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NON-AMENABLE GROUPS WITH AMENABLE ACTION AND SOME PARADOXICAL DECOMPOSITIONS IN THE PLANE

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A finitely additive non-negative (not necessarily finite) measure is called *universal* iff it is defined over all subsets of the underlying space. A group G is called *amenable* iff there exists a universal left invariant measure μ over G with $\mu(G) = 1$. If an amenable group G acts on a space X, then there exists a universal G-invariant measure ϱ over X with $\varrho(X) = 1$. Indeed, we pick $x_0 \in X$, define $\nu(Y) = 0$ if $x_0 \notin Y$ and $\nu(Y) = 1$ if $x_0 \in Y$ for all $Y \subseteq X$ and define

$$\varrho(Y) = \int_G \nu(g(Y))\,\mu(dg),$$

where μ is given by the amenability of G. It is clear that ϱ has the required properties. In a similar way one can show that if G is amenable, then there exists a left and right invariant universal measure μ in G with $\mu(G) = 1$.

When G is not amenable, the theory of Hausdorff–Banach–Tarski paradoxical decompositions gives many examples of actions of G for which no universal invariant measures exist (see [W]). However, in the present paper we will give natural examples of non-amenable group actions which are faithful and transitive and nevertheless such that universal invariant measures, positive and finite on appropriate sets, do exist (Theorems 1, 2 and 3). Moreover, we will prove or conjecture several facts on the existence of Hausdorff–Banach–Tarski paradoxical decompositions of sets in the plane $\mathbb{R}^2$ which preclude the existence of other universal measures (Corollaries $1,\dots,5$ and Theorem 4). These are related to a well-known theorem of von Neumann about paradoxical decompositions of sets in $\mathbb{R}^2$ (see [W], Thm. 7.3) which will be proved again in the present paper as Corollary 3. For related work concerning the hyperbolic plane see [M_1].

We recall some results of the Banach–Tarski theory of equivalence by finite decomposition which will be used below. If a group G acts on a space X, then a set $Y \subseteq X$ will be called *paradoxical* iff there exists a partition of

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Y into $2n$ disjoint subsets

$$Y = U_1 \cup \ldots \cup U_n \cup V_1 \cup \ldots \cup V_n,$$

and there exist $2n$ elements $g_1, \ldots, g_n$, $h_1, \ldots, h_n \in G$ such that

$$Y = g_1(U_1) \cup \ldots \cup g_n(U_n) = h_1(V_1) \cup \ldots \cup h_n(V_n).$$

Two sets $Y_1, Y_2 \subseteq X$ are said to be *equivalent by finite decomposition*, in symbols $Y_1 \equiv Y_2$, iff there exist partitions of Y_1 and Y_2 into the same number n of disjoint sets,

$$Y_1 = U_1 \cup \ldots \cup U_n \quad \text{and} \quad Y_2 = V_1 \cup \ldots \cup V_n,$$

and there exist n transformations $g_1, \ldots, g_n \in G$ such that $g_i(U_i) = V_i$ for $i = 1, \ldots, n$.

We will use the following two theorems of Banach and Tarski (see [W]).

THEOREM A (A variant of the Cantor–Bernstein Theorem). *If* $Y_1 \subseteq Y_2 \subseteq Y_3 \subseteq X$ *and* $Y_1 \equiv Y_3$, *then* $Y_1 \equiv Y_2$.

THEOREM B (A Cancellation Theorem). *If* $Y_1 \cup \ldots \cup Y_n = Y \subseteq X$, $Y_1 \equiv Y_2 \equiv \ldots \equiv Y_n$ *and* Y *is paradoxical*, *then each* Y_i *is paradoxical.*

Theorem A does not require the Axiom of Choice, but Theorem B apparently does (see [W], Corollary 8.8). The Axiom of Choice will be freely used in the present paper.

