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SOLUTIONS WITH BIG GRAPH OF ITERATIVE FUNCTIONAL EQUATIONS OF THE FIRST ORDER

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Abstract. We obtain a result on the existence of a solution with big graph of functional equations of the form $g(x, \varphi(x), \varphi(f(x))) = 0$ and we show that it is applicable to some important equations, both linear and nonlinear, including those of Abel, Böttcher and Schröder. The graph of such a solution φ has some strange properties: it is dense and connected, has full outer measure and is topologically big.

1. Introduction. Let X and Y be two sets and $\mathcal{R}$ be a family of subsets of $X \times Y$. We say that $\varphi : X \to Y$ has a *big graph* with respect to $\mathcal{R}$ if the graph $\operatorname{Gr} \varphi$ of φ meets every set of $\mathcal{R}$. We are interested in finding conditions under which the iterative functional equation of the form

$$g(x, \varphi(x), \varphi(f(x))) = 0 \tag{1}$$

has a solution φ with big graph with respect to a sufficiently large family. Well known results on solutions of the Cauchy equation with big graph are due to F. B. Jones [8] (see also [11]). Observe, however, that the latter equation is not of the iterative type. What concerns iterative functional equations, solutions with big graph were obtained in [9], [2] and [4] for equations of invariant curves, in [1] for some homogeneous equations, and in [3] for the equation of iterative roots.

2. Main result. Let X and Y be two nonempty sets, let T be a set with a distinguished element 0 and let $g : X \times Y \times Y \to T$, $f : X \to X$ be two given functions. The set of all *periodic* points of f with *period* p will be denoted by $\operatorname{Per}(f, p)$, i.e.,

$$\operatorname{Per}(f, p) = \{x \in X : f^p(x) = x,\ f^k(x) \neq x \text{ for } k \in \{1, \ldots, p-1\}\};$$

moreover we put

$$\operatorname{Per} f = \bigcup_{p=1}^{\infty} \operatorname{Per}(f, p).$$

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Our general assumptions read:

(H_1) The set X is uncountable.

(H_2) For every $x \in X$ the set $f^{-1}(\{x\})$ is countable and

$$\operatorname{card} \operatorname{Per} f < \operatorname{card} X.$$

(H_3) For every $p \in \mathbb{N}$ and $x \in \operatorname{Per}(f,p)$ there exists $(y_1,\ldots,y_p) \in Y^p$ such that for every $k \in \{1,\ldots,p-1\}$ we have

$$g(f^k(x), y_k, y_{k+1}) = 0 \tag{2}$$

and

$$g(x, y_p, y_1) = 0. \tag{3}$$

(H_4) For every $x \in X$ and $y \in Y$ there exists a $z \in Y$ such that

$$g(x,y,z) = 0, \tag{4}$$

and for every $x \in X$ and $z \in Y$ there exists a $y \in Y$ such that (4) holds.

Note that if $\varphi\colon X \to Y$ is a solution of (1) and $x \in X$ is periodic with period p, then putting

$$y_k = \varphi(f^k(x))$$

for $k \in \{1,\ldots,p\}$ we have (2) and (3). Hence (H_3) is necessary for (1) to have a solution.

Let $\pi\colon X \times Y \to X$ be the projection. The following is the main result of this paper.

THEOREM 1. *Assume* (H_1)–(H_4) *and let* $\mathcal{R}$ *be a family of subsets of* $X \times Y$ *such that*

$$\operatorname{card} \mathcal{R} \le \operatorname{card} X \tag{5}$$

and

$$\operatorname{card} \pi(R) = \operatorname{card} X \quad \textit{for every } R \in \mathcal{R}. \tag{6}$$

Then there exists a solution $\varphi : X \to Y$ *of* (1) *with big graph with respect to* $\mathcal{R}$.

Proof. Let $\sim$ be the standard equivalence relation defining orbits of f, i.e. (cf. [10, p. 14], [16, (1.1.2)]),

$$x \sim y \Leftrightarrow f^m(x) = f^n(y) \text{ for some } m,n \in \mathbb{N}_0,$$

and denote by $C(x)$ the equivalence class (orbit) of $x \in X$, i.e.,

$$C(x) = \bigcup_{n=0}^{\infty} \bigcup_{m=0}^{\infty} f^{-n}(\{f^m(x)\}).$$

The family $\mathcal{C}$ of all orbits is a partition of X and a function $\varphi\colon X \to Y$ is a solution of (1) iff so is $\varphi|_C$ for every $C \in \mathcal{C}$. This allows us to define a solution of (1) by defining it on each orbit.

