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K-CONTACT $\mathcal{A}$-MANIFOLDS

BY

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The aim of this paper is to give a characterization of regular K-contact $\mathcal{A}$-manifolds.

1. Introduction. The class of $\mathcal{A}$-manifolds (for a definition see [G]) is an important class of Riemannian manifolds which has appeared in a natural way during the investigation of spaces with volume preserving local geodesic symmetries. In our paper [J-2] we have constructed a family (P_c, g_c) of $\mathcal{A}$-manifolds on a circle bundle P over an arbitrary Kähler–Einstein manifold M. If M is locally non-homogeneous, then P is also locally non-homogeneous. Since there are many locally non-homogeneous Einstein–Kähler manifolds, we have given in this way an answer to the open question stated in the book [B]. The family P_c is parametrized by a real number $c > 0$. For an appropriate choice of c we obtain a K-contact metric structure on P (even Sasakian structure) which at the same time is an $\mathcal{A}$-manifold. In the present paper we consider a related problem and give necessary and sufficient conditions for a regular K-contact metric manifold to be an $\mathcal{A}$-manifold (in the compact case our definition of regular contact structure coincides with that in [B-W]).

2. Preliminaries. Throughout the paper we use the notations and terminology of [O-1], [Bl]. Let P be a $(2n+1)$-dimensional C^∞-manifold. We say that a Riemannian manifold (P, g) admits a *contact metric structure* (g, ξ, ϕ, η) if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η such that

$$\phi^2 = -\operatorname{id} + \eta\otimes\xi, \quad \phi(\xi) = 0, \quad \eta\circ\phi = 0, \quad \eta(\xi) = 1,$$
$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad g(X,\xi) = \eta(X),$$
$$g(X, \phi Y) = d\eta(X,Y).$$

A contact structure (g, ξ, ϕ, η) is called a *K-contact structure* if the characteristic vector field ξ is Killing, i.e. $L_\xi g = 0$, L being the operator of Lie dif-

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ferentiation. This is equivalent to $L_\xi\phi = 0$ (see [Bl]) and then $\phi X = -\nabla_X\xi$. In the sequel we shall assume that P is a circle bundle over a $2n$-manifold M and ξ is (up to a constant factor) the fundamental vector field of the action of the group S^1 on P (we shall say in that case that the structure (g, ξ, ϕ, η) is *regular*). By H (resp. V) we denote the horizontal (resp. vertical) distribution:

$$H = \{X \in TP : \eta(X) = 0\} \quad \text{and} \quad V = \{X \in TP : X \| \xi\}.$$

It is clear that $TP = V \oplus H$ and we denote by $\mathcal{V}$, $\mathcal{H}$ the projections $\mathcal{V} : TP \to V$ and $\mathcal{H} : TP \to H$. By $X^* \in \mathfrak{X}(P)$ we denote the horizontal lift of a vector field $X \in \mathfrak{X}(M)$. A K-contact metric structure (g, ξ, ϕ, η) on P induces an almost Hermitian structure (J_*, g_*) on the manifold M such that (M, g_*, J_*) is an almost Kähler manifold. The metric g_* and the almost complex structure J_* satisfy the conditions

$$g_*(X, Y) = g(X^*, Y^*), \qquad dp \circ \phi = J_* \circ dp$$

for any $X, Y \in \mathfrak{X}(M)$. The existence of such g_* and J_* follows from the fact that ξ is a Killing field and that $L_\xi\phi = 0$. The Kähler form Ω of (M, g_*, J_*) satisfies the relation

$$p^*\Omega(X, Y) = d\eta(X, Y) = g(X, \phi Y).$$

Note also that the mapping $p : P \to M$ is a Riemannian submersion (see [O'N]) and if P is a Sasakian manifold, then (M, g_*, J_*) is a Kähler manifold. In the sequel we shall use O'Neill's tensors T and A. They are defined as follows:

$$A_X Y = \mathcal{V}(\nabla_{\mathcal{H}X}\mathcal{H}Y) + \mathcal{H}(\nabla_{\mathcal{H}X}\mathcal{V}Y),$$
$$T_X Y = \mathcal{H}(\nabla_{\mathcal{V}X}\mathcal{V}Y) + \mathcal{V}(\nabla_{\mathcal{V}X}\mathcal{H}Y).$$

