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ON THE FORMAL INVERSE OF POLYNOMIAL ENDOMORPHISMS

BY

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Let k be a field of characteristic 0. We begin by recalling some facts about the Jacobian Conjecture. We denote by $J(F)$ the Jacobian matrix of a polynomial map F.

CONJECTURE 1 (Jacobian Conjecture). If $F : k^n \to k^n$ is a polynomial map such that $\det J(F) \in k \setminus \{0\}$, then F is a polynomial automorphism, that is, there exists a polynomial map $G : k^n \to k^n$ satisfying $F(G) = X$.

Yagzhev [9] and Bass, Connell and Wright [1] showed that, if the Jacobian Conjecture is true for all $n \ge 2$ and all polynomial maps of the form $F = X - H$ with H homogeneous of degree 3, then it is true for all polynomial maps. For the Jacobian matrix of a polynomial map F the hypothesis $\det J(F) \in k \setminus \{0\}$ is equivalent to the nilpotence of $J(H)$.

Let $G = (G_1, \ldots, G_n)$ with $G_i \in k[[X_1, \ldots, X_n]]$ be the formal inverse of $F = X - H$, that is, $F(G) = X$. It is obvious that F is an automorphism if and only if $G_1, \ldots, G_n$ are polynomials.

Since in $X - H$ all the non-zero homogeneous components have odd degree, G has the same property. Let $G_i = \sum_{d\ge 0} G_i^{(d)}$, where each $G_i^{(d)}$ is homogeneous of degree $2d+1$ and $i = 1, \ldots, n$. Several formulas for $G_i^{(d)}$ are known. In those given by Bass, Connell and Wright [1] and Drużkowski and Rusek [2], the components $G_i^{(d)}$ are expressed as $\mathbb{Q}$-linear combinations of polynomials indexed by rooted trees. Our aim is to prove that the polynomials, corresponding in the above mentioned expansions to the same rooted tree, differ by a rational factor depending on the structure of the rooted tree.

1. Rooted trees. If T is a non-directed tree, then $V(T)$ denotes the set of its vertices, and the set of its edges is a symmetric subset $E(V) \subseteq V(T) \times V(T)$. A tree T with a distinguished vertex $\mathrm{rt}_T \in V(T)$, a *root*, is called a *rooted tree*. By induction we define the sets $V_j(T)$ of vertices of

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height j. Let $V_0(T) = \{\mathrm{rt}_T\}$. For $j > 0$ let $v \in V_j(T)$ iff for some $w \in V_{j-1}(T)$ there exists an edge $(w, v) \in E(T)$ and $v \notin V_i(T)$ for all $i < j$. Moreover, let $\mathrm{ht}(T) = \max\{j : V_j(T) \neq \emptyset\}$.

If $v \in V_j(T)$, then let

$$v^+ = \{w \in V_{j+1}(T) : (w, v) \in E(T)\}.$$

By a *leaf* of a rooted tree T we mean a vertex $v \in V(T)$ such that $v^+ = \emptyset$.

The rooted trees form a category; a morphism $T \to T'$ is a map $f : V(T) \to V(T')$ such that $f(\mathrm{rt}_T) = \mathrm{rt}_{T'}$ and $(f \times f)(E(T)) \subseteq E(T')$. If T is a rooted tree, then $\mathrm{Aut}(T)$ denotes the group of all automorphisms of T and $\alpha(T) = |\mathrm{Aut}(T)|$. ($|X|$ is the cardinality of the set X.)

For a rooted tree T and a vertex $v \in V(T)$ we define a rooted tree T_v to be a subtree of T such that $\mathrm{rt}_{T_v} = v$ and $w \in V(T_v)$ if v belongs to a path from w to the root.

Let T be a rooted tree and

$$\mathrm{rt}_T^+ = \{v_{11}, \ldots, v_{1m_1}, \ldots, v_{s1}, \ldots, v_{sm_s}\}.$$

Moreover, let $\{T_{v_{11}}, \ldots, T_{v_{1m_1}}\}, \ldots, \{T_{v_{s1}}, \ldots, T_{v_{sm_s}}\}$ be the isomorphism classes of the rooted trees $T_{v_{ij}}$. It is easy to see that

$$\alpha(T) = \prod_{j=1}^{s} (\alpha(T_{v_{j1}})^{m_j} \cdot m_j!) \tag{1}$$

(cf. [6]).

