

A CONSEQUENCE OF AN EFFECTIVE FORM  
OF THE *abc*-CONJECTURE

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**Abstract.** T. Cochrane and R. E. Dressler [CD] proved that the *abc*-conjecture implies that, for every  $\varepsilon > 0$ , the gap between two consecutive numbers  $A < C$  having the same prime factors is  $\gg A^{1/2-\varepsilon}$ . In the present paper, from a weak effective form of the *abc*-conjecture we deduce that  $C - A > A^{0.4}$  with two exceptions given in Table 2.

**1. Introduction.** T. Cochrane and R. E. Dressler [CD] asked if the gap between consecutive positive integers  $A, C$  with the same sets of prime divisors can be small, e.g. less than  $A^{1/3}$ . They deduced from the *abc*-conjecture that for every  $\varepsilon > 0$  the inequality  $C - A < A^{1/2-\varepsilon}$  has only a finite number of solutions in  $A, C$  as above.

In the present paper assuming an effective weak variant of the *abc*-conjecture we prove that  $C - A > A^{0.4}$  holds with two exceptions given explicitly.

**2. Notations.** For a positive integer  $n$  let  $r(n)$  be its radical, i.e. the product of distinct prime divisors of  $n$ .

We shall consider triples  $(a, b, c)$  of positive integers satisfying

$$(1) \quad a + b = c, \quad a < b, \quad \gcd(a, b, c) = 1.$$

For such a triple let

$$L = L(a, b, c) = \frac{\log c}{\log r(abc)}.$$

Then  $c = r(abc)^L$ .

We shall also consider pairs  $(A, C)$  of positive integers satisfying

$$C - A < A < C, \quad r(A) = r(C).$$

We call such a pair *admissible*.

For an admissible pair  $(A, C)$  we define

$$\alpha = \alpha(A, C) = \frac{\log(C - A)}{\log A}.$$

Then  $C - A = A^\alpha$ .

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For a prime number  $p$  and positive integers  $k, n$  the notation  $p^k \parallel n$  means that  $p^k \mid n$  and  $p^{k+1} \nmid n$ . If  $p^k \parallel n$  we write  $v_p(n) = k$ .

**3. Lemmas.** First we prove several lemmas on admissible pairs.

LEMMA 1. *If  $(A, C)$  is an admissible pair then, for every positive integer  $d$ , the pair  $(Ad, Cd)$  is also admissible, and*

$$\alpha(A, C) < \alpha(Ad, Cd) \quad \text{for } d > 1.$$

Moreover,  $\lim_{d \rightarrow \infty} \alpha(Ad, Cd) = 1$ .

PROOF. From  $r(A) = r(C)$  it follows that, for every prime number  $p$ , we have  $p \mid Ad$  if and only if  $p \mid Cd$ . Thus the pair  $(Ad, Cd)$  is admissible.

Moreover, since  $d > 1$  and  $C - A < A$  we get

$$\alpha(Ad, Cd) = \frac{\log(Cd - Ad)}{\log(Ad)} = \frac{\log d + \log(C - A)}{\log d + \log A} > \frac{\log(C - A)}{\log A} = \alpha(A, C).$$

The last part of the lemma follows at once taking  $d \rightarrow \infty$  in the above formula. ■

We say that the admissible pair  $(A, C)$  is *reduced* if, for every prime  $p \mid A$ , the pair  $(A/p, C/p)$  is not admissible.

LEMMA 2. (i) *The admissible pair  $(A, C)$  is reduced if and only if for every prime number  $p \mid A$  we have*

$$(2) \quad p \parallel A, p^2 \mid C \quad \text{or} \quad p \parallel C, p^2 \mid A.$$

(ii) *For every admissible pair  $(A, C)$  there exists a unique  $d$  such that  $d \mid \gcd(A, C)$  and the pair  $(A/d, C/d)$  is admissible and reduced.*

PROOF. (i) Suppose that the admissible pair  $(A, C)$  is reduced and  $p \mid A$ .

If  $p \parallel A$  and  $p \parallel C$ , then the pair  $(A/p, C/p)$  is admissible, since  $r(A/p) = r(A)/p = r(C)/p = r(C/p)$ .

If  $p^2 \mid A$  and  $p^2 \mid C$ , then the pair  $(A/p, C/p)$  is admissible, since  $r(A/p) = r(A) = r(C) = r(C/p)$ .

Thus in both cases we get a contradiction with the assumption that the pair  $(A, C)$  is reduced.

Conversely, suppose that the admissible pair  $(A, C)$  is not reduced. Then for some prime number  $p \mid A$  the pair  $(A/p, C/p)$  is admissible, i.e.  $r(A/p) = r(C/p)$ .

If  $p \parallel A$ , then  $p \nmid r(A/p) = r(C/p)$ , hence  $p \parallel C$ .

If  $p^2 \mid A$ , then  $p \mid r(A/p) = r(C/p)$ , hence  $p^2 \mid C$ .

Thus in both cases we get a contradiction with (2).