In view of the results of D. Blair [Bl] we have:

PROPOSITION 1. *A contact metric structure (g, ξ, ϕ, η) on a Riemannian manifold (P, g) is a K-contact structure if and only if the characteristic vector field ξ is an eigenfield of the Ricci tensor ϱ of (P, g) with constant eigenvalue $\lambda = 2n$.*

Let us recall that a *Hodge manifold* is a Kähler manifold whose Kähler form is a multiple of an integral class. We shall use the following:

DEFINITION. An almost Kähler manifold (M, g, J) whose Kähler form $\Omega(X, Y) = g(X, JY)$ is a multiple of an integral class $\omega \in H^2(M, \mathbb{Z})$ is said to be an *almost Hodge manifold*.

Remark. Note that there are many almost Hodge manifolds which are not Hodge manifolds, i.e. which are not Kähler manifolds. Many examples of such non-Kähler manifolds are given in [J-1].

Finally, recall that a Riemannian manifold (M,g) is called an $\mathcal{A}$-*manifold* (see [G]) (we write $M\in\mathcal{A}$) if the Ricci tensor of (M,g) satisfies the condition

$$\nabla_X\varrho(X,X)=0$$

for all local vector fields $X\in\mathfrak{X}(M)$. On Sasakian manifolds, this condition is equivalent to the so-called η-parallelity of the Ricci tensor (see [O-2]).

3. K-contact regular $\mathcal{A}$-manifolds. Assume that $p:P\to M$ is a principal circle bundle admitting a K-contact metric structure (g,ξ,ϕ,η) such that the fundamental vector field of the action of S^1 on P coincides with the characteristic vector field of the structure. Hence (g,ξ,ϕ,η) induces on M an almost Hermitian structure (g_*,J_*) such that (M,g_*,J_*) is an almost Hodge manifold. We denote by $\overline{\nabla}$ the Levi-Civita connection of the metric g_* and by ∇ that of g. We prove the following theorem:

THEOREM 1. *A K-contact regular metric space (P,g) is an $\mathcal{A}$-manifold if and only if the almost Kähler manifold (M,g_*,J_*) is an $\mathcal{A}$-manifold whose Ricci tensor ϱ_* satisfies the additional condition*

$$\varrho_*(J_*X,J_*Y)=\varrho_*(X,Y). \tag{A}$$

Proof. Denote by ϱ the Ricci tensor of (P,g). We have to show that condition (A) implies

$$\mathfrak{C}_{X,Y,Z}\nabla_X\varrho(Y,Z)=0$$

for all $X,Y,Z\in\mathfrak{X}(P)$ where $\mathfrak{C}$ denotes the cyclic sum. We consider three cases. Note first that ξ is a Killing vector field of constant length, hence $\phi(\xi)=-\nabla_\xi\xi=0$, which means that the fibres $p^{-1}(x)$ of P are totally geodesic submanifolds. Hence the O'Neill tensor T vanishes (see [O'N]). It is easy to compute the tensor A (see [B]). Note that $A_XY=A_{\mathcal{H}X}\mathcal{H}Y+A_{\mathcal{H}X}\mathcal{V}Y$ and $\mathcal{H}X=X-\eta(X)\xi$, $\mathcal{H}Y=Y-\eta(Y)\xi$, $\mathcal{V}Y=\eta(Y)\xi$. Hence we obtain

$$A_XY=A_{X-\eta(X)\xi}(Y-\eta(Y)\xi)+A_{X-\eta(X)\xi}\eta(Y)\xi.$$

Assume now that $X,Y\in\mathfrak{X}(P)$ are two horizontal vector fields, i.e. $X,Y\in\Gamma(H)$ (where we denote by $\Gamma(H)$ the set of all local sections of the vector bundle H). Then $A_XY=\eta(\nabla_XY)\xi$. We also have in this case

$$\begin{aligned}g(\xi,\nabla_XY)=\eta(\nabla_XY)&=Xg(\xi,Y)-g(Y,\nabla_X\xi)\\&=-g(\nabla_X\xi,Y)=g(\phi X,Y).\end{aligned}$$