$\mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ denote the rings of integers, rational numbers and real number respectively; $J = \{x \in \mathbb{R} : 0 < x \leq 1\}$; $\omega = \{k \in \mathbb{Z} : k \geq 0\}$. For any commutative ring R with unity, $SL_n(R)$ denotes the group of $n \times n$ matrices with entries in R and determinant 1.

THEOREM 1. *There exists a finitely additive measure* ϱ *over all bounded subsets of* $\mathbb{Q}^n$ *satisfying* $\varrho((J \cap \mathbb{Q})^n) = 1$, *invariant under* $SL_n(\mathbb{Z})$ *and under the group* $\mathbb{Q}^n$ *of rational translations. Moreover*, $\varrho(\alpha Y) = |\alpha|^n \varrho(Y)$ *for all* $\alpha \in \mathbb{Q}$.

Proof. Let F be any non-principal ultrafilter of subsets of ω. For any bounded function $f : \omega \to \mathbb{R}$ we define the generalized limit $\lim_{k \to F} f(k)$ to be the unique real number λ such that for every open neighborhood V of λ we have

$$\{k : f(k) \in V\} \in F.$$

Now we define an auxiliary measure ν over all bounded sets $Y \subset \mathbb{Q}^n$:

$$\nu(Y) = \lim_{k \to F} (k!)^{-n} \left| Y \cap \frac{1}{k!} \mathbb{Z}^n \right|,$$

where $|U|$ denotes the cardinality of U. Since the lattice $\frac{1}{k!}\mathbb{Z}^n$ is invariant under $SL_n(\mathbb{Z})$, it follows that ν is invariant under $SL_n(\mathbb{Z})$.

Now, the multiplicative group $\mathbb{Q}^*$ of rational numbers different from zero is abelian and hence amenable. Let μ be an invariant universal measure in $\mathbb{Q}^*$ given by its amenability. For all bounded $Y \subset \mathbb{Q}^n$ we define

$$\varrho(Y) = \int_{\mathbb{Q}^*} |q|^{-n} \nu(qY)\, \mu(dq).$$

It is easy to check that the integrated function is bounded and hence the integral exists. The finite additivity of ϱ is obvious. Since ν is invariant under $SL_n(\mathbb{Z})$ so is ϱ. Notice that if $v \in \mathbb{Q}^n$ and k is large enough such that $k!v \in \mathbb{Z}^n$, then

$$\left|(Y+v) \cap \frac{1}{k!}\mathbb{Z}^n\right| = |k!(Y+v) \cap \mathbb{Z}^n| = |k!Y \cap \mathbb{Z}^n| = \left|Y \cap \frac{1}{k!}\mathbb{Z}^n\right|.$$

Hence ν is invariant under the action of $\mathbb{Q}^n$. Thus it is easy to check that ϱ is also invariant under $\mathbb{Q}^n$. It is also clear that

$$\nu((J \cap \mathbb{Q})^n) = 1,$$

and hence the same is true for ϱ. Finally, since $\mu(\alpha dq) = \mu(dq)$ for $\alpha \in \mathbb{Q}^*$, we get $\varrho(\alpha Y) = |\alpha|^n \varrho(Y)$. ∎

Remark 1. Of course one can extend ϱ from $\mathbb{Q}^n$ to $\mathbb{R}^n$ putting $\widetilde{\varrho}(Y) = \varrho(Y \cap \mathbb{Q}^n)$ and $\widetilde{\varrho}$ still has the same invariance and homogeneity properties. Such a $\widetilde{\varrho}$ is an extension of the *Jordan measure*, i.e., the Lebesgue measure restricted to bounded sets whose boundaries have measure zero.

Remark 2. We conjecture that there exists no finitely additive measure ϱ over all bounded subsets of $\mathbb{R}^2 - \{(0,0)\}$ invariant under $SL_2(\mathbb{R})$ with $\varrho(J^2) = 1$. Compare with Theorem 4 below.