(ii) For every  $p \mid A$  let

$$s_p = \begin{cases} \min(v_p(A), v_p(C)) - 1 & \text{if } v_p(A) \neq v_p(C), \\ v_p(A) & \text{if } v_p(A) = v_p(C), \end{cases}$$

and define

$$d = \prod_{p|A} p^{s_p}.$$

We shall prove that this  $d$  satisfies the conditions of the lemma.

Evidently, for every prime number  $p|A$ , we have

$$v_p(d) = s_p \leq \min(v_p(A), v_p(C)),$$

hence  $d | \gcd(A, C)$ .

From the definition of  $d$  we get, for  $p|A$ :

If  $v_p(A) < v_p(C)$ , then  $v_p(d) = v_p(A) - 1$ . Hence  $v_p(A/d) = 1$  and  $v_p(C/d) \geq 2$ . Similarly, if  $v_p(A) > v_p(C)$ , then  $v_p(C/d) = 1$  and  $v_p(A/d) \geq 2$ .

If  $v_p(A) = v_p(C)$ , then  $v_p(d) = v_p(A)$ . Hence  $v_p(A/d) = v_p(C/d) = 0$ .

Thus we have proved that the pair  $(A/d, C/d)$  is admissible and it is reduced by the first part of the lemma.

The uniqueness of  $d$  is evident. ■

**4. Admissible reduced pairs and triples.** For an admissible and reduced pair  $(A, C)$  we define the triple  $(a, b, c)$  as follows:

$$(3) \quad a = (C - A)/r(A), \quad b = A/r(A), \quad c = C/r(A).$$

For a triple  $(a, b, c)$  satisfying (1) we define the pair  $(A, C)$  as follows:

$$(4) \quad A = br(bc), \quad C = cr(bc).$$

Then  $C - A = ar(bc)$ .

LEMMA 3. (i) *The triple  $(a, b, c)$  defined by (3) satisfies (1).*

(ii) *The pair  $(A, C)$  defined by (4) is admissible and reduced.*

(iii) *The formulas (3) and (4) give 1-1 correspondence between the set of admissible and reduced pairs and the set of triples satisfying (1).*

PROOF. (i) From (3) we get  $a + b = c$ , and  $a < b$  since  $C - A < A$  by assumption.

Suppose that a prime number  $p$  divides  $b$  and  $c$ . Then  $p^2 | A$  and  $p^2 | C$  in view of (3). This contradicts the fact that the pair  $(A, C)$  is reduced, in view of Lemma 2(i). Hence  $\gcd(b, c) = 1$  and consequently  $\gcd(a, b, c) = 1$ .

(ii) Since  $a < b$  and  $a + b = c$ , we get  $c - b < b < c$ , and hence  $C - A < A < C$  by (4).

From (4) it follows that  $r(A) = r(br(bc)) = r(bc) = r(cr(bc)) = r(c)$ , i.e. the pair  $(A, C)$  is admissible.

To prove that the pair  $(A, C)$  is reduced let us observe that for every prime  $p$  we have in view of (4):

$$p|b \quad \text{iff} \quad p^2|A \quad \text{and} \quad p||C$$

and

$$p \mid c \quad \text{iff} \quad p^2 \mid C \text{ and } p \parallel A.$$

Thus the claim follows from Lemma 2(i).

(iii) We shall prove that from (3) (as well from (4)) it follows that

$$(5) \quad r(A) = r(bc).$$

Assume (3). Then for every prime number  $p \mid A$  we have  $p^3 \mid AC$  in view of Lemma 2(i), and evidently  $p \parallel r(A)$ . Consequently, in view of (3),

$$r(bc) = r\left(\frac{AC}{r(A)^2}\right) = r(AC) = r(A).$$

Similarly, from (4) we get

$$r(A) = r(br(bc)) = r(bc).$$

Now, from (3)–(5) the claim follows easily. ■

## 5. Conjectures

CONJECTURE 1 (The *abc*-conjecture). *For every real number  $q > 1$  there are only a finite number of triples  $(a, b, c)$  satisfying (1) such that  $L(a, b, c) > q$ .*

E.g., for  $q = 1.5$ , only 11 such triples are known (see [B]):

Table 1

No.	$a$	$b$	$c$	$L(a, b, c)$
1.	2	$3^{10} \cdot 109$	$23^5$	1.629912
2.	$11^2$	$3^2 \cdot 5^6 \cdot 7^3$	$2^{21} \cdot 23$	1.625991
3.	$19 \cdot 1307$	$7 \cdot 29^2 \cdot 31^8$	$2^8 \cdot 3^{22} \cdot 5^4$	1.623490
4.	283	$5^{11} \cdot 13^2$	$2^8 \cdot 3^8 \cdot 17^3$	1.580756
5.	1	$2 \cdot 3^7$	$5^4 \cdot 7$	1.567887
6.	$7^3$	$3^{10}$	$2^{11} \cdot 29$	1.547075
7.	$7^2 \cdot 41^2 \cdot 311^3$	$11^{16} \cdot 13^2 \cdot 79$	$2 \cdot 3^3 \cdot 5^{23} \cdot 953$	1.544434
8.	$5^3$	$2^9 \cdot 3^{17} \cdot 13^2$	$11^5 \cdot 17 \cdot 31^3 \cdot 137$	1.536714
9.	$13 \cdot 19^6$	$2^{30} \cdot 5$	$3^{13} \cdot 11^2 \cdot 31$	1.526999
10.	$3^{18} \cdot 23 \cdot 2269$	$17^3 \cdot 29 \cdot 31^8$	$2^{10} \cdot 5^2 \cdot 7^{15}$	1.522160
11.	239	$5^8 \cdot 17^3$	$2^{10} \cdot 37^4$	1.502839