In the sequel we shall consider two families of orbits:

$$\mathcal{C}_1 = \{C \in \mathcal{C} : C \cap \operatorname{Per} f = \emptyset\}, \qquad \mathcal{C}_2 = \{C \in \mathcal{C} : C \cap \operatorname{Per} f \neq \emptyset\}.$$

Since $\operatorname{card} \mathcal{C}_2 \leq \operatorname{card} \operatorname{Per} f$, from the second part of (H_2) it follows that

$$\operatorname{card} \mathcal{C}_2 < \operatorname{card} X. \tag{7}$$

Let γ be the smallest ordinal such that its cardinal $\overline{\gamma}$ equals that of $\mathcal{R}$ and let $(R_\alpha : \alpha < \gamma)$ be a one-to-one transfinite sequence of all elements of $\mathcal{R}$. Using transfinite induction we shall define a sequence $((x_\alpha, y_\alpha) : \alpha < \gamma)$ of elements of $X \times Y$ such that, for all $\alpha < \gamma$,

$$(x_\alpha, y_\alpha) \in R_\alpha \tag{8}$$

and

$$x_\alpha \in \Big(\bigcup \mathcal{C}_1 \cap \pi(R_\alpha)\Big) \setminus \bigcup\{C \in \mathcal{C} : x_\beta \in C \text{ for some } \beta < \alpha\}. \tag{9}$$

Suppose $\alpha < \gamma$ and that we have already defined (x_β, y_β) for $\beta < \alpha$. It follows from (6) and (7) that $\operatorname{card}(\pi(R_\alpha) \cap \bigcup \mathcal{C}_1) = \operatorname{card} X$ whereas (H_1) and (5) give

$$\operatorname{card} \bigcup\{C \in \mathcal{C} : x_\beta \in C \text{ for some } \beta < \alpha\} \leq \aleph_0 \cdot \overline{\alpha} = \max\{\aleph_0, \overline{\alpha}\} < \operatorname{card} X.$$

Consequently, the set in (9) is nonempty; choose a point x_α from it. In particular, $x_\alpha \in \pi(R_\alpha)$ and so there exists a y_α such that (8) holds.

Now we start to define, for each $C \in \mathcal{C}$, a solution $\varphi_C : C \to Y$ of (1). To this end we shall decompose the orbit depending on whether it is in $\mathcal{C}_1$ or in $\mathcal{C}_2$. However, we begin with the general case. Fix $x \in X$ and put

$$A_{-1} = \bigcup_{k=0}^{\infty} f^{-k}(\{x\}), \qquad A_0 = \{f^k(x)\colon k \in \mathbb{N}\},$$
$$A_n = \bigcup_{k=0}^{\infty} f^{-k}(f^{-1}(\{f^n(x)\}) \setminus \{f^{n-1}(x)\}) \quad \text{for } n \in \mathbb{N}.$$

Then

$$C(x) = \bigcup_{n=-1}^{\infty} A_n. \tag{10}$$

Assume that $C(x) \in \mathcal{C}_1$. We show that

$$A_m \cap A_n = \emptyset \tag{11}$$

for $m \neq n$. Suppose that m and n are positive integers, $m < n$ and $z \in A_m \cap A_n$. Then $f^{k+1}(z) = f^n(x)$, $f^k(z) \neq f^{n-1}(x)$ and $f^{l+1}(z) = f^m(x)$, for some nonnegative integers k, l, whence $f^{k+1}(z) = f^n(x) = f^{l+1+n-m}(z)$. Consequently, since z, as a member of $C(x)$, is aperiodic and so is any of its iterates, $k = l + n - m$ and $f^{n-1}(x) = f^{k+m-l-1}(x) = f^k(z)$, a contradiction. In the remaining cases we argue similarly. Analogously, for every $n \in \mathbb{N}$ and $k, l \in \mathbb{N}_0$ with $k \neq l$ we have

$$f^k(x) \neq f^l(x), \quad f^{-k}(\{x\}) \cap f^{-l}(\{x\}) = \emptyset \tag{12}$$

and

$$f^{-k}(f^{-1}(\{f^n(x)\}) \setminus \{f^{n-1}(x)\}) \cap f^{-l}(f^{-1}(\{f^n(x)\}) \setminus \{f^{n-1}(x)\}) = \emptyset. \tag{13}$$