Remark 3. There exists no finitely additive measure ϱ over all bounded subsets of $\mathbb{R}^2$ with $\varrho(J^2) = 1$ invariant under $SL_2(\mathbb{Z})$, under the group of integer translations $\mathbb{Z}^2$ and any single translation τ such that $\tau(\mathbb{Q}^2) \cap \mathbb{Q}^2 = \emptyset$. This follows from the fact that J^2 is paradoxical relative to the group generated by the above transformations (a theorem of von Neumann, see [W], Thm. 7.3). For another proof see Corollary 3 below.

Remark 4. The set $\mathbb{Z}^2 - \{(0,0)\}$ has a paradoxical decomposition relative to the group $SL_2(\mathbb{Z})$. Thus it has no universal finitely additive measure ϱ invariant under $SL_2(\mathbb{Z})$ satisfying $\varrho(\mathbb{Z}^2 - \{(0,0)\}) = 1$. This follows from the observation that infinitely many disjoint copies of a quadrant of $\mathbb{Z}^2 - \{(0,0)\}$ can be packed into $\mathbb{Z}^2 - \{(0,0)\}$ by means of this group (see [W], Addendum to Second Printing, p. 235). For related assertions see Corollary 4 and Theorem 4 below.

Remark 4 is related to the following problems which were already raised in [MW], §9.

Problem 1. Does the group of transformations of $\mathbb{Z}^2$ generated by $SL_2(\mathbb{Z})$ and by $\mathbb{Z}^2$ have a free non-abelian subgroup F such that for any $x \in \mathbb{Z}^2$ the subgroup $\{\varphi \in F : \varphi(x) = x\}$ is cyclic?

If the answer was positive, then $\mathbb{Z}^2$ would be paradoxical relative to the action of that free group. This follows from a general theorem of T. J. Dekker (see [W], Cor. 4.12).

Problem 2. Does the group of transformations of $\mathbb{R}^3$ generated by $SL_3(\mathbb{Z})$ and $\mathbb{Z}^3$ have a free non-abelian subgroup whose elements different from the identity have no fixed point in $\mathbb{R}^3$?

The following remark shows that for $\mathbb{R}^2$ no such free group is possible.

Remark 5. If $A, B \in SL_2(\mathbb{R})$, $AB \neq BA$, $\varphi(x) = A(x)+u$ and $\psi(x) = B(x)+v$, where $u, v \in \mathbb{R}^2$, then at least one of the four equations $\varphi(x) = x$, $\psi(x) = x$, $\varphi\psi(x) = x$, $\varphi\psi^{-1}(x) = x$ has a solution $x \in \mathbb{R}^2$. Indeed, if neither $\varphi(x) = x$ nor $\psi(x) = x$ can be solved, then $\det(A - I) = \det(B - I) = 0$. Hence $\mathrm{tr}(A) = \mathrm{tr}(B) = 2$, i.e., A and B are parabolic. Then it follows by an easy calculation that since $AB \neq BA$ either AB or AB^{-1} is hyperbolic, i.e., has trace larger than 2, and that $\varphi\psi(x) = x$ or $\varphi\psi^{-1}(x) = x$ has a solution.

For other remarks about Problems 1 and 2, see [MW], §9. See also [K], [B] and [S].

Remark 6. Let $SA_n(\mathbb{R})$ denote the group of transformations of $\mathbb{R}^n$ generated by $SL_n(\mathbb{R})$ and by $\mathbb{R}^n$. Then a generic element of $SA_n(\mathbb{R})$ has exactly one fixed point in $\mathbb{R}^n$. The proof is similar to the argument in Remark 8 below.

For generic isometries of $\mathbb{R}^n$ and of the spheres S^{n-1} the situation may be different depending on the parity of n. The following remarks describe this situation.

Remark 7. All the elements of $SO_{2n+1}(\mathbb{R})$ have eigenvectors in $\mathbb{R}^{2n+1}$, but the generic orientation-preserving isometries of $\mathbb{R}^{2n+1}$ have no fixed points. Indeed, for all $A \in SO_{2n+1}(\mathbb{R})$ we have $\det(A - I) = 0$, whence $A(x) + v$ has no fixed points unless v is in the (proper) linear subspace $(A - I)[\mathbb{R}^{2n+1}]$ of $\mathbb{R}^{2n+1}$.