In this note we assume that there exists an empty rooted tree $\emptyset$ with $V(\emptyset) = \emptyset$ and $E(\emptyset) = \emptyset$.

2. Bass–Connell–Wright formal inverse expansion. Let $H = (H_1, \ldots, H_n)$, where $H_1, \ldots, H_n \in k[X_1, \ldots, X_n]$ are homogeneous of degree 3. Let $\mathbf{n} = \{1, \ldots, n\}$. For $i \in \mathbf{n}$, a rooted tree T and a function $f : V(T) \to \mathbf{n}$ such that $f(\mathrm{rt}_T) = i$, in [1] there are defined polynomials

$$P_{T,f} = \prod_{v \in V(T)} \Big(\Big(\prod_{w \in v^+} D_{f(w)} \Big) H_{f(v)} \Big)$$

and

$$\sigma_i(T) = \sum_{\substack{f : V(T) \to \mathbf{n} \\ f(\mathrm{rt}_T) = i}} P_{T,f}.$$

In [1, Ch. III, 5.(4)] it is shown that if T contains a vertex such that $|v^+| > 3$, then $\sigma_i(T) = 0$. We denote by $\mathbb{T}'_d$ a fixed set of representatives of the isomorphism classes of rooted trees with d vertices and with $|v^+| \le 3$ for each $v \in V(T)$. Note that $\mathbb{T}'_0 = \{\emptyset\}$. Using these observations, we can quote the following theorem.

THEOREM 2 (Bass–Connell–Wright [1]). *Let* $F = X - H : k^n \to k^n$, *where* H *is homogeneous of degree* 3 *and the matrix* $J(H)$ *is nilpotent. Then* $G_i^{(0)} = X_i$ *and*

$$G_i^{(d)} = \sum_{T \in \mathbb{T}'_d} \frac{1}{\alpha(T)} \sigma_i(T) \quad \text{for } d \ge 1.$$

Theorem 2 suggests the following definition: $\sigma_i(\emptyset) = X_i$ for $i \in \mathbf{n}$.

In the sequel we use the below description of the numbers $\alpha(T)$.

DEFINITION 3. For a rooted tree T and a vertex $v \in V(T)$ let

$$\alpha(v, T) = \prod_{j=1}^{s} m_j!$$

where $m_1, \ldots, m_s$ are the cardinalities of the isomorphism classes of the rooted trees from $\{T_w : w \in v^+\}$. Note that $\alpha(v, T) = 1$ for each leaf v.

REMARK. One can rewrite the formula (1) in the form

$$\alpha(T) = \alpha(\mathrm{rt}_T, T) \prod_{v \in \mathrm{rt}_T^+} \alpha(T_v). \tag{2}$$

LEMMA 4. *If* T *is a rooted tree*, *then*

$$\alpha(T) = \prod_{v \in V(T)} \alpha(v, T).$$

Proof. Use (2) and induction with respect to the height of T. ■

3. Drużkowski–Rusek formal inverse. In [2] we can find another description of the formal inverse. We suppose that $F = X - H$, where H is homogeneous of degree 3. It is well known that there exists a unique 3-linear symmetric polynomial map $\widetilde{H} : k^n \times k^n \times k^n \to k^n$ such that $\widetilde{H}(X, X, X) = H(X)$.

THEOREM 5 (Drużkowski–Rusek [2]). *If* $G = \sum_{d \ge 0} G^{(d)}$ *is the formal inverse of* $F = X - H$, *then* $G^{(0)} = X$ *and*

$$G^{(d)} = \sum_{p+q+r=d-1} \widetilde{H}(G^{(p)}, G^{(q)}, G^{(r)}) \quad \text{for } d \ge 1.$$

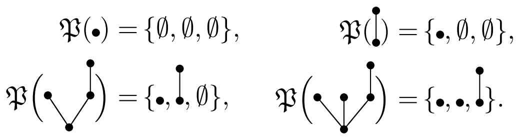
For small indices we have:

$$\begin{aligned} G^{(0)} &= X, \\ G^{(1)} &= \widetilde{H}(X, X, X), \\ G^{(2)} &= 3\widetilde{H}(X, X, \widetilde{H}(X, X, X)), \\ G^{(3)} &= 9\widetilde{H}(X, X, \widetilde{H}(X, X, \widetilde{H}(X, X, X))) \\ &\quad + 3\widetilde{H}(X, \widetilde{H}(X, X, X), \widetilde{H}(X, X, X)). \end{aligned} \tag{3}$$

We shall see that each $G^{(d)}$ is a linear combination of polynomial maps corresponding to rooted trees.