Fix now an orbit $C \in \mathcal{C}_1$. If the set

$$C \cap \{x_\alpha : \alpha < \gamma\} \tag{14}$$

is nonempty, then, according to (9), it consists of exactly one point x_α and we put

$$(x, y) = (x_\alpha, y_\alpha). \tag{15}$$

Otherwise we choose $(x, y) \in C \times Y$ arbitrarily. In both cases $C = C(x)$ and we can use all the facts established in the preceding paragraph.

The decomposition (10) jointly with (11)–(13) allows us to define a solution $\varphi_C : C \to Y$ of (1) by putting

$$\varphi_C(x) = y \tag{16}$$

and defining it on each A_n's inductively using the following observation. Having a $u \in C$ and φ_C defined at u or at $f(u)$, according to (H_4) we can define it at the other element in such a manner that

$$g(u, \varphi_C(u), \varphi_C(f(u))) = 0. \tag{17}$$

Hence for every orbit $C \in \mathcal{C}_1$ we have a solution $\varphi_C : C \to Y$ of (1) such that if $x_\alpha \in C$, then $\varphi_C(x_\alpha) = y_\alpha$. But, according to (9), for every $\alpha < \gamma$ we have $C(x_\alpha) \in \mathcal{C}_1$. Consequently, by (15) and (16),

$$\varphi_{C(x_\alpha)}(x_\alpha) = y_\alpha \quad \text{for } \alpha < \gamma. \tag{18}$$

Consider now an orbit $C \in \mathcal{C}_2$. Thus $C = C(x)$ with $x \in \mathrm{Per}(f, p)$ for some $p \in \mathbb{N}$. In this case $A_0 = \{f(x), \ldots, f^p(x)\}$ and

$$C(x) = \bigcup_{n=0}^{p} A_n. \tag{19}$$

By standard calculations the summands $A_0, A_1, \ldots, A_p$ of (19) are pairwise disjoint and (13) holds for $n \in \{1, \ldots, p\}$ and $k, l \in \mathbb{N}_0$ with $k \neq l$. A solution $\varphi_C : C \to Y$ of (1) may now be defined as follows. Fix a sequence $(y_1, \ldots, y_p)$ of elements of Y satisfying (2) and (3) and put

$$\varphi_C(f^k(x)) = y_k$$

for $k \in \{1, \ldots, p\}$. Then define φ_C on each of $A_1, \ldots, A_p$ inductively (in such a manner that (17) holds).

Hence for every orbit C a suitable solution $\varphi_C : C \to Y$ of (1) has been constructed. Put $\varphi = \bigcup_{C \in \mathcal{C}} \varphi_C$. Clearly, φ is a solution of (1). According to (18) we also have $\varphi(x_\alpha) = y_\alpha$ for $\alpha < \gamma$, which jointly with (8) shows that φ has a big graph with respect to $\mathcal{R}$ and ends the proof.

Remark 1. Instead of equation (1) we can consider a relation

$$g(x, \varphi(x), \varphi(f(x))) \in T_0 \tag{20}$$

where T_0 is a fixed subset of T. Replacing, in the hypotheses (H_3) and (H_4), every expression of the form $g(u, v, w) = 0$ by $g(u, v, w) \in T_0$ we can obtain an analogue of Theorem 1 on existence of a solution $\varphi : X \to Y$ of (20) which has a big graph with respect to the family $\mathcal{R}$.

In order to apply the above analogue of Theorem 1 to the equation

$$\varphi(f(x)) = g(x, \varphi(x)) \tag{21}$$

with given $f : X \to Y$ and $g : X \times Y \to Y$ we make the following hypotheses.

