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A NOTE ON SCHRÖDINGER OPERATORS WITH POLYNOMIAL POTENTIALS

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1. Introduction. In [DHJ] the authors apply methods of harmonic analysis on nilpotent Lie groups to study certain Schrödinger operators. This article is a continuation of that work. Our aim is to investigate Schrödinger operators with nonnegative polynomial potentials on $\mathbb{R}^d$.

Let A be a Schrödinger operator on $\mathbb{R}^d$ which has the form

$$A = -\Delta + P, \tag{1.1}$$

where $P(x) = \sum_{\gamma\le\alpha} a_\gamma x^\gamma$ is a nonnegative nonzero polynomial on $\mathbb{R}^d$, $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_+^d$, $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. Without loss of generality we can assume that $\min_j \alpha_j \ge 2$. Let $\int_0^\infty \lambda\, dE_A(\lambda)$ be the spectral resolution of A. For a bounded function ϕ on $\mathbb{R}_+$ we define the operator $\phi(A)$ by

$$\phi(A) = \int_0^\infty \phi(\lambda)\, dE_A(\lambda).$$

The most important part of this paper is to derive estimates for the integral kernels of the operators $\phi(A)$ and the kernels of the semigroup generated by $-A$. In order to obtain the estimates we use the idea which relates the operator $A = -\Delta + P$ to an operator Π_H, where Π is a unitary representation of a nilpotent Lie group and H is a special left-invariant homogeneous operator on the group.

The estimates we obtain here enable us to prove the following result: For all $\gamma, \gamma' \in \mathbb{Z}_+^d$ the operator

$$D^\gamma A^{-(|\gamma|+|\gamma'|)/2} D^{\gamma'}$$

originally defined on $C_c^\infty(\mathbb{R}^d)$ is a Calderón–Zygmund operator; here $D^\gamma = D_1^{\gamma_1} \ldots D_d^{\gamma_d}$, $D_j = \partial/\partial x_j$. This result was obtained, using different methods, by Zhong in the case where $|\gamma| + |\gamma'| \le 2$ (cf. [Z]).

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Moreover, for every $q > 0$ the operator

$$P^q(x)D^\gamma A^{-q-(|\gamma|+|\gamma'|)/2}D^{\gamma'}$$

can be extended to a bounded operator on $L^p(\mathbb{R}^d)$ for $1 \le p < \infty$ (cf. [Sh]).

In [E] the author considered the Hermite operator

$$\mathcal{H} = -\frac{\partial^2}{\partial x^2} + x^2$$

and for $\kappa \in \mathbb{R}$, $1 < p < \infty$, and $1 < q \le \infty$ defined Triebel–Lizorkin norms $\| \ \|_{\mathcal{H}_p^{\kappa,q}}$ associated with the Hermite expansions by setting

$$\|f\|_{\mathcal{H}_p^{\kappa,q}} = \Big\| \Big(\sum_{\mu\in\mathbb{Z}} (2^{\mu\kappa}|\phi(2^{-\mu}\mathcal{H})f|)^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R})},$$

where $\phi(2^{-\mu}\mathcal{H})f = \sum_{k=0}^\infty \phi(2^{-\mu}(2k+1))\langle f, h_k\rangle h_k$, h_k is the kth orthogonal Hermite function, and ϕ is an appropriate bump function. He proved using Mehler's formula that the definition of the Triebel–Lizorkin space is independent of ϕ. In this paper we show that the result holds in the case of Schrödinger operators with nonnegative polynomial potentials, that is, for $\kappa \in \mathbb{R}$, $0 < p, q < \infty$, and suitable bump functions ϕ_1 and ϕ_2 the norms

$$\|f\|_{A_p^{\kappa,q}(\phi_i)} = \Big\| \Big[\sum_{\mu\in\mathbb{Z}} (2^{\mu\kappa}|\phi_i(2^{-\mu}A)f|)^q \Big]^{1/q} \Big\|_{L^p(\mathbb{R}^d)}, \qquad i = 1, 2,$$

are equivalent (see Section 5).

In a subsequent paper we shall study the Hardy spaces H_A^p associated with $A = -\Delta + P$. We shall present several characterizations of these spaces.

