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A HILBERT CUBE COMPACTIFICATION OF THE SPACE OF RETRACTIONS OF THE INTERVAL

BY

SHIGENORI UEHARA (TSUKUBA)

Introduction. In this paper, let all maps be continuous and $\mathbf{I} = [0,1]$ be the closed interval. In [BS], it was proved that the space $R(\mathbf{I})$ of retractions $f : \mathbf{I} \to \mathbf{I}$ (i.e. $f \circ f = f$) is homeomorphic ($\approx$) to the pseudo-interior $s = (-1,1)^\omega$ of the Hilbert cube $Q = [-1,1]^\omega$, where $R(\mathbf{I})$ has the sup-metric. Thus the Hilbert cube Q is a compactification of $R(\mathbf{I})$. Here we consider such a natural compactification of $R(\mathbf{I})$.

Equip the product $\mathbf{I}^2 = \mathbf{I} \times \mathbf{I}$ with the following metric:

$$d((x,y),(x',y')) = \max\{|x-x'|, |y-y'|\},$$

and let $\exp(\mathbf{I}^2)$ be the hyperspace of nonempty compact subsets of $\mathbf{I}^2$ endowed with the Hausdorff metric:

$$d_{\mathrm{H}}(E,F) = \inf\{\varepsilon > 0 \mid E \subset N_d(F,\varepsilon),\ F \subset N_d(E,\varepsilon)\},$$

where $N_d(F,\varepsilon)$ is the ε-neighborhood of F in $\mathbf{I}^2$ with metric d. In this paper, we always identify a map $f : \mathbf{I} \to \mathbf{I}$ with its graph $\mathrm{Gr}(f) \in \exp(\mathbf{I}^2)$. So we can regard $R(\mathbf{I})$ as a subset of $\exp(\mathbf{I}^2)$. Moreover, as is easily observed, the space $R(\mathbf{I})$ (with the sup-norm) is a subspace of $\exp(\mathbf{I}^2)$. Define $\overline{R}(\mathbf{I})$ as the closure of $R(\mathbf{I})$ in $\exp(\mathbf{I}^2)$ (cf. [Fe]). The following is our main result.

MAIN THEOREM. *The pair* $(\overline{R}(\mathbf{I}), R(\mathbf{I}))$ *is homeomorphic to the pair* (Q,s).

A related result is shown in [SU_2]: $(\overline{H}_\partial(\mathbf{I}), H_\partial(\mathbf{I})) \approx (Q,s)$, where $H_\partial(\mathbf{I})$ is the space of orientation preserving homeomorphisms of $\mathbf{I}$, and $\overline{H}_\partial(\mathbf{I})$ is the closure of $H_\partial(\mathbf{I})$ in $\exp(\mathbf{I}^2)$. Concerning the space $C(X,\mathbf{I})$ of maps from a compactum X to $\mathbf{I}$, it is shown in [SU_1] that $(\overline{C}(X,\mathbf{I}), C(X,\mathbf{I})) \approx (Q,s)$ if X is locally connected and infinite, where $\overline{C}(X,\mathbf{I})$ is the closure of $C(X,\mathbf{I})$ in $\exp(X \times \mathbf{I})$. Moreover, in case X has no isolated points, $\overline{C}(X,\mathbf{I})$ coincides

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with the space $\mathrm{USCC}(X,\mathbf{I})$ of upper semi-continuous (u.s.c.) multi-valued functions $\varphi : X \to \mathbf{I}$ such that each $\varphi(x)$ is a closed interval (see [Fe]).