CONJECTURE 2. *The above table contains all triples  $(a, b, c)$  satisfying (1) and  $L(a, b, c) > 1.5$ .*

Evidently neither Conjecture 1 implies Conjecture 2, nor conversely. From Conjecture 1 it follows only that there is a finite set of triples  $(a, b, c)$  with  $L(a, b, c) > 1.5$ . Thus Conjecture 2 can be considered as an effective weak version of Conjecture 1.

CONJECTURE 3. For every real number  $t$  with  $0 < t < 1/2$  there are only a finite number of admissible pairs  $(A, C)$  such that  $\alpha(A, C) < t$ .

CONJECTURE 4. The only admissible pairs  $(A, C)$  satisfying  $\alpha(A, C) < 0.4$  are given in the following table.

Table 2

No.	$A$	$C$	$\alpha(A, C)$
1.	$3^{11} \cdot 23 \cdot 109^2$	$3 \cdot 23^6 \cdot 109$	0.390953
2.	$2^2 \cdot 3^8 \cdot 5 \cdot 7$	$2 \cdot 3 \cdot 5^5 \cdot 7^2$	0.389431

## 6. Main results

THEOREM 1. For an admissible and reduced pair  $(A, C)$  let  $(a, b, c)$  be the triple defined by (3). If  $\alpha = \alpha(A, C) < t < 1/2$ , then

$$L = L(a, b, c) > \frac{1-t}{t} = 1 + \frac{2}{t} \left( \frac{1}{2} - t \right) > 1.$$

Proof. Let us recall the basic relations:

$$c = r(abc)^L, \quad C - A = A^\alpha, \quad r(A) = r(bc).$$

Hence in view of (3) we get

$$ar(A) = C - A = A^\alpha = (br(A))^\alpha,$$

i.e.  $a^{1-\alpha}r(A)^{1-\alpha} < ar(A)^{1-\alpha} = b^\alpha$ , so  $ar(A) \leq b^{\alpha/(1-\alpha)}$ .

Consequently,

$$c^{1/L} = r(abc) = r(a)r(bc) = r(a)r(A) \leq ar(A) \leq b^{\alpha/(1-\alpha)} < c^{\alpha/(1-\alpha)}.$$

Therefore

$$\frac{1}{L} < \frac{\alpha}{1-\alpha}, \quad \text{i.e. } L > \frac{1-\alpha}{\alpha}. \quad \blacksquare$$

REMARK 1. If we put  $t = 1/2 - \varepsilon > 0$  in Theorem 1, we get  $L > 1 + 2\varepsilon/t$ . Therefore from the *abc*-conjecture and Lemma 3 it follows that there are only a finite number of admissible reduced pairs  $(A, C)$  satisfying  $\alpha(A, C) < 1/2 - \varepsilon$ . The number of all admissible pairs  $(A, C)$  (not necessarily reduced) satisfying the above inequality is also finite in view of the last part of Lemma 1.

Thus we proved once more the result of Cochrane and Dressler (see [CD]) that the *abc*-conjecture implies Conjecture 3.

COROLLARY 1. Conjecture 2 implies Conjecture 4.

Proof. Put  $t = 0.4$  in Theorem 1. Then we get  $L > 3/2$ . Thus the admissible reduced pairs  $(A, C)$  satisfying  $\alpha(A, C) < 0.4$  lead to triples  $(a, b, c)$  given in Table 1.

One can verify that only triples of No. 1 and No. 5 in the table give the pairs  $(A, C)$  satisfying this inequality. They are listed in Table 2.

One should also consider the nonreduced admissible pairs  $(Ad, Cd)$  corresponding to these two triples, where  $d = 2, 3, \dots$

Yet

$$\alpha(2A, 2C) < \alpha(3A, 3C) < \dots,$$

and one can easily verify that  $\alpha(2A, 2C) > 0.4$  for the two pairs  $(A, C)$  in question. ■

**COROLLARY 2.** *If there exists an admissible pair  $(A', C')$  such that  $\alpha(A', C') < 1/3$ , then one can find a triple  $(a, b, c)$  satisfying (1) such that  $L(a, b, c) > 2$ .*

**Proof.** In view of Lemma 2(ii) for some  $d \mid \gcd(A', C')$  the pair  $(A, C) = (A'/d, C'/d)$  is admissible and reduced. Then  $\alpha(A, C) < \alpha(A', C') < 1/3$ .

Let  $(a, b, c)$  be the triple corresponding to the pair  $(A, C)$  defined by (4). Then by Theorem 1 we have

$$L(a, b, c) > \frac{1 - 1/3}{1/3} = 2. \quad \blacksquare$$

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