DEFINITION 6. For any rooted tree $T \in \mathbb{T}'_d$ with $d \geq 1$ we define $\mathfrak{P}(T)$ to be a multiset (i.e., a set with repeated elements; see [7]) containing representatives of the isomorphism classes of the rooted trees T_v for $v \in \mathrm{rt}_T^+$ and $3 - |\mathrm{rt}_T^+|$ empty trees. Thus the multiset $\mathfrak{P}(T)$ has exactly 3 elements.

EXAMPLE 7. (Always the lowest vertex is the root.)



DEFINITION 8. For a rooted tree $T \in \mathbb{T}'_d$ we define, by induction on $d \geq 0$, a polynomial homogeneous map $\tau(T) : k^n \to k^n$ of degree $2d+1$ as follows:

$$\begin{aligned}
&\tau(\emptyset) = X \qquad \text{(the identity map)},\\
&\tau(T) = \widetilde{H}(\tau(T_1), \tau(T_2), \tau(T_3)) \qquad \text{for } d \geq 1 \text{ and } \mathfrak{P}(T) = \{T_1, T_2, T_3\}.
\end{aligned}$$

Now, let us describe the coefficients in linear combinations like (3).

DEFINITION 9. For a rooted tree T and a vertex $v \in V(T)$ we define $\beta(v, T)$ to be the number of different sequences of elements of the multiset $\mathfrak{P}(T_v)$.

LEMMA 10. *If* $d \geq 0$, *then*

$$G^{(d)} = \sum_{T \in \mathbb{T}'_d} \beta(T)\tau(T), \quad \textit{where} \quad \beta(T) = \prod_{v \in V(T)} \beta(v, T). \tag{4}$$

Proof. We prove this lemma by induction on d.

For $d = 0$ the equality (4) is obvious. Note that $\beta(\emptyset) = 1$.

Let $d > 0$. Then

$$\begin{aligned}
G^{(d)} &= \sum_{p+q+r=d-1} \widetilde{H}(G^{(p)}, G^{(q)}, G^{(r)})\\
&= \sum_{p+q+r=d-1} \widetilde{H}\Big(\sum_{T_1 \in \mathbb{T}'_p} \beta(T_1)\tau(T_1), \sum_{T_2 \in \mathbb{T}'_q} \beta(T_2)\tau(T_2), \sum_{T_3 \in \mathbb{T}'_r} \beta(T_3)\tau(T_3)\Big)\\
&= \sum_{p+q+r=d-1} \sum_{T_1 \in \mathbb{T}'_p} \sum_{T_2 \in \mathbb{T}'_q} \sum_{T_3 \in \mathbb{T}'_r} \beta(T_1)\beta(T_2)\beta(T_3) \cdot \widetilde{H}(\tau(T_1), \tau(T_2), \tau(T_3)).
\end{aligned}$$

All maps of the form $\widetilde{H}(\tau(T_1), \tau(T_2), \tau(T_3))$ are homogeneous of degree $2p + 1 + 2q + 1 + 2r + 1 = 2(p+q+r) + 3 = 2d + 1$. Collecting summands with the same map $\tau(T)$, for $T \in \mathbb{T}'_d$, we see that the coefficient of $\tau(T)$ is equal to

$$\beta(\mathrm{rt}_T, T) \cdot \beta(T_1)\beta(T_2)\beta(T_3) = \beta(T),$$

where $\mathfrak{P}(T) = \{T_1, T_2, T_3\}$. ■

4. Main theorem. We are going to compare the expressions for $G^{(d)}$ given in the previous subsections.

DEFINITION 11. For a rooted tree $T \in \mathbb{T}'_d$ and a vertex $v \in V(T)$ we define numbers $\varrho(v,T)$ and $\varrho(T)$ as

$$\varrho(v,T) = \alpha(v,T)\beta(v,T) \quad \text{and} \quad \varrho(T) = \prod_{v\in V(T)} \varrho(v,T).$$

In particular, $\varrho(\emptyset) = 1$.