Remark 8. For even dimensions the situation is the opposite. The generic elements of $SO_{2n}(\mathbb{R})$ have no eigenvectors in $\mathbb{R}^{2n}$, but generic isometries of $\mathbb{R}^{2n}$ have single fixed points. Indeed, for generic $A \in SO_{2n}(\mathbb{R})$ we have $\det(A - I) \neq 0$. Hence $A(x) + v$ has one fixed point in $\mathbb{R}^{2n}$.

Remarks 7 and 8 suggest further problems.

Problem 3. Does the group $SO_{2n}(\mathbb{Q})$ $(n > 1)$ have a free non-abelian subgroup whose elements other than unity have no fixed points in $\mathbb{Q}^{2n} - \{0\}$?

For n even the answer is yes. This was shown recently by Kenzi Satô, by an adaptation of a proof of T. J. Dekker (see [W], proof of Theorem 5.2). Thus it is easy to see that Problem 3 fully reduces to the case $n = 3$.

PROBLEM 4. Does the group $SO_{2n+1}(\mathbb{Q})$ $(n \geq 1)$ have a free non-abelian subgroup F whose elements other than unity have no fixed points in the rational unit sphere in $\mathbb{Q}^{2n+1}$ and such that for all $x \in \mathbb{Q}^{2n+1} - \{0\}$ the subgroup $\{\varphi \in F : \varphi(x) = x\}$ is cyclic?

For $n = 1$ the answer is yes. This follows easily from a recent theorem of Kenzi Satô [S]. And if the answer to Problem 3 is yes, then the answer to Problem 4 is also yes with the only possible exception for the case $n = 2$.

PROBLEM 5. Does the group of isometries of $\mathbb{Q}^3$ have a free non-abelian subgroup whose elements other than unity have no fixed points in $\mathbb{Q}^3$?

Problems 3 and 5 have positive solutions if $\mathbb{Q}$ is replaced by $\mathbb{R}$, see [DS], [B] and a theorem of Dekker, Mycielski and Świerczkowski ([W], Thm. 5.7).

THEOREM 2. *There exists a finitely additive measure ϱ over all bounded subsets of $\mathbb{R}^n$ which is invariant under $SL_n(\mathbb{Z})$, satisfies $\varrho(J^n) = 1$ and $\varrho(\alpha Y) = |\alpha|^n \varrho(Y)$ for all $\alpha \in \mathbb{R}$.*

Proof. The proof is very similar to that of Theorem 1, only integration over $\mathbb{Q}^*$ should be replaced by integration over $\mathbb{R}^*$ (the multiplicative group of non-zero real numbers). ∎

Remark 9. The measure ϱ of Theorem 2 is an extension of the Jordan measure.

PROBLEM 6. Unlike in Theorem 1 we do not know if Theorem 2 can be strengthened by requiring also the invariance of ϱ under some group of translations, e.g., under $\mathbb{Z}^n$.

THEOREM 3. *There exists a universal measure ϱ over the rational torus $(\mathbb{Q}/\mathbb{Z})^n$ which is invariant under the natural action of $SL_n(\mathbb{Z})$ and of $\mathbb{Q}^n$, and such that $\varrho((\mathbb{Q}/\mathbb{Z})^n) = 1$.*

Proof. This follows from Theorem 1. It suffices to identify $(\mathbb{Q}/\mathbb{Z})^n$ with $(J \cap \mathbb{Q})^n$ and to treat the transformations of $SL_n(\mathbb{Z})$ and $\mathbb{Q}^n$ over $(\mathbb{Q}/\mathbb{Z})^n$ as unions of finitely many restrictions of appropriate transformations of the space $\mathbb{Q}^n$ to appropriate disjoint subsets of $(J \cap \mathbb{Q})^n$. ∎