Proof of Main Theorem. We identify each $\varphi \in \mathrm{USCC}(\mathbf{I},\mathbf{I})$ with its graph $\mathrm{Gr}(\varphi) \subset \mathbf{I}\times\mathbf{I}$, and assume the first and second factor of the product $\mathbf{I}\times\mathbf{I}$ to be the domain and the range of φ, respectively. Let $p_1, p_2 : \mathbf{I}\times\mathbf{I} \to \mathbf{I}$ be the projections onto the first and second factor, respectively. Define maps $a, b : \exp(\mathbf{I}^2) \to [0,1]$ by

$$a(\varphi) = \min p_2(\varphi), \qquad b(\varphi) = \max p_2(\varphi).$$

Observe that $\varphi|_{[a(\varphi),b(\varphi)]} = \mathrm{id}$ for every $\varphi \in R(\mathbf{I})$. Moreover, put

$$P = \{\varphi \in \mathrm{USCC}(\mathbf{I},\mathbf{I}) \mid a(\varphi) \neq b(\varphi) \Rightarrow \varphi|_{(a(\varphi),b(\varphi))} = \mathrm{id}\}.$$

Note that $\overline{R}(\mathbf{I}) \subset P$. In fact, P is closed in $\mathrm{USCC}(\mathbf{I},\mathbf{I})$, $R(\mathbf{I}) \subset P$ and $\mathrm{USCC}(\mathbf{I},\mathbf{I})$ is closed in $\exp(\mathbf{I}^2)$.

For $i \in \{0,1\}$, put

$$\mathrm{USCC}^i(\mathbf{I},\mathbf{I}) = \{\varphi \in \mathrm{USCC}(\mathbf{I},\mathbf{I}) \mid i \in \varphi(1-i)\},$$
$$C^i(\mathbf{I},\mathbf{I}) = C(\mathbf{I},\mathbf{I}) \cap \mathrm{USCC}^i(\mathbf{I},\mathbf{I}).$$

For any $\varphi \in \mathrm{USCC}^i(\mathbf{I},\mathbf{I})$ and $\varepsilon > 0$, similarly to Theorem 1.9 in [Fe], we can take a map $f \in C^i(\mathbf{I},\mathbf{I})$ such that $d_{\mathrm{H}}(\varphi, f) < \varepsilon$ and $f(1) = 0$, that is, the subset $C^i(\mathbf{I},\mathbf{I})$ is dense in $\mathrm{USCC}^i(\mathbf{I},\mathbf{I})$. By the same method as in [SU_1, Lemma 2], we can construct a homotopy $G^i : \mathrm{USCC}^i(\mathbf{I},\mathbf{I}) \times [0,1] \to \mathrm{USCC}^i(\mathbf{I},\mathbf{I})$ such that $G^i_0 = \mathrm{id}$ and $G^i_t(\mathrm{USCC}^i(\mathbf{I},\mathbf{I})) \subset C^i(\mathbf{I},\mathbf{I})$ for each $t > 0$.

Each $\varphi \in \mathrm{USCC}([a,b],[c,d])$ is linearly transferred to an element of $\mathrm{USCC}(\mathbf{I},\mathbf{I})$ by the function $\mathrm{T}^{[c,d]}_{[a,b]} : \mathrm{USCC}([a,b],[c,d]) \to \mathrm{USCC}(\mathbf{I},\mathbf{I})$, that is,

$$\mathrm{T}^{[c,d]}_{[a,b]}(\varphi) = \left\{\left(\frac{x-a}{b-a}, \frac{y-c}{d-c}\right) \,\middle|\, (x,y) \in \varphi\right\}.$$

The inverse of $\mathrm{T}^{[c,d]}_{[a,b]}$ is denoted by $\mathrm{T}^{-1}{}^{[c,d]}_{[a,b]} : \mathrm{USCC}(\mathbf{I},\mathbf{I}) \to \mathrm{USCC}([a,b],[c,d])$.

These functions will be used in the following lemma.