Hence

$$A_{X-\eta(X)\xi}(Y-\eta(Y)\xi)=g(\phi X,Y)\xi$$

for any $X,Y\in\mathfrak{X}(P)$. Since for any $X,Y\in\mathfrak{X}(P)$,

$$A_X(\eta(Y)\xi)=\mathcal{H}(\eta(Y)\nabla_X\xi)=\eta(Y)\nabla_X\xi=-\eta(Y)\phi(X),$$

we finally get the formula

$$A_XY = -g(X, \phi Y)\xi - \eta(Y)\phi(X), \tag{3.1}$$

where $X, Y \in \mathfrak{X}(P)$. Hence we obtain, in view of the results of O'Neill [O'N], the following formulas for the sectional curvatures:

$$K(U \wedge V) = K_*(U_* \wedge V_*) - 3\|A_U V\|^2 \quad \text{for } U, V \in H,$$
$$K(U \wedge \xi) = \|A_U\xi\|^2 = \|\phi U\|^2 \quad \text{for } U \in H,$$

where $U_* = p(U)$, $V_* = p(V)$, $g(U,U) = g(V,V) = 1$, $g(U,V) = 0$ and by K (resp. K_*) we denoted the sectional curvature of P (resp. M). Let $\{E_1, \ldots, E_n\}$ be an orthonormal local frame on M and $U \in \mathfrak{X}(M)$. Then $\{E_1^*, \ldots, E_n^*, \xi\}$ is a local orthonormal frame on P. Then

$$\begin{aligned}\varrho(U^*, U^*) &= \sum_{i=1}^n K(U^* \wedge E_i^*) + K(U^* \wedge \xi)\\ &= \sum_{i=1}^n (K_*(U \wedge E_i) - 3g(\phi U, E_i)^2) + g(\phi U, \phi U)\\ &= \varrho_*(U,U) - 2g(\phi U, \phi U).\end{aligned}$$

Consequently, if $U, V \in \mathfrak{X}(M)$, then

$$\varrho_*(U,V) = \varrho(U^*, V^*) + 2g(\phi U^*, \phi V^*) = \varrho(U^*, V^*) - 2g(\phi^2 U^*, V^*), \tag{3.2}$$

which implies

$$\varrho_*(U,V) = \varrho(U^*, V^*) + 2g(U^*, V^*). \tag{3.3}$$

From (3.3) we get

$$\overline{\nabla}_U \varrho_*(U,U) = \nabla_{U^*} \varrho(U^*, U^*). \tag{3.4}$$

Hence, for all $U \in \mathfrak{X}(M)$, the equality $\nabla_{U^*}\varrho(U^*, U^*) = 0$ holds if and only if $\overline{\nabla}_U \varrho_*(U,U) = 0$. The last equality is equivalent to $(M, g_*) \in \mathcal{A}$.

From Proposition 1 we have $S\xi = 2n\xi$, where S is the Ricci endomorphism of (P, g). Taking account of $\phi\xi = 0$ we obtain

$$\nabla S(\xi, \xi) = 0. \tag{3.5}$$

Hence

$$\nabla \varrho(\xi, \xi, V^*) = 0 \tag{3.6}$$

for any $V \in \mathfrak{X}(M)$. On the other hand, $\varrho(\xi, \xi) = 2n$ is constant and consequently

$$\nabla_{V^*}\varrho(\xi, \xi) - 2\varrho(\phi V^*, \xi) = 0.$$

The equation $\varrho(\phi V^*, \xi) = 2ng(\xi, \phi V^*) = 0$ implies

$$\nabla_{V^*}\varrho(\xi, \xi) = 0. \tag{3.7}$$

As a result, $\mathfrak{C}_{V^*,\xi,\xi}\nabla_{V^*}\varrho(\xi,\xi) = 0$. Let now $U, V \in \mathfrak{X}(M)$. We compute the cyclic sum