LEMMA 1. (i) *If $A \in SL_2(\mathbb{R})$ and $\mathrm{tr}(A) \neq 2$, then $A(x) \neq x$ for all $x \in \mathbb{R}^2 - \{(0,0)\}$.*

(ii) *If $A \in SL_2(\mathbb{Z})$ and $\mathrm{tr}(A) \neq 2$, then $A(x) \neq x$ for all $x \in (\mathbb{R}/\mathbb{Z})^2 - (\mathbb{Q}/\mathbb{Z})^2$.*

Proof. (i) It is easy to check that if $A \in SL_2(\mathbb{R})$ and $\mathrm{tr}(A) \neq 2$, then $\det(A-I) \neq 0$. Hence, if $A(x) = x$, then $(A-I)x = 0$ and $x = (0,0)$ follows.

(ii) We show in the same way that $\det(A-I)\neq 0$. Thus if $A(x)=x$ for $x=\widetilde{x}/\mathbb{Z}^2$, then $A(\widetilde{x})-\widetilde{x}\in\mathbb{Z}^2$. Hence $(A-I)\widetilde{x}\in\mathbb{Z}^2$ and $\widetilde{x}\in(A-I)^{-1}[\mathbb{Z}^2]\subseteq\mathbb{Q}^2$. Thus $x\in\mathbb{Q}^2/\mathbb{Z}^2$. ∎

COROLLARY 1. *$SL_2(\mathbb{Z})$ has free non-abelian subgroups whose elements other than unity act without fixed point on $\mathbb{R}^2-\{(0,0)\}$ and on $(\mathbb{R}/\mathbb{Z})^2-(\mathbb{Q}/\mathbb{Z})^2$.*

Proof. It is known that $SL_2(\mathbb{Z})$ has free non-abelian subgroups all of whose elements other than unity are hyperbolic, i.e., have traces larger than 2. The pair of matrices

$$\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}5 & 2\\ 2 & 1\end{pmatrix}$$

generates such a subgroup (a theorem of B. H. Neumann, see [W], p. 86 and references therein). Hence Corollary 1 follows from Lemma 1. ∎

COROLLARY 2. *$J^2-\mathbb{Q}^2$ is paradoxical relative to the group of transformations of $\mathbb{R}^2$ generated by $SL_2(\mathbb{Z})$ and by $\mathbb{Z}^2$, and also relative to $SL_2(\mathbb{Z})$ acting on $(\mathbb{R}/\mathbb{Z})^2$.*

Proof. This follows from the second conclusion of Corollary 1 and the general theory of equivalence by finite decompositions (see [W], Corollary 4.12). ∎

COROLLARY 3. *J^2 is paradoxical relative to the group of transformations of $\mathbb{R}^2$ generated by $SL_2(\mathbb{Z})$, by $\mathbb{Z}$ and by any single translation τ of $\mathbb{R}^2$ such that $\tau(\mathbb{Q}^2)\cap\mathbb{Q}^2=\emptyset$.*

Proof. This follows from Corollary 2 and the fact that $\tau(J^2)\equiv J^2$ relative to the group $\mathbb{Z}^2$. ∎

COROLLARY 4. *If $A,B\subseteq\mathbb{R}^2$ are two bounded sets with non-empty interiors, then $A\equiv B$ relative to the group of transformations of $\mathbb{R}^2$ generated by $SL_2(\mathbb{Z})$, by $\mathbb{Q}^2$ and by any single translation τ of $\mathbb{R}^2$ such that $\tau(\mathbb{Q}^2)\cap\mathbb{Q}^2=\emptyset$.*