2. A nilpotent Lie algebra and Schrödinger operators. Let G be a homogeneous group, that is, a nilpotent Lie group equipped with a family of dilations δ_t (cf. [FS]), and let $\mathbf{g}$ be the Lie algebra of G. We say that a distribution H on G is a *regular kernel of order* $r \in \mathbb{R}$ if H coincides with a smooth function away from the origin and

$$\langle H, f \circ \delta_t \rangle = t^r \langle H, f \rangle \quad \text{for } f \in C_c^\infty(G),\ t > 0. \tag{2.1}$$

For a given Schrödinger operator A as above we shall build a homogeneous group G, a unitary representation Π of G, and a symmetric kernel H of order 2 such that $\Pi_H = A$. In our construction we shall use ideas from [DHJ] and the results of W. Hebisch [He]. The following theorem proved in [He] plays an essential role in our construction.

THEOREM 2.2. *Let G be a homogeneous Lie group with dilations δ_t, and let Γ be a closed subset of $\mathbf{g}^*$ such that $\mathrm{Ad}^*(G)\Gamma \subset \Gamma$, and $\delta_t^*\Gamma \subset \Gamma$ for every $t > 0$. Then for every $r > 0$ there exists a regular symmetric kernel R*

of order r such that

$$\pi_R^l = 0 \quad \text{for all } l \in \Gamma$$

and the operator $\overline{\pi_R^l}$ is positive definite and injective on its domain for all $l \notin \Gamma$. Here π^l denotes an irreducible unitary representation of G which corresponds to the functional l via the Kirillov correspondence.

Let $V_P = \{x \in \mathbb{R}^d : D_x P \equiv 0\}$, $D_x = \sum_{j=1}^d x_j D_j$. There is no loss of generality in assuming that $V_P = \{(x', 0) : x' \in \mathbb{R}^k\}$, $0 \le k < d$. Therefore, $\mathbb{R}^d = V_P \oplus \mathbb{R}^m = \mathbb{R}^k \oplus \mathbb{R}^m$, $m = d - k$. For $\varepsilon > 0$ we set

$$P_\varepsilon(x) = \begin{cases} P(x) + \varepsilon(x_1^2 + \ldots + x_k^2) & \text{if } V_P \ne \{0\}, \\ P(x) & \text{if } V_P = \{0\}. \end{cases}$$

We define a nilpotent Lie algebra $\mathbf{g}$ as follows. Let $\alpha \in \mathbb{Z}_+^d$. As a vector space, $\mathbf{g}$ has a basis $\{X_1, \ldots, X_d, Y^{[\beta]} : 0 \le \beta \le \alpha\}$. Let $\mathcal{X}$, $\mathcal{Y}$ denote the spans of X_j's and $Y^{[\beta]}$'s respectively. The nontrivial commutators are

$$[X_k, Y^{[\beta]}] = \begin{cases} Y^{[\beta - e_k]} & \text{if } \beta - e_k \ge 0, \\ 0 & \text{otherwise,} \end{cases} \tag{2.3}$$

where e_k is the d-tuple consisting of zeros except for a 1 in the kth position.

For α as above we define

$$\mathcal{P}_\alpha = \Big\{\omega : \omega(x) = \sum_{\beta \le \alpha} c_\beta x^\beta,\ c_\beta \in \mathbb{R}\Big\}. \tag{2.4}$$

For $\omega \in \mathcal{P}_\alpha$ we set $V_\omega = \{x \in \mathbb{R}^d : D_x \omega \equiv 0\}$. Let $C_c^\infty(\mathbb{R}^d/V_\omega)$ denote the smooth functions on $\mathbb{R}^d$ that are invariant under translations by elements of V_ω and compactly supported on any subspace complementary to V_ω. Denote by $\mathbf{g}_\omega(\mathbb{R}^d/V_\omega)$ (respectively $\mathbf{g}_\omega(\mathbb{R}^d)$) for $\omega \in \mathcal{P}_\alpha$ the Lie algebra of operators on $C_c^\infty(\mathbb{R}^d/V_\omega)$ (respectively $C_c^\infty(\mathbb{R}^d)$) generated by the D_j's and multiplication by $i\omega$, denoted by $M_{i\omega}$. Define the mappings $\pi^\omega : \mathbf{g} \to \mathbf{g}_\omega(\mathbb{R}^d/V_\omega)$ and $\Pi^\omega : \mathbf{g} \to \mathbf{g}_\omega(\mathbb{R}^d)$ by

$$\pi^\omega, \Pi^\omega : \begin{cases} X_j \mapsto D_j, \\ Y^{[\alpha]} \mapsto M_{i\omega}, \\ \text{and, inductively, if } Y^{[\beta]} \mapsto M_{i\omega_\beta} \text{ then} \\ [X_j, Y^{[\beta]}] \mapsto M_{i(D_j \omega_\beta)}, \end{cases} \tag{2.5}$$

and extend linearly to $\mathbf{g}$.