$$\mathfrak{C}_{\xi,V^*,U^*}\nabla_\xi\varrho(U^*,V^*).$$

Since ξ is a Killing vector field we have $L_\xi\varrho = 0$, and consequently

$$\nabla_\xi\varrho(U^*,V^*) = (\nabla_\xi - L_\xi)\varrho(U^*,V^*) = \varrho(\phi U^*,V^*) + \varrho(U^*,\phi V^*). \tag{3.8}$$

Similarly from the equality $S\xi = 2n\xi$ it follows that

$$\begin{aligned}\nabla_{U^*}\varrho(\xi,V^*) &= g(\nabla S(U^*,\xi),V^*) = g((2nI - S)(\nabla_{U^*}\xi),V^*)\\ &= -2ng(\phi U^*,V^*) + \varrho(\phi U^*,V^*).\end{aligned} \tag{3.9}$$

Consequently, (3.8) and (3.9) give

$$\mathfrak{C}_{\xi,U^*,V^*}\nabla_\xi\varrho(U^*,V^*) = 2(\varrho(\phi U^*,V^*) + \varrho(U^*,\phi V^*)). \tag{3.10}$$

In view of (3.3) we conclude that

$$\varrho(\phi U^*,V^*) + \varrho(U^*,\phi V^*) = \varrho_*(J_*U,V) + \varrho_*(U,J_*V). \tag{3.11}$$

Hence $\mathfrak{C}_{\xi,U^*,V^*}\nabla_\xi\varrho(U^*,V^*) = 0$ if and only if equation (A) is satisfied. From the above considerations it is clear that if (P,g) is an $\mathcal{A}$-manifold, then (M,g_*,J_*) is an $\mathcal{A}$-manifold satisfying (A), and if M with the induced almost Kähler structure (g_*,J_*) is an $\mathcal{A}$-manifold satisfying (A), then (P,g) is an $\mathcal{A}$-manifold. ∎

THEOREM 2. *If (M,g_*,J_*) is an almost Hodge $\mathcal{A}$-manifold satisfying condition* (A) *then there exists a circle bundle $p : P \to M$ such that P admits a K-contact metric structure (g,ξ,ϕ,η), (P,g) is an $\mathcal{A}$-manifold and p is a Riemannian submersion.*

To prove Theorem 2 recall the following theorem by S. Kobayashi ([K]):

PROPOSITION 2. *Let $\omega \in H^2(M,\mathbb{R})$ be an integral cohomology class and let a 2-form $\Omega \in \mathcal{A}^2(M)$ belong to ω. Then there exists an S^1-principal fibre bundle $p : P \to M$ and a connection Γ on P with connection form $\theta \in \mathcal{A}^1(P)$ such that*

$$d\theta = 2\pi p^*\Omega.$$

Proof of Theorem 2. Assume that (M,g_*,J_*) is an almost Kähler manifold and $\Omega_*(X,Y) = g_*(X,J_*Y)$ is its Kähler form. Assume that $\Omega_* = c\Omega$, where $c \in \mathbb{R}$ and $\{\Omega\} \in H^2(M,\mathbb{Z})$. By Proposition 2 there exists an S^1-principal fibre bundle $p : P \to M$ with connection form θ such that

$$d\theta = 2\pi p^*\Omega.$$

Define $\eta := (2\pi/c)\theta$. Then

$$d\eta = p^*\Omega_*. \tag{3.12}$$

Next we define the metric g on P by $g = \eta\otimes\eta + p^*g_*$. It is clear that p is a Riemannian submersion $p : (P,g) \to (M,g_*)$. If $\overline{\xi}$ is the fundamental vector field of the action of S^1 on P, then write $\xi = (2\pi/c)\overline{\xi}$. Then $\eta(\xi) = 1$ and $\eta(X) = g(\xi, X)$. Note also that $L_\xi \eta = 0$ (since $\eta = (2\pi/c)\theta$ and θ is a connection form, which implies $L_{\overline{\xi}}\theta = 0$). Define a $(1,1)$-tensor field ϕ on P by

$$\phi(X^*) = (J_* X)^* \quad \text{if } X \in TM \text{ and } \phi(X) = 0. \tag{3.13}$$

