COLLOQUIUM MATHEMATICUM

VOL. 82 1999 NO. 1

ON A PROBLEM OF MATKOWSKI

BY

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Abstract. We solve Matkowski's problem for strictly comparable quasi-arithmetic means.

1. Introduction. Let $I \subset \mathbb{R}$ be an open interval and let $\mathrm{CM}(I)$ denote the class of all *continuous* and *strictly monotone* real functions defined on I. A function $M : I^2 \to I$ is called a *quasi-arithmetic mean* on I if there exists $\psi \in \mathrm{CM}(I)$ such that

$$M(x,y) = \psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right) =: A_\psi(x,y) \tag{1.1}$$

for all $x, y \in I$. In this case, $\psi \in \mathrm{CM}(I)$ is called the *generating function* of the quasi-arithmetic mean $A_\psi : I^2 \to I$.

We recall the following result ([1], [4], [5]):

If $\varphi, \chi \in \mathrm{CM}(I)$ then $A_\varphi(x,y) = A_\chi(x,y)$ for all $x, y \in I$ if, and only if, there exist real constants $a \neq 0$ and b such that

$$\varphi(x) = a\chi(x) + b \quad \text{for all } x \in I. \tag{1.2}$$

If for the (generating) functions $\varphi, \chi \in \mathrm{CM}(I)$, (1.2) holds for some constants $a \neq 0$ and b then we say that φ is *equivalent* to χ; and, in this case, we write $\varphi \sim \chi$ or $\varphi(x) \sim \chi(x)$ if $x \in I$.

Matkowski ([6], [7]) proposed the following problem: For which pairs of functions $\varphi, \psi \in \mathrm{CM}(I)$ does the functional equation

$$A_\varphi(x,y) + A_\psi(x,y) = x + y \tag{1.3}$$

hold for all $x, y \in I$? The problem has not been solved yet in this general form. Obviously, it is enough to solve (1.3) disregarding the equivalence of the generating functions φ and ψ.

A pair $(\varphi, \psi) \in \mathrm{CM}(I)^2$ is called *equivalent* to $(\Phi, \Psi) \in \mathrm{CM}(I)^2$ if $\varphi \sim \Phi$ and $\psi \sim \Psi$. We then write $(\varphi, \psi) \sim (\Phi, \Psi)$.

1991 *Mathematics Subject Classification*: 39B22, 26A51.

Key words and phrases: quasi-arithmetic mean, functional equation, convexity.

This work was supported by a grant from the National Foundation for Scientific Research OTKA (no. T-030082).

We introduce the following one-parameter family of functions belonging to $\mathrm{CM}(I)$:

$$\chi_p(x) := \begin{cases} x & \text{if } p = 0, \\ e^{px} & \text{if } p \neq 0 \end{cases} \qquad (x \in I). \tag{1.4}$$

Using the notions and notations above, Matkowski's result can be formulated as follows ([6]).

THEOREM 1. *If a pair $(\varphi, \psi) \in \mathrm{CM}(I)^2$ is a solution of the functional equation* (1.3) *for all $x, y \in I$ and the functions φ and ψ are twice continuously differentiable on I then there exists $p \in \mathbb{R}$ such that $(\varphi, \psi) \sim (\chi_p, \chi_{-p})$, where χ_p is the function defined in* (1.4)*.*

Daróczy and Páles ([3], see also [2]) improved Matkowski's result by proving the following theorem.

THEOREM 2. *If a pair $(\varphi, \psi) \in \mathrm{CM}(I)^2$ is a solution of the functional equation* (1.3) *for all $x, y \in I$ and either φ or ψ is continuously differentiable on I then there exists $p \in \mathbb{R}$ such that $(\varphi, \psi) \sim (\chi_p, \chi_{-p})$.*

These results suggest the following conjecture.

CONJECTURE. *If a pair $(\varphi, \psi) \in \mathrm{CM}(I)^2$ is a solution of the functional equation* (1.3) *for all $x, y \in I$ then there exists $p \in \mathbb{R}$ such that $(\varphi, \psi) \sim (\chi_p, \chi_{-p})$.*

In this paper we try to give support to our conjecture from a different approach.