With each $\omega \in \mathcal{P}_\alpha$, we associate the linear functional ξ_ω on $\mathbf{g}$ by setting

$$\begin{cases} \langle \xi_\omega, X_j \rangle = 0 & \text{for each } 1 \le j \le d, \\ \langle \xi_\omega, Y^{[\beta]} \rangle = \omega_\beta(0) & \text{if } \pi^\omega(Y^{[\beta]}) = M_{i\omega_\beta}. \end{cases} \tag{2.6}$$

Clearly $\langle \xi_\omega, Y^{[\beta]} \rangle = D^{\alpha-\beta}\omega(0)$. We set $\mathcal{X}_\omega = \{X \in \mathcal{X} : \pi^\omega([X, Y^{[\alpha]}]) = 0\}$.

The following lemma was proved in [DHJ].

LEMMA 2.7. *$\mathcal{X}_\omega + \mathcal{Y}$ is the maximal subalgebra subordinate to the functional ξ_ω, and π^ω is the infinitesimal representation associated with ξ_ω via the Kirillov correspondence. In particular, if $V_\omega \neq \{0\}$, then Π^ω is reducible.*

On the Lie algebra $\mathbf{g}$ let δ_t be the one-parameter group of dilations determined by $\delta_t X_i = tX_i$, $\delta_t Y^{[\alpha]} = t^2 Y^{[\alpha]}$, and, inductively, $\delta_t[X, Y^{[\beta]}] = [\delta_t X, \delta_t Y^{[\beta]}]$. The corresponding dilations δ_t^* on $\mathbf{g}^*$ are given by duality, that is, $\langle \delta_t^* \xi, Z\rangle = \langle \xi, \delta_t Z\rangle$. Let G be the connected simply connected Lie group with Lie algebra $\mathbf{g}$. Throughout this paper we shall identify G with its Lie algebra $\mathbf{g}$ with the Campbell–Hausdorff multiplication (cf. [FS]). Topologically $G = \mathbb{R}^d \times \mathbb{R}^D$, where $D = \dim \mathcal{Y} = (\alpha_1+1)(\alpha_2+1)\ldots(\alpha_d+1)$. We shall use the same symbol π^ω to denote the representation of G that corresponds to the functional ξ_ω via the Kirillov correspondence. Since δ_t is an automorphism of G, $\pi^\omega \circ \delta_t$ is the representation associated with $\delta_t^* \xi_\omega$. Moreover, $\delta_t^* \xi_\omega = \xi_{\omega^t}$, where $\omega^t(x) = t^2 \omega(tx)$.

We choose and fix a *homogeneous norm* on G, that is, a continuous, positive and symmetric function $G \ni g \mapsto |g|$ which is smooth on $G \setminus \{0\}$, homogeneous of degree 1, and vanishes only at the origin.

The *homogeneous dimension* of G is the number Q defined by $d(\delta_t g) = t^Q\,dg$, where dg is a bi-invariant Haar measure on G.

Let $x = (x_j) \in \mathbb{R}^d$ and $X = \sum_{j=1}^d x_j X_j$. It was shown in [DHJ] that if $\Pi^\omega_Y = M_{iV}$ for $\omega \in \mathcal{P}_\alpha$, then $\langle \mathrm{Ad}^*(\exp X)\xi_\omega,\ Y\rangle = V(x)$, and, consequently,

$$\mathrm{Ad}^*(\exp X)\xi_\omega = \xi_{\omega_x}, \quad \text{where} \quad \omega_x(x') = \omega(x+x'), \qquad x, x' \in \mathbb{R}^d. \tag{2.8}$$