The tensor field ϕ is well defined and satisfies the following conditions:

$$\phi^2 = -\operatorname{id} + \eta \otimes \xi, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X).$$

We show that

$$d\eta(X, Y) = g(X, \phi Y) \tag{3.14}$$

for all $X, Y \in \mathfrak{X}(M)$. Note that if X, Y are horizontal, the equality follows from formulas (3.12), (3.13) and definition of g. Since $L_\xi \eta = 0$ and $0 = L_\xi \eta(X) = (d \circ i_\xi \eta)(X) + (i_\xi d\eta)(X) = d\eta(\xi, X)$, it is clear that (3.14) holds for all $X, Y \in \mathfrak{X}(M)$. Note also that since $L_\xi \eta = 0$ and consequently $L_\xi g = 0$, the field ξ is a Killing vector field on P. Hence (g, ξ, ϕ, η) is a K-contact structure. From the previous theorem it follows that if (M, g_*) is an $\mathcal{A}$-manifold, then so is (P, g), which concludes the proof. ∎

Remark. Note that every Kähler manifold (M, g_*, J_*) satisfies condition (A). On the other hand, a general almost Kähler manifold does not satisfy this condition. It is clear that every almost Kähler Einstein manifold satisfies both conditions; however, the Goldberg conjecture says that every such (compact) manifold is Kähler (see [S-1], [S-2]).

Taking account of the result of A. Gray [G] and K. Sekigawa and L. Vanhecke [S-V] we obtain:

COROLLARY. *A K-contact metric regular manifold (P, g) is a Sasakian $\mathcal{A}$-manifold if and only if (M, g_*, J_*) is a Kähler manifold with parallel Ricci tensor $\overline{\nabla}\varrho_* = 0$. If M is complete and simply connected, then the last condition is equivalent to the fact that $M = M_1 \times \ldots \times M_r$, where each M_i is a Kähler–Einstein manifold.*

Moreover, if $M = M_1 \times \ldots \times M_r$, where each (M_i, g_i, J_i) is a Kähler–Einstein manifold with non-zero scalar curvature (or more generally, which is a Hodge manifold), then there exists a circle bundle $p : P \to M$ such that P admits a Sasakian structure (g, ξ, ϕ, η), (P, g) is an $\mathcal{A}$-manifold and $p : (P, g) \to (M, g_*)$ is a Riemannian submersion, where $g_* = \sum \alpha_i g_i$ for some $\alpha_i \in \mathbb{R}_+$.

Proof. Recall that in our case P is a Sasakian manifold if and only if (M, g_*, J_*) is Kähler manifold ([Bl]). Hence in view of the results of

A. Gray [G] and K. Sekigawa and L. Vanhecke [S-V], $(M, g_*, J_*) \in \mathcal{A}$ implies $\overline{\nabla}\varrho_* = 0$. If M is complete and simply connected, then (as is well known) the de Rham theorem gives $M = M_1 \times \ldots \times M_r$, where each M_i is a Kähler–Einstein manifold.

On the other hand, if $M = M_1 \times \ldots \times M_r$, where each (M_i, g_i, J_i) is a Kähler–Einstein manifold with nonzero scalar curvature (or Hodge manifold if it has zero scalar curvature), then for some $\alpha_i \in \mathbb{R}_+$ the manifold (M, g_*, J_*), where $g_* = \sum_i \alpha_i g_i$, $J_* = \sum_i J_i$, is a Hodge manifold. Hence using the result of S. Kobayashi [K] and our theorem we get a principal S^1 fibre bundle over M which is a Sasakian $\mathcal{A}$-manifold. ∎

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