COROLLARY. *If $T \in \mathbb{T}'_d$, then $\varrho(T) = \alpha(T)\beta(T)$.*

LEMMA 12. *If $T \in \mathbb{T}'_d$ and $v \in V(T)$, then*

$$\varrho(v,T) = \frac{3!}{(3-|v^+|)!} = \begin{cases} 1 & \textit{for } |v^+| = 0, \\ 3 & \textit{for } |v^+| = 1, \\ 6 & \textit{for } |v^+| \in \{2,3\}. \end{cases}$$

Proof. It is sufficient to collect the numbers $\alpha(v,T)$, $\beta(v,T)$ and $\varrho(v,T) = \alpha(v,T)\beta(v,T)$ in a table. In the second column we assume that the rooted trees $T_1, T_2, T_3, \emptyset$ are all distinct.

$\|v^+\|$	$\mathfrak{P}(T_v)$	$\alpha(v,T)$	$\beta(v,T)$	$\varrho(v,T)$
0	$\{\emptyset,\emptyset,\emptyset\}$	1	1	1
1	$\{T_1,\emptyset,\emptyset\}$	1	3	3
2	$\{T_1,T_1,\emptyset\}$	2	3	6
	$\{T_1,T_2,\emptyset\}$	1	6	6
3	$\{T_1,T_1,T_1\}$	6	1	6
	$\{T_1,T_1,T_2\}$	2	3	6
	$\{T_1,T_2,T_3\}$	1	6	6

Now, compare the first and last columns. The last column is obviously equal to $3!/(3-|v^+|)!$. ■

COROLLARY. *For $T \in \mathbb{T}'_d$ we have*

$$\varrho(T) = 2^{|\{v\in V(T)\,:\,|v^+|\ge 2\}|} \cdot 3^{|V(T)\setminus \mathrm{Leaf}(T)|},$$

where $\mathrm{Leaf}(T)$ *is the set of all leaves of the rooted tree T.*

In the proof of Theorem 13 we make use of the following polarization formula:

$$\widetilde{H}_i(U,V,W) = \frac{1}{3!}\sum_{p,q,r=1}^{n} U_pV_qW_r \frac{\partial^3 H_i}{\partial X_p \partial X_q \partial X_r} \tag{5}$$

(see [5, p. 251]). We also recall Euler's formula: if $F(X)$ is homogeneous of degree p, then

$$\sum_{p=1}^{n} X_p \frac{\partial F(X)}{\partial X_p} = p \cdot F(X).$$

We are now in a position to formulate and prove the main theorem of our paper.

THEOREM 13. *If* $i \in \mathbf{n}$ *and* $T \in \mathbb{T}'_d$ *for* $d \geq 0$, *then*

$$\sigma_i(T) = \varrho(T)\tau_i(T), \tag{6}$$

where $\tau(T) = (\tau_1(T), \ldots, \tau_n(T))$.

Proof. We argue by induction on the number of vertices of T.

The case $d = 0$ is obvious:

$$\sigma_i(\emptyset) = X_i = 1 \cdot X_i = \varrho(\emptyset)\tau_i(\emptyset).$$

Suppose now that $T \in \mathbb{T}'_d$ ($d \geq 1$) and (6) is true for all rooted trees with less than d vertices. Let $\mathfrak{P}(T) = \{T_1, \ldots, T_s, \emptyset, \ldots, \emptyset\}$, where $0 \leq s \leq 3$ and $T_1, \ldots, T_s$ are non-empty. For $i \in \mathbf{n}$, using a "tree surgery" (see [1], [8]), we can write

$$\sigma_i(T) = \sigma_i\left(\begin{smallmatrix} T_1 & \cdots & T_s \\ & \bullet & \end{smallmatrix}\right) = \sum_{j_1, \ldots, j_s = 1}^{n} \sigma_{j_1}(T_1) \ldots \sigma_{j_s}(T_s) \frac{\partial^s H_i}{\partial X_{j_1} \ldots \partial X_{j_s}}.$$