Set $\Gamma = \{\xi_\omega : \omega \in \mathcal{P}_\alpha,\ \omega(x) \geq 0 \text{ for all } x \in \mathbb{R}^d\} + \mathcal{Y}^\perp \subset \mathbf{g}^*$, where $\mathcal{Y}^\perp = \{\xi \in \mathbf{g}^* : \langle \xi, Y\rangle = 0 \quad \text{for every} \quad Y \in \mathcal{Y}\}$. One can check using Lemma 1.5 of [DHJ] that Γ satisfies the assumptions of Theorem 2.2. Let

$$W = -X_1^2 - X_2^2 - \ldots - X_d^2 - iY^{[\alpha]}. \tag{2.9}$$

Note that W is a regular symmetric kernel of order 2 and

$$\Pi^\omega_W f(x) = -\Delta f(x) + \omega(x) f(x) \quad \text{for } \omega \in \mathcal{P}_\alpha. \tag{2.10}$$

Theorem 2.2 guarantees that there is a regular symmetric kernel R of order 4 such that $\pi^\xi_R = 0$ for $\xi \in \Gamma$ and π^l_R is positive definite and injective on its domain for all $l \in \mathbf{g}^* \setminus \Gamma$. Set $H = \sqrt{R + W^2}$. We can verify that H is a regular symmetric kernel of order 2 that satisfies the Rockland condition, that is, π_H is injective for every nontrivial irreducible unitary representation π of G. Moreover,

$$\Pi^{P_\varepsilon}_H = \pi^{P_\varepsilon}_H = -\Delta + P_\varepsilon(x), \qquad x \in \mathbb{R}^d.$$

One can check that

$$\lim_{\varepsilon \to 0} \pi^{P_\varepsilon}_H f = \Pi^P_H f = Af \quad \text{for } f \in C_c^\infty(\mathbb{R}^d). \tag{2.11}$$

Applying [G1, Theorem 3.1 and Remark 3.14], we conclude that the following maximal subelliptic estimates hold for H: for every regular kernel U of order r and a positive integer k such that $r \le 2k$ there is a constant C such that

$$\|f * U\|_{L^2} \le C(\|f * H^k\|_{L^2} + \|f\|_{L^2}) \quad \text{for all } f \in C_c^\infty(G). \tag{2.12}$$

Let $\int_0^\infty \lambda\, dE_H(\lambda)$ be the spectral resolution of the essentially self-adjoint positive operator $f \mapsto Hf = f * H$. For a bounded function ϕ on $\mathbb{R}$ we define the operator $\phi(H)f = \int_0^\infty \phi(\lambda)\, dE_H(\lambda) f$. Obviously, by (2.11),

$$\phi(A) = \Pi^P_{\phi(H)} \quad \text{for } \phi \in C_c^\infty((0,\infty)). \tag{2.13}$$

3. Estimates of kernels. Let $\{S_t\}_{t>0}$ be the semigroup of linear operators on $L^2(G)$ generated by $-H$. The homogeneity of H and (2.12) imply that the semigroup has the form

$$S_t f = f * q_t, \quad q_t(g) = t^{-Q/2} q_1(\delta_{t^{-1/2}} g), \quad \text{where } q_t \in C^\infty(G) \cap L^2(G). \tag{3.1}$$

The results of P. Głowacki [G] (see also [D]) assert that for every homogeneous left-invariant (or right-invariant) differential operator ∂ on G and for every nonnegative integer j there are constants C_∂, $C_{j,\partial}$ such that

$$\begin{gathered} |\partial q_t(g)| \le C_\partial t (t^{1/2} + |g|)^{-Q-|\partial|-2}, \\ |\partial H^j q_t(g)| \le C_{j,\partial} (t^{1/2} + |g|)^{-Q-|\partial|-2j}, \end{gathered} \tag{3.2}$$

where $|\partial|$ is the degree of homogeneity of ∂.

Let us denote by $\mathcal{S}_0([0,\infty))$ the subspace of all functions ϕ from the Schwartz class $\mathcal{S}([0,\infty))$ such that

$$\frac{d^k}{d\lambda^k}\phi(0^+) = 0 \quad \text{for } k = 1, 2, \ldots \tag{3.3}$$

The following lemma was proved in [D1].