Proof. It follows from Theorem B and Corollary 3 that for every positive integer k the square $\left(\frac{1}{k}J\right)^2$ is paradoxical relative to this group. Since A and B have interior points, there are translations $\tau_1,\tau_2\in\mathbb{Q}^2$ and a k such that $\left(\frac{1}{k}J\right)^2\subseteq\tau_1(A)\cap\tau_2(B)$. Hence there are sets A' and B' $\left[\text{containing sufficiently many disjoint translates of } \left(\frac{1}{k}J\right)^2\right]$ such that $A\equiv A'\supseteq B$ and $B\equiv B'\supseteq A$. Thus, by the Cantor–Bernstein theorem (Theorem A at the beginning of this paper), we have $A\equiv B$. ∎

LEMMA 2. *If $A,B\in SL_2(\mathbb{R})$ and $A(x)=B(x)=x$ for some $x\in\mathbb{R}^2-\{(0,0)\}$, then $AB=BA$.*

P r o o f. Choose an orthonormal basis x_0, x_1 in $\mathbb{R}^2$ such that $A(x_0) = B(x_0) = x_0$. Then, relative to this basis,

$$A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},$$

for some $a, b \in \mathbb{R}$. Thus

$$AB = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix} = BA.$$

COROLLARY 5. $\mathbb{R}^2 - \{(0,0)\}$ *is paradoxical relative to every free non-abelian subgroup of* $SL_2(\mathbb{R})$.

P r o o f. By Lemma 2 and the general decomposition theorem of Dekker (see [W], Thm. 4.12). ∎

For related theorems about $\mathbb{R}^n$, see [M$_2$] and [W].

Corollaries 2–5 suggest the problem if there are any natural bounded sets in $\mathbb{R}^2 - \{(0,0)\}$ which are paradoxical relative to the group $SL_2(\mathbb{R})$. The problem remains unsolved but I will reduce it to a certain conjecture (C) and will explain why I believe that (C) is true. (The idea of the reduction is similar to that in [M$_1$].)

Let $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le r^2\}$, and $f{\restriction}Y = f \cap (Y \times X)$ for $Y \subseteq X$, $f : X \to X$.

LEMMA 3. *For any* $\varphi \in SL_2(\mathbb{R})$ *there exists a rotation* $\varrho_\varphi \in SO_2(\mathbb{R})$ *such that*

$$D - \varphi(D) = \varrho_\varphi(D - \varphi^{-1}(D)).$$

P r o o f. This follows since the ellipses $\varphi(D)$ and $\varphi^{-1}(D)$ are congruent. ∎

From now on our arguments are incomplete in the sense that they depend on the following conjecture.

(C) *There exists a free non-abelian group* F *acting on* D *such that if* $f \in F - \{e\}$ *and* $x \in D - \{(0,0)\}$, *then* $f(x) \ne x$, *and for every* $f \in F$ *there exists a finite partition* $D = D_1 \cup \ldots \cup D_n$ *and* $\varphi_1, \ldots, \varphi_n \in SL_2(\mathbb{R})$ *such that* $f{\restriction}D_i = \varphi_i{\restriction}D_i$ *for* $i = 1, \ldots, n$.

An incomplete argument supporting this conjecture (on the basis of Lemma 1(i) and Lemma 3) will be given at the end of this paper.

LEMMA 4 (Assuming (C)). *The punctured disk* $D - \{(0,0)\}$ *is paradoxical relative to the group* $SL_2(\mathbb{R})$.

P r o o f. This follows by (C) and the general decomposition theorem (see [W], Cor. 4.12). ∎

LEMMA 5 (Assuming (C)). *If* D_1 *and* D_2 *are two disks both with center* $(0,0)$, *then* $D_1 \equiv D_2$ *relative to* $SL_2(\mathbb{R})$.

P r o o f. Let radius $D_1 \le$ radius D_2. A transformation in $SL_2(\mathbb{R})$ can turn D_1 into an ellipse E whose long axis is longer than the diameter of D_2. Then finitely many rotations of E can cover D_2. Hence, by Lemma 4, $D_2 \equiv D_2'$ for some set $D_2' \subseteq D_1$. Of course, $D_1 \subseteq D_2$. Hence by the Cantor–Bernstein Theorem (Theorem A) we have $D_1 \equiv D_2$. ∎

LEMMA 6 (Assuming (C)). *If* $A \subseteq \mathbb{R}^2$ *is a bounded set which contains a neighborhood of* $(0,0)$, *then* $A \equiv D$ *relative to* $SL_2(\mathbb{R})$.