Let us apply Euler's formula $3 - s$ times and let $T_j = \emptyset$ for $j = s + 1, \ldots, 3$:

$$\begin{aligned}
\sigma_i(T) &= \frac{1}{(3-s)!} \sum_{j_1, j_2, j_3 = 1}^{n} \sigma_{j_1}(T_1) \ldots \sigma_{j_s}(T_s) X_{j_{s+1}} \ldots X_{j_3} \frac{\partial^3 H_i}{\partial X_{j_1} \partial X_{j_2} \partial X_{j_3}} \\
&= \frac{1}{(3-s)!} \sum_{j_1, j_2, j_3 = 1}^{n} \sigma_{j_1}(T_1) \sigma_{j_2}(T_2) \sigma_{j_3}(T_3) \frac{\partial^3 H_i}{\partial X_{j_1} \partial X_{j_2} \partial X_{j_3}}.
\end{aligned}$$

Hence by assumption,

$$\sigma_i(T) = \frac{\varrho(T_1)\varrho(T_2)\varrho(T_3)}{(3-s)!} \sum_{j_1, j_2, j_3 = 1}^{n} \tau_{j_1}(T_1) \tau_{j_2}(T_2) \tau_{j_3}(T_3) \frac{\partial^3 H_i}{\partial X_{j_1} \partial X_{j_2} \partial X_{j_3}},$$

and by (5),

$$\sigma_i(T) = \frac{3! \cdot \varrho(T_1)\varrho(T_2)\varrho(T_3)}{(3-s)!} \widetilde{H}_i(\tau(T_1), \tau(T_2), \tau(T_3)).$$

Finally, we apply Lemma 12, Definition 8 and Definition 11 to get

$$\sigma_i(T) = \varrho(\mathrm{rt}_T, T)\varrho(T_1)\varrho(T_2)\varrho(T_3)\tau_i(T) = \varrho(T)\tau_i(T),$$

and the proof is complete. ∎

COROLLARY. *If* $T \in \mathbb{T}'_d$ *for* $d \geq 0$, *then*

$$\sigma(T) = \varrho(T)\tau(T),$$

where $\sigma(T) = (\sigma_1(T), \ldots, \sigma_n(T))$.

5. Remarks. Theorem 13 and Lemma 10 give us an alternative proof of Theorem 2. Indeed,

$$G^{(d)} = \sum_{T \in \mathbb{T}'_d} \beta(T)\tau(T) = \sum_{T \in \mathbb{T}'_d} \frac{1}{\alpha(T)} \varrho(T)\tau(T) = \sum_{T \in \mathbb{T}'_d} \frac{1}{\alpha(T)} \sigma(T).$$

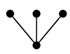
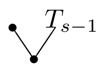
This proof looks simpler than the original one in [1].

It is well known that a polynomial map $F = X - H : k^n \to k^n$ with H homogeneous of degree 3 and $J(H)$ nilpotent has a polynomial inverse iff $G^{(d)} = 0$ for $\deg G^{(d)} = 2d + 1 > 3^{n-1}$. Bass, Connell and Wright [1] conjectures that not only $G^{(d)} = 0$ but also $\sigma(T) = 0$ in the case $T \in \mathbb{T}'_d$ and $2d + 1 > 3^{n-1}$. A counterexample was given in [4]. On the other hand, Gorni and Zampieri [3] showed that there is a polynomial automorphism of the form $X - H$ as above such that for any n there exists a rooted tree T with $\tau(T) \neq 0$ and with the number of vertices of T greater than n. In both papers, the counterexample is the same polynomial map:

$$F = (X_1 + X_4(X_1X_3 + X_2X_4), X_2 - X_3(X_1X_3 + X_2X_4), X_3 + X_4^3, X_4),$$

given by van den Essen for other reasons. In view of Theorem 13, $\tau(T) \neq 0$ iff $\sigma(T) \neq 0$, and problems solved in [3] and [4] are equivalent. Moreover, we can exhibit rooted trees T for which $\tau(T) \neq 0$ (Gorni and Zampieri have not done it).

If

$T_0 =$  $\in \mathbb{T}'_4$, $T_s =$ 

$\in \mathbb{T}'_{2s+4}$ for $s \geq 1$,

then (see [4]) $\sigma(T_s) \neq 0$ for all $s \geq 0$ and therefore the polynomial maps

$$\tau(T_0) = \widetilde{H}(\widetilde{H}(X, X, X), \widetilde{H}(X, X, X), \widetilde{H}(X, X, X)),$$

$$\tau(T_s) = \widetilde{H}(X, \widetilde{H}(X, X, X), \tau(T_{s-1})) \quad \text{for } s \geq 1$$

are non-zero.

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