2. A preliminary result: The solution of a functional equation. We need the following result.

LEMMA. *Let $J \subset \mathbb{R}$ be an open interval. If the strictly decreasing functions $f, g : J \to \mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\}$ satisfy the functional equation*

$$\frac{1}{2} f\left(\frac{u+v}{2}\right)(g(u) - g(v)) = f(v)g(u) - f(u)g(v) \tag{2.1}$$

for all $u, v \in J$ then there exist real constants $p > 0$, b, and $c > 0$ such that

$$f(u) = \frac{1}{pu + b} > 0 \quad \text{and} \quad g(u) = cf^2(u) \tag{2.2}$$

for all $u \in J$.

Proof. (i) First we prove that f and g are continuous functions on J. Let $t \in J$. Then $2t - J$ is an open interval and $t \in 2t - J$. Thus $U := J \cap (2t - J)$ is an open interval containing t. If $u \in U$ ($\subset J$) and $u \neq t$ then let $v := 2t - u \in U$ $(\subset J)$ in (2.1). Then, since g is strictly monotone, (2.1) implies

$$f(t) = 2\frac{f(2t-u)g(u) - f(u)g(2t-u)}{g(u) - g(2t-u)} \tag{2.3}$$

for all $u \in U,\ u \neq t$. Because of the monotonicity of f and g, there exists $u_0 \neq t$ such that f and g are continuous at $2t - u_0$. Therefore, by (2.3), with the substitution $u := u_0$, we find that f is continuous at t.

Now let $v \in J$ be fixed. Then by the continuity of f, there exists $\delta > 0$ for which $\frac{1}{2}f((u+v)/2) - f(v) \neq 0$ if $u \in \,]v-\delta, v+\delta[\, \subset J$. Thus from (2.1) we deduce that for the values $u \in \,]v-\delta, v+\delta[\, \subset J$ we have

$$g(u) = g(v)\frac{\frac{1}{2}f((u+v)/2) - f(u)}{\frac{1}{2}f((u+v)/2) - f(v)},$$

which implies

$$\lim_{u\to v} g(u) = g(v)\lim_{u\to v}\frac{\frac{1}{2}f((u+v)/2) - f(u)}{\frac{1}{2}f((u+v)/2) - f(v)} = g(v),$$

that is, g is continuous at v.

(ii) Let $F := f \circ g^{-1} : g(J) \to \mathbb{R}_+$, where, by the previous results, $g(J) \subset \mathbb{R}_+$ is an open interval. F is obviously continuous on $g(J)$ and

$$F(g(u)) = f(u) \quad \text{for all } u \in J.$$

Then from equation (2.1), for any $s, t \in g(J)$ with $s \neq t$, with the substitutions $u = g^{-1}(s),\ v = g^{-1}(t)$, we have

$$\frac{1}{2}f\left(\frac{g^{-1}(s) + g^{-1}(t)}{2}\right) = \frac{F(t)s - F(s)t}{s-t} = F(t) - t\frac{F(s) - F(t)}{s-t}.$$

By the continuity of f and g, the limit of the left hand side exists as $s \to t$, thus the right hand side also has a limit. Therefore F is differentiable and

$$\frac{1}{2}F(t) = \frac{1}{2}f \circ g^{-1}(t) = \lim_{s\to t}\frac{1}{2}f\left(\frac{g^{-1}(s) + g^{-1}(t)}{2}\right) = F(t) - tF'(t)$$

for all $t \in g(J)$. Since $t > 0$ and $F(t) > 0$, this implies

$$(\log F(t) - \log\sqrt{t})' = 0.$$

Therefore, there exists $d > 0$ such that $F(t) = d\sqrt{t}$. This yields, by the definition of F, that $f \circ g^{-1}(t) = d\sqrt{t}$, i.e., $f(u) = d\sqrt{g(u)}$ for $u \in J$, which gives

$$g(u) = cf^2(u) \quad \text{for } u \in J, \tag{2.4}$$

where $c = 1/d^2 > 0$. Putting (2.4) back in (2.1), for $u \neq v$ we have

$$f\left(\frac{u+v}{2}\right) = \frac{2f(u)f(v)}{f(u)+f(v)},$$

which obviously also holds for $u = v$. This implies that the function h defined by $h(u) := 1/f(u)$ $(u \in J)$ satisfies Jensen's functional equation

$$h\left(\frac{u+v}{2}\right) = \frac{h(u)+h(v)}{2} \qquad (u, v \in J)$$

([1], [5]), thus, by the continuity and strict monotonicity of f, we have $h(u) = pu + b$, where $p > 0$ and b are constants. From this we have

$$f(u) = \frac{1}{pu+b} > 0 \qquad \text{for } u \in J,$$

and so, by (2.4), the statement of the lemma is proved. ■

3. Comparable quasi-arithmetic means and the main result. The notion of comparability forms the basis of the different approach mentioned in the introduction ([4], [5]). Let $(\varphi, \psi) \in \mathrm{CM}(I)^2$. We say that the quasi-arithmetic means A_φ and A_ψ are *strictly comparable* in I if

$$A_\varphi(x,y) \lhd A_\psi(x,y) \qquad \text{for all } x \neq y,\ x, y \in I, \tag{3.1}$$

where $\lhd$ is one of the relations $=$, $<$, $>$ on the real numbers. With this natural notion, our main result is the following:

THEOREM 3. *If a pair* $(\varphi, \psi) \in \mathrm{CM}(I)^2$ *is a solution of the functional equation* (1.3), *and the quasi-arithmetic means* A_φ *and* A_ψ *are strictly comparable in* I, *then there exists* $p \in \mathbb{R}$ *such that* $(\varphi, \psi) \sim (\chi_p, \chi_{-p})$.

Proof. (i) If the relation $\lhd$ is $=$ then, by (1.3) and $A_\varphi = A_\psi$,

$$A_\varphi(x,y) = \frac{x+y}{2} = A_\psi(x,y) \qquad \text{if } x, y \in I,\ x \neq y.$$

This implies that φ and ψ satisfy Jensen's functional equation, thus, by the continuity and strict monotonicity, $\varphi(x) = ax + b$ and $\psi(x) = Ax + B$ for all $x \in I$, where $aA \neq 0$, b, B are constants ([1], [5]). Therefore $(\varphi, \psi) \sim (\chi_0, \chi_0)$, that is, the conclusion holds with $p = 0$.

(ii) If the relation $\lhd$ is $<$ or $>$ then, since φ and ψ can be interchanged, it is enough to investigate only one direction. Suppose that it is $>$, i.e.,

$$A_\varphi(x,y) > A_\psi(x,y) \qquad \text{for } x, y \in I,\ x \neq y. \tag{3.2}$$

Then, by (3.2), (1.3) implies

$$A_\varphi(x,y) > \frac{x+y}{2} \quad \text{and} \quad \frac{x+y}{2} < A_\psi(x,y) \tag{3.3}$$

for all $x, y \in I$ with $x \neq y$. Since we disregard the equivalence of the generating functions φ and ψ, we can assume that φ and ψ are strictly increasing functions on I. Then, by (3.3), φ is strictly Jensen convex and ψ is strictly Jensen concave. Since φ and ψ are continuous and strictly increasing, φ is strictly convex and ψ is strictly concave on I. Therefore φ^{-1} is strictly concave on $\varphi(I)$, ψ^{-1} is strictly convex on $\psi(I)$, and $\gamma := \psi \circ \varphi^{-1}$ is strictly increasing and strictly concave on $\varphi(I)$ ([5], [8]). Thus the left and right derivatives of the functions φ^{-1} and γ exist on the open interval $J := \varphi(I)$, as well as those of ψ^{-1} on the open interval $\psi(I)$. If $u, v \in J = \varphi(I)$ and $x = \varphi^{-1}(u),\ y = \varphi^{-1}(v)$ in (1.3) then

$$\psi^{-1}\left(\frac{\gamma(u)+\gamma(v)}{2}\right) = \varphi^{-1}(u) + \varphi^{-1}(v) - \varphi^{-1}\left(\frac{u+v}{2}\right) \tag{3.4}$$

for all $u, v \in J$.

By the previous results, the right derivative (denoted by h'_+ for a function h) of each function in (3.4) exists at all the points of the domain. Since γ is strictly increasing, both sides of (3.4) can be differentiated from the right with respect to $u \in J$ and $v \in J$; and by the well-known rules, we have the following equations for all $u, v \in J$:

$$\psi_+^{-1'}\left(\frac{\gamma(u)+\gamma(v)}{2}\right)\frac{1}{2}\gamma'_+(u) = \varphi_+^{-1'}(u) - \frac{1}{2}\varphi_+^{-1'}\left(\frac{u+v}{2}\right),$$

$$\psi_+^{-1'}\left(\frac{\gamma(u)+\gamma(v)}{2}\right)\frac{1}{2}\gamma'_+(v) = \varphi_+^{-1'}(v) - \frac{1}{2}\varphi_+^{-1'}\left(\frac{u+v}{2}\right).$$

These two equations imply, as $\big(\varphi_+^{-1'}(u) - \frac{1}{2}\varphi_+^{-1'}((u+v)/2)\big)\gamma'_+(v) =: u \circ v = v \circ u$, that

$$\frac{1}{2}\varphi_+^{-1'}\left(\frac{u+v}{2}\right)(\gamma'_+(u) - \gamma'_+(v)) = \varphi_+^{-1'}(v)\gamma'_+(u) - \varphi_+^{-1'}(u)\gamma'_+(v) \tag{3.5}$$

for all $u, v \in J$.