P r o o f. There are disks D_1 and D_2 centered at $(0,0)$ such that $D_1 \subseteq A \subseteq D_2$. Thus Lemma 6 follows from Lemma 5 and the Cantor–Bernstein Theorem. ∎

LEMMA 7 (Assuming (C)). *If* T *is an open triangle in* $\mathbb{R}^2$ *which has a vertex at* $(0,0)$, *then* $T \equiv D - \{(0,0)\}$ *relative to* $SL_2(\mathbb{R})$.

P r o o f. A union of finitely many rotations of T covers a punctured disk $D_0 - \{(0,0)\}$. Hence, by the Cancellation Theorem (Theorem B) and Lemmas 4 and 5, T is paradoxical. Thus there exists a set $S \equiv T$ such that S contains $D_0 - \{(0.0)\}$. And by Lemma 6, $T \equiv D - \{(0,0)\}$. ∎

THEOREM 4 (Assuming (C)). *If* $A, B \subseteq \mathbb{R}^2 - \{(0,0)\}$ *are bounded sets and either* (α) *both* A *and* B *include open triangles with one vertex at* $(0,0)$, *or* (β) *both* A *and* B *have non-empty interior and both distances from* $(0,0)$ *to* A *and from* $(0,0)$ *to* B *are positive*, *then* $A \equiv B$ *relative to the group* $SL_2(\mathbb{R})$.

P r o o f. *Case* (α). This case follows immediately from Lemmas 5 and 7, and the Cantor–Bernstein Theorem.

Case (β). Instead of disks we have to work with annuli $\{x \in \mathbb{R}^2 : r_1 \le \|x\| \le r_2\}$, and prove for them lemmas similar to Lemmas $4, \ldots, 7$. We omit these proofs as they are quite similar to the previous ones. ∎

Incomplete argument for the conjecture (C). Using Lemma 3 for all $\varphi \in SL_2(\mathbb{R})$ and all $x \in D$ we define

$$\widehat{\varphi}(x) = \begin{cases} \varphi(x) & \text{if } \varphi(x) \in D, \\ \varrho_\varphi(x) & \text{if } \varphi(x) \notin D. \end{cases}$$

Thus $\widehat{\varphi} : D \to D$ is a piecewise linear bijection. It is easy to check that there are three nonempty open sets $A, B, C \subseteq SL_2(\mathbb{R})$ such that if $(\varphi, \psi, \chi) \in A \times B \times C$ then the composed map $\widehat{\varphi}\widehat{\psi}\widehat{\chi}$ has the following property:

(P) *For every* $x \in D$, $\widehat{\varphi}\widehat{\psi}\widehat{\chi}(x) = fgh(x)$, *where*

$$(f, g, h) \in \{\varphi, \varrho_\varphi\} \times \{\psi, \varrho_\psi\} \times \{\chi, \varrho_\chi\} - \{(\varrho_\varphi, \varrho_\psi, \varrho_\chi)\}.$$

Thus out of the eight possible forms of $\widehat{\varphi}\widehat{\psi}\widehat{\chi}(x)$ only seven involving at least one of the functions φ, ψ or χ may actually occur (although those forms which occur depend on φ, ψ, χ and on x).

Now the conjecture (C) reduces to the following more specific conjecture: There exist two triples $(\varphi_1, \psi_1, \chi_1), (\varphi_2, \psi_2, \chi_2) \in A \times B \times C$ such that the pair of transformations $\widehat{\varphi}_1\widehat{\psi}_1\widehat{\chi}_1, \widehat{\varphi}_2\widehat{\psi}_2\widehat{\chi}_2 : D \to D$ generates a free group as required in (C). I feel that (P) and Lemma 1(i) suggest that (C) is true.

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