LEMMA 3.4. *If* $\phi \in \mathcal{S}_0([0,\infty))$, *then* $\phi(H)f = f * \Phi$, *where* $\Phi \in \mathcal{S}(G)$. *Moreover*, *if* $\phi^t(\lambda) = \phi(t\lambda)$, *then*

$$\phi^t(H) f = f * \Phi_t, \quad \textit{where} \quad \Phi_t(g) = t^{-Q/2}\Phi(\delta_{t^{-1/2}} g).$$

From (2.8) and (2.13) we deduce that for every $F \in L^1(G)$ and a polynomial $\omega \in \mathcal{P}_\alpha$ the kernel $F^\omega(x,u)$ of the operator Π_F^ω on $L^2(\mathbb{R}^d)$ is expressed by

$$\begin{aligned} F^\omega(x,u) &= \int_{\mathcal{Y}} F(u-x, y) \exp(i\langle \mathrm{Ad}_x^* \xi_\omega, y\rangle)\, dy \\ &= (\mathcal{F}_{\mathcal{Y}} F)(u-x, \omega(x), \ldots, D^\beta \omega(x), \ldots), \end{aligned} \tag{3.5}$$

where $\mathcal{F}_{\mathcal{Y}} F$ is the Fourier transform of F with respect to $\mathcal{Y}$.

Consequently, by (2.13) and Lemma 3.4, the kernels $Q_\mu(x,u)$ of the operators $Q_\mu = \phi(2^{-\mu}A)$, where $\phi \in \mathcal{S}_0([0,\infty))$, are given by

$$(3.6)\quad Q_\mu(x,u) = 2^{d\mu/2}\Psi(2^{\mu/2}(u-x), 2^{-\mu}P(x), \ldots, 2^{-\mu(|\beta|+2)/2}D^\beta P(x), \ldots),$$

with $\Psi = \mathcal{F}_{\mathcal{Y}}\Phi \in \mathcal{S}(\mathbb{R}^d \times \widehat{\mathcal{Y}})$, $|\beta| = |(\beta_1, \ldots, \beta_d)| = \sum \beta_j$.

For a multi-index $\gamma \in \mathbb{Z}^d$ we set $X^\gamma = X_1^{\gamma_1}X_2^{\gamma_2}\ldots X_d^{\gamma_d}$. Then

$$(3.7)\quad D^\gamma = \frac{\partial^{|\gamma|}}{\partial x^\gamma} = \Pi^P_{X^\gamma}.$$

Let us observe that $\frac{\partial^{|\gamma|}}{\partial x^\gamma}\frac{\partial^{|\gamma'|}}{\partial u^{\gamma'}}Q_\mu(x,u)$ is the kernel corresponding to the operator

$$(3.8)\quad (-1)^{|\gamma'|}D^\gamma\phi(2^{-\mu}A)D^{\gamma'}f = (-1)^{|\gamma'|}\Pi^P_{X^\gamma}\Pi^P_{\Phi_{2^{-\mu}}}\Pi^P_{X^{\gamma'}}f = (-1)^{|\gamma'|}2^{(|\gamma|+|\gamma'|)\mu/2}\Pi^P_{(X^\gamma*\Phi*X^{\gamma'})_{2^{-\mu}}}f$$

for $f \in C_c^\infty(\mathbb{R}^d)$. Therefore

$$(3.9)\quad \frac{\partial^{|\gamma|}}{\partial x^\gamma}\frac{\partial^{|\gamma'|}}{\partial u^{\gamma'}}Q_\mu(x,u) = (-1)^{|\gamma'|}2^{(d+|\gamma|+|\gamma'|)\mu/2} \times \Psi_{(\gamma,\gamma')}(2^{\mu/2}(u-x), 2^{-\mu}P(x), \ldots, 2^{-\mu(|\beta|+2)/2}D^\beta P(x), \ldots),$$

where $\Psi_{(\gamma,\gamma')} = \mathcal{F}_{\mathcal{Y}}(X^\gamma * \Phi * X^{\gamma'})$.

Thus we have proved

PROPOSITION 3.10. *For every $b > 0$ and every $\phi \in \mathcal{S}_0([0,\infty))$ the kernels $Q_\mu(x,u)$ of the operators $Q_\mu = \phi(2^{-\