(H_3') For every $p \in \mathbb{N}$ and $x \in \mathrm{Per}(f, p)$ there exists a $y \in Y$ such that for the sequence $y_0, \ldots, y_{p-1}$ defined by $y_0 = y$, $y_{k+1} = g(f^k(x), y_k)$, we have

$$y_0 = g(f^{p-1}(x), y_{p-1}).$$

(H_4') For every $x \in X$ the function $g(x, \cdot)$ maps Y onto Y.

Theorem 2. *Assume* (H_1), (H_2), (H_3') *and* (H_4') *and let $\mathcal{R}$ be a family of subsets of $X \times Y$ such that* (5) *and* (6) *hold. Then there exists a solution $\varphi : X \to Y$ of* (21) *with big graph with respect to $\mathcal{R}$.*

Since many important equations, e.g., Abel's, Böttcher's, Schröder's, are of the form (21) with g depending only on the second variable, we also formulate a suitable corollary concerning the equation

$$\varphi(f(x)) = g(\varphi(x)). \tag{22}$$

Corollary 1. *Assume* (H_1), (H_2), *let g map Y onto Y, and suppose that for every $p \in \mathbb{N}$ we have*

$$\mathrm{Per}(f, p) \neq \emptyset \Rightarrow \mathrm{Per}(g, k) \neq \emptyset \textit{ for some } k \,|\, p.$$

Let $\mathcal{R}$ be a family of subsets of $X \times Y$ such that (5) *and* (6) *hold. Then there exists a solution $\varphi : X \to Y$ of* (22) *with big graph with respect to $\mathcal{R}$.*

The following remark gives some sufficient conditions for (H_2) to hold.

Remark 2. If X is a real interval, then each of the following two conditions (i), (ii) guarantees that (H_2) holds:

(i) f is piecewise polynomial and the degree of each polynomial is greater than 1,

(ii) f is piecewise monotonic and on each monotonicity interval we have $|f(x)-f(y)|>|x-y|$ for $x\neq y$ (or $|f(x)-f(y)|<|x-y|$ for $x\neq y$).

Using Sharkovskiĭ's Theorem on cycles ([16, (8.2.1)], [12, Theorem 1.1.3]) we obtain the following.

REMARK 3. Let X be a real interval and f be a continuous self-mapping of X. If $\mathrm{Per}(f,1)$ is countable and $f^2(x)\neq x$ for $x\in X\setminus \mathrm{Per}(f,1)$, then $\mathrm{Per}\, f=\mathrm{Per}(f,1)$; consequently, $\mathrm{Per}\, f$ is countable.

3. Properties of functions with big graph. Given two topological spaces X and Y, consider the family

$$\{R\in\mathcal{B}(X\times Y):\pi(R) \text{ is uncountable}\}, \tag{23}$$

where $\mathcal{B}(X\times Y)$ denotes the σ-algebra of all Borel subsets of $X\times Y$. The following simple observation (cf. [11, p. 289]) shows that if a function $\varphi: X\to Y$ has a big graph with respect to the family (23), then its graph is big from the topological point of view.

PROPOSITION 1. *Assume X is a* T_1-space *and has no isolated point. If $\varphi: X\to Y$ has a big graph with respect to the family* (23), *then* $(X\times Y)\setminus \mathrm{Gr}\,\varphi$ *contains no subset of $X\times Y$ of second category having the property of Baire*.

Such a graph is also big from the point of view of measure theory:

PROPOSITION 2. *Assume X is a* T_1-space *and λ is a measure on $\mathcal{B}(X\times Y)$ vanishing on all vertical lines $\{x\}\times Y$, $x\in X$. If $\varphi: X\to Y$ has a big graph with respect to the family* (23), *then* $(X\times Y)\setminus \mathrm{Gr}\,\varphi$ *contains no Borel subset of $X\times Y$ of positive λ-measure.*

In other words $\lambda_*((X\times Y)\setminus \mathrm{Gr}\,\varphi)=0$ and, consequently, $\lambda^*(B\cap \mathrm{Gr}\,\varphi)=\lambda(B)$ for every $B\in\mathcal{B}(X\times Y)$. Here λ_* and λ^* denote the inner and outer measures, respectively, generated by the Borel measure λ; cf. [7, Sec. 14].

