the action of ψ on $\mathbb{S}$. The matrix of A can be written as

$$\begin{bmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$$

with $a < 0$, $b < 0$ and $c < 0$. Consequently,

$$e^{tA}u = e^{at} \begin{bmatrix} 1 & bt & ct \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u^{[1]} \\ u^{[2]} \\ u^{[3]} \end{bmatrix} = e^{at} \begin{bmatrix} u^{[1]} + btu^{[2]} + ctu^{[3]} \\ u^{[2]} \\ u^{[3]} \end{bmatrix}.$$

As, by Lemma 3.2, $u^{[2]} \geq 0$ and $u^{[3]} \geq 0$, and by hypothesis, one of these components is positive (recall that $u \notin \operatorname{span} e$), one has $(e^{-tA}u)^{[1]} > 0$ for t sufficiently large, which contradicts Lemma 3.2.

It would be perhaps interesting to look at the action of ψ in the last case. There is a two-dimensional vector subspace $W = \operatorname{span}\{e, \widetilde{w}\}$, $\widetilde{w} = (0, 1, -b/c)$, such that $W \cap \mathbb{S}$ consists of the fixed points for the flow ψ. For any $v \in \mathbb{S} \setminus W$ one finds that $\psi_t v$ converges to e (or $-e$) as $t \to \infty$ (and similarly as $t \to -\infty$, with changed sign).

4. Lotka–Volterra systems. Now we apply our Main Theorem to three-dimensional systems (S_3) of Lotka–Volterra type, that is, to systems

$$\dot{x}^{[i]} = b_i x^{[i]} \Big(1 - \sum_{j=1}^{3} a_{ij} x^{[j]}\Big) \tag{4.1}$$

where $a_{ij} > 0$ and $b_i > 0$.

It is straightforward that for system (4.1),

$$y_1 = (1/a_{11}, 0, 0), \quad y_1 = (0, 1/a_{22}, 0), \quad y_1 = (0, 0, 1/a_{33}).$$

At y_i the internal eigenvalue equals $-b_i$, whereas the external eigenvalue in V^j is equal to $b_k(1 - a_{ki}/a_{ii})$, with $k \neq i$, $k \neq j$. As a consequence of the Main Theorem we obtain the following.

THEOREM 4.1. *For system* (4.1) *the carrying simplex Σ has a peak singularity at y_i if and only if*

$$a_{ii}(b_i + b_j) \leq b_j a_{ji}$$

for both $j \neq i$.

REFERENCES

[1] E. Akin, *The General Topology of Dynamical Systems*, Grad. Stud. Math. 1, Amer. Math. Soc., Providence, RI, 1993.

[2] M. Benaïm, *On invariant hypersurfaces of strongly monotone maps*, J. Differential Equations 137 (1997), 302–319.

[3] P. Brunovský, *Controlling nonuniqueness of local invariant manifolds*, J. Reine Angew. Math. 446 (1994), 115–135.

[4] C. Conley, *Isolated Invariant Sets and the Morse Index*, CBMS Regional Conf. Ser. in Math. 38, Amer. Math. Soc., Providence, RI, 1978.

[5] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Math. Surveys Monographs 25, Amer. Math. Soc., Providence, RI, 1988.

[6] M. W. Hirsch, *Systems of differential equations which are competitive or cooperative. III. Competing species*, Nonlinearity 1 (1988), 51–71.

[7] M. W. Hirsch, C. C. Pugh and M. Shub, *Invariant Manifolds*, Lecture Notes in Math. 583, Springer, Berlin, 1977.

[8] J. Mierczyński, *The C^1 property of carrying simplices for a class of competitive systems of ODEs*, J. Differential Equations 111 (1994), 385–409.

[9] —, *On smoothness of carrying simplices*, Proc. Amer. Math. Soc. 127 (1999), 543–551.

[10] —, *Smoothness of carrying simplices for three-dimensional competitive systems: A counterexample*, Dynam. Contin. Discrete Impuls. Systems 6 (1999), 149–154.

[11] —, *Smoothness of unordered invariant curves for two-dimensional strongly competitive systems*, Appl. Math. (Warsaw) 25 (1999), 449–455.

[12] I. Tereščák, *Dynamics of C^1 smooth strongly monotone discrete-time dynamical systems*, preprint.

[13] M. L. Zeeman, *Hopf bifurcations in competitive three-dimensional Lotka–Volterra systems*, Dynam. Stability Systems 8 (1993), 189–217.

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