We recall that the right derivatives of strictly concave functions are positive and strictly decreasing ([5], [8]). Therefore the functions $f, g : J \to \mathbb{R}_+$ defined by

$$f(u) := \varphi_+^{-1'}(u) \quad \text{and} \quad g(u) := \gamma'_+(u) \qquad (u \in J) \tag{3.6}$$

are strictly decreasing on I and satisfy (2.1) for all $u, v \in J$. Thus the Lemma implies that there exist real constants $p > 0,\ b$, and $c > 0$ such that

$$f(u) = \frac{1}{pu+b} > 0 \quad \text{and} \quad g(u) = cf^2(u) \tag{3.7}$$

for all $u \in J$.

Therefore, by (3.6), the functions $\varphi_+^{-1'}$ and γ'_+ are continuous on J. Thus, since φ^{-1} and γ are concave, φ^{-1} and γ are differentiable on J ([5]).

Therefore (3.7) and (3.6) show that

$$\varphi^{-1'}(u) = \frac{1}{pu+b} \quad \text{and} \quad \gamma'(u) = \frac{c}{(pu+b)^2} \qquad (u \in J), \tag{3.8}$$

where $p > 0,\ b,\ c > 0$ are constants. From (3.8) we have

$$\varphi(x) = \frac{1}{p}(e^{p(x-d)} - b) \sim e^{px} \qquad \text{for } x \in I, \tag{3.9}$$

where $p > 0$ (d is a constant of integration). On the other hand, as $\gamma = \psi \circ \varphi^{-1}$, we have $\psi = \gamma \circ \varphi$, and therefore (3.8) and (3.9) imply

$$\begin{aligned} \psi(x) &= \frac{c}{-p(p\varphi(x)+b)} + D \\ &= \frac{c}{-pe^{p(x-d)}} + D \sim e^{-px} \qquad \text{for } x \in I, \end{aligned} \tag{3.10}$$

where $p > 0$ (D is a constant of integration). Relations (3.9) and (3.10) prove the statement of the theorem, namely, $(\varphi, \psi) \sim (\chi_p, \chi_{-p})$ for some $p > 0$. If the reverse inequality holds in (3.2) then $(\varphi, \psi) \sim (\chi_p, \chi_{-p})$ for some $p < 0$. ∎

4. Concluding remarks. Theorem 3 suggests proving the Conjecture concerning Matkowski's problem stated in the introduction in the following way. From the functional equation (1.3) we should conclude that A_φ and A_ψ are strictly comparable in I. But this leads to the following, still open, problem, which, as shown below, is equivalent to the Conjecture.

Open Problem. *Is the following statement true or false? If $\varphi, \psi \in$ CM(I) and*

$$A_\varphi(x,y) + A_\psi(x,y) = x + y$$

for all $x, y \in I$, and there exist $a, b \in I$ such that $a \neq b$ and $A_\varphi(a,b) = A_\psi(a,b)$, then $A_\varphi(x,y) = A_\psi(x,y) = (x+y)/2$ for all $x, y \in I$.

Proof of the equivalence of the Problem and the Conjecture. Consider the continuous function

$$D(x,y) := A_\varphi(x,y) - A_\psi(x,y) \qquad (x, y \in I).$$

If the answer to the Problem is "yes" then either $D(x,y) = 0$ for all $x, y \in I$ or $D(x,y) \neq 0$ for all $x, y \in I$ with $x \neq y$. This implies, by the symmetry and continuity of D, that $D(x,y) > 0$ (or $D(x,y) < 0$) for all $x, y \in I$ with $x \neq y$. Thus the quasi-arithmetic means A_φ and A_ψ are strictly comparable in I. Therefore, applying Theorem 3, we conclude that the Conjecture is true.

If the answer to the Problem is "no" then, by Theorem 2, ψ (and of course φ) cannot be continuously differentiable. ∎

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Received 13 April 1999;
revised 17 May 1999