It is worth-while to mention that if X is a Polish space and has no isolated point then there are a lot of measures on $\mathcal{B}(X)$ vanishing on all singletons [15, p. 55, Corollary 8.1] and if μ is such a measure and ν is any measure on $\mathcal{B}(Y)$ then the product measure $\mu\times\nu$ vanishes on all vertical lines.

Assume now that X and Y are abelian Polish groups. Following J. P. R. Christensen ([5], [6, p. 115]) we say that a Borel subset R of $X\times Y$ is a *Haar zero set* if there exists a probability measure λ on $\mathcal{B}(X\times Y)$ such that $\lambda(R+z)=0$ for every $z\in X\times Y$. We have the following analogue of the above propositions.

PROPOSITION 3. *Assume X and Y are abelian Polish groups and X has no isolated point. If $\varphi: X\to Y$ has a big graph with respect to the* *family* (23), *then* $(X \times Y) \setminus \mathrm{Gr}\,\varphi$ *contains no Borel subset of* $X \times Y$ *which is not a Haar zero set.*

Finally, we return to topological properties of functions with big graph. Applying Lemmas 1 and 2 of [13] we obtain

PROPOSITION 4. *Assume that* X *and* Y *are connected topological spaces and every non-empty open subset of* X *is uncountable. If* $\varphi\colon X \to Y$ *has a big graph with respect to the family* (23) *then* $\mathrm{Gr}\,\varphi$ *is dense and connected in* $X \times Y$.

REMARK 4. If X and Y are Polish spaces and X is uncountable, then according to [7, Sec. 5, Exercise 9] and to the theorem of Alexandrov–Hausdorff ([14, p. 427]) we have

$$\operatorname{card} \mathcal{B}(X \times Y) \le \mathfrak{c} = \operatorname{card} X$$

and $\operatorname{card} \pi(R) = \mathfrak{c}$ for every Borel subset R of $X \times Y$ with uncountable vertical projection; in particular, the family (23) satisfies all the requirements of the theorems.

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REFERENCES

[1] L. Bartłomiejczyk, *Solutions with big graph of homogeneous functional equations in a single variable*, Aequationes Math. 56 (1998), 149–156.

[2] —, *Solutions with big graph of the equation of invariant curves*, submitted.

[3] —, *Iterative roots with big graph*, submitted.

[4] —, *Solutions with big graph of an equation of the second iteration*, submitted.

[5] J. P. R. Christensen, *On sets of Haar measure zero in abelian Polish groups*, Israel J. Math. 13 (1972), 255–260.

[6] J. P. R. Christensen, *Topology and Borel Structure*, North-Holland Math. Stud. 10, North-Holland, Amsterdam, 1974.

[7] P. R. Halmos, *Measure Theory*, Grad. Texts in Math. 18, Springer, New York, 1974.

[8] F. B. Jones, *Connected and disconnected plane sets and the functional equation* $f(x) + f(y) = f(x + y)$, Bull. Amer. Math. Soc. 48 (1942), 115–120.

[9] P. Kahlig and J. Smítal, *On the solutions of a functional equation of Dhombres*, Results Math. 27 (1995), 362–367.

[10] M. Kuczma, *Functional Equations in a Single Variable*, Monografie Mat. 46, PWN–Polish Sci. Publ., Warszawa, 1968.

[11] —, *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality*, Prace Nauk. Uniw. Śląskiego 489, PWN & Uniw. Śląski, Warszawa–Kraków–Katowice, 1985.

[12] M. Kuczma, B. Choczewski and R. Ger, *Iterative Functional Equations*, Encyclopedia Math. Appl. 32, Cambridge Univ. Press, Cambridge, 1990.

[13] W. Kulpa, *On the existence of maps having graphs connected and dense*, Fund. Math. 76 (1972), 207–211.
[14] K. Kuratowski and A. Mostowski, *Set Theory*, Stud. Logic Found. Math. 86, PWN and North-Holland, Warszawa–Amsterdam, 1976.
[15] K. R. Parthasarathy, *Probability Measures on Metric Spaces*, Academic Press, New York, 1967.
[16] Gy. Targonski, *Topics in Iteration Theory*, Vandenhoeck & Ruprecht, Göttingen, 1981.

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