LEMMA 1. *There exists a homotopy $F : P\times[0,1] \to P$ such that $F_0 = \mathrm{id}$ and $F_t(P) \subset R(\mathbf{I})$ for $t > 0$ (that is, $R(\mathbf{I})$ is homotopy co-negligible in P).*

Proof. First, we will define a homotopy $H : P\times[0,1] \to P$ such that $H_0 = \mathrm{id}$ and

$$H_t(P) \subset \{\varphi \in P \mid p_2(\varphi) \cap \{0,1\} = \emptyset \text{ or } a(\varphi) = b(\varphi)\}.$$

For each $\varphi \in P$ and each $t \in [0,1]$, define numbers

$$a_t(\varphi) = \left(1 - \frac{t}{2}\right)a(\varphi) + \frac{t}{2}b(\varphi), \qquad b_t(\varphi) = \left(1 - \frac{t}{2}\right)b(\varphi) + \frac{t}{2}a(\varphi)$$

and a retraction $r_{(\varphi,t)} : \mathbf{I} \to \mathbf{I}$ by

$$r_{(\varphi,t)}(y) = \begin{cases} a_t(\varphi) & \text{if } y \in [0, a_t(\varphi)], \\ y & \text{if } y \in [a_t(\varphi), b_t(\varphi)], \\ b_t(\varphi) & \text{if } y \in [b_t(\varphi), 1], \end{cases}$$

for every $y \in \mathbf{I}$. The homotopy $H : P \times [0,1] \to P$ is defined by

$$H_t(\varphi)(x) = r_{(\varphi,t)}(\varphi(x)) \subset \mathbf{I}$$

for every $\varphi \in P$, $t \in [0,1]$ and $x \in \mathbf{I}$. Observe that $H_t(\varphi) = \varphi$ for each $t \in [0,1]$ and each $\varphi \in P$ such that $a(\varphi) = b(\varphi)$.

Next, by using the homotopy G^i, we approximate the multi-valued functions $H_t(\varphi)$ by (single-valued) continuous maps. Let

$$L_t(\varphi) = (\mathrm{T}^{-1}{}^{[a_t(\varphi),b_t(\varphi)]}_{[0,a_t(\varphi)]} \circ G^0_t \circ \mathrm{T}^{[a_t(\varphi),b_t(\varphi)]}_{[0,a_t(\varphi)]})(H_t(\varphi)|_{[0,a_t(\varphi)]}) \subset \mathbf{I}^2,$$

$$R_t(\varphi) = (\mathrm{T}^{-1}{}^{[a_t(\varphi),b_t(\varphi)]}_{[b_t(\varphi),1]} \circ G^1_t \circ \mathrm{T}^{[a_t(\varphi),b_t(\varphi)]}_{[b_t(\varphi),1]})(H_t(\varphi)|_{[b_t(\varphi),1]}) \subset \mathbf{I}^2,$$

for every $t \in (0,1]$ and for every $\varphi \in P$ such that $a(\varphi) \neq b(\varphi)$. Now we define the desired homotopy $F : P \times [0,1] \to P$ as follows:

$$F_t(\varphi) = \begin{cases} \varphi & \text{if } t = 0 \text{ or } a(\varphi) = b(\varphi), \\ L_t(\varphi) \cup \mathrm{id}|_{[a_t(\varphi),b_t(\varphi)]} \cup R_t(\varphi) & \text{otherwise. } \blacksquare \end{cases}$$

We call a closed set A in Y a *Z-set* if any map $f : Q \to Y$ can be approximated by maps $g : Q \to Y \setminus A$. A countable union of Z-sets is called a *Z_σ-set*. To prove the Main Theorem, we use the following characterization of the pseudo-boundary $B(Q) = Q \setminus s$ of Q (cf. [An], [Ch, Lemma 8.1]).

LEMMA 2. *For a subset $M \subset Q$, we have $(Q, M) \approx (Q, B(Q))$ if and only if M is a Z_σ-set in Q and satisfies the following condition*:

(∗) *for any pair (A, B) of compacta in Q such that $B \subset M$ and for any $\varepsilon > 0$, there exists a closed embedding $h : A \to M$ such that $h|_B = \mathrm{id}$ and h is ε-close to* id. ■

Proof of Main Theorem. Since P is closed in $\exp(\mathbf{I}^2)$, it follows from Lemma 1 that $\overline{R}(\mathbf{I}) = P$. Because $R(\mathbf{I})$ is homotopy co-negligible in $\overline{R}(\mathbf{I})$ (Lemma 1) and $R(\mathbf{I}) \approx s$ ([BS]), we can easily verify that $\overline{R}(\mathbf{I})$ is an AR and has the disjoint cells property, hence $\overline{R}(\mathbf{I}) \approx Q$ by Toruńczyk's [To] characterization of Q. For convenience, we identify $\overline{R}(\mathbf{I})$ with Q, and assume $R(\mathbf{I}) \subset Q$. It is easily seen by Lemma 1 that $Q \setminus R(\mathbf{I})$ is a Z_σ-set in Q.

We will prove that $R(\mathbf{I})$ satisfies condition (∗). Let $\alpha : A \to [0,1]$ be the map defined by $\alpha(\varphi) = \frac{1}{3}\min\{\varepsilon, d_{\mathrm{H}}(\varphi, B)\}$. By using Lemma 1, we can define a map $f : A \to Q$ such that $f(A \setminus B) \subset R(\mathbf{I})$, $f|_B = \mathrm{id}$ and $d_{\mathrm{H}}(f(\varphi), \varphi) < \alpha(\varphi)$ for each $\varphi \in A \setminus B$. Since $R(\mathbf{I}) \approx s$ and $A \setminus B$ is completely metrizable, we have a closed embedding $g : A \setminus B \to R(\mathbf{I})$ such that $d_{\mathrm{H}}(g(\varphi), f(\varphi)) < \alpha(\varphi)$ for each $\varphi \in A \setminus B$. We may assume that $g(A)$

does not intersect the subset $R_c(\mathbf{I})$ consisting of all constant maps because $R_c(\mathbf{I}) \approx \mathbf{I}$ is a compact subset of $R(\mathbf{I}) \approx s$, whence it is a Z-set in $R(\mathbf{I})$. Now we define $h : A \setminus B \to Q \setminus R(\mathbf{I})$ as follows:

$$h(\varphi)(x) = \begin{cases} [g(\varphi)(x), \min\{b(\varphi), g(\varphi)(x) + \alpha(\varphi)\}] & \text{if } x = a(\varphi), \\ g(\varphi)(x) & \text{otherwise} \end{cases}$$

(recall that $a(\varphi) = \min \bigcup_{x\in\mathbf{I}} \varphi(x)$ and $b(\varphi) = \max \bigcup_{x\in\mathbf{I}} \varphi(x)$). As is easily observed, h is continuous and injective. For each $\varphi \in A \setminus B$,

$$\begin{aligned} d_{\mathrm{H}}(h(\varphi), \varphi) &\le d_{\mathrm{H}}(h(\varphi), g(\varphi)) d_{\mathrm{H}}(g(\varphi), f(\varphi)) + d_{\mathrm{H}}(f(\varphi), \varphi) \\ &< \alpha(\varphi) + \alpha(\varphi) + \alpha(\varphi) = 3\alpha(\varphi) \le d_{\mathrm{H}}(\varphi, B). \end{aligned}$$

Hence we can extend h to a map $\widetilde{h} : A \to Q$ by $\widetilde{h}|_B = \mathrm{id}$. Since $d_{\mathrm{H}}(\varphi, h(\varphi)) < d_{\mathrm{H}}(\varphi, B)$ for each $\varphi \in A \setminus B$, we see that $\widetilde{h}(A \setminus B) = h(A \setminus B)$ does not meet $h(B)$. Then it follows that $\widetilde{h}$ is injective, whence it is an embedding since A is compact. Thus we have the desired embedding $\widetilde{h}$. By Lemma 2, we have the result. ■

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Institute of Mathematics
University of Tsukuba
Tsukuba 305-8571, Japan
E-mail: uehara@math.tsukuba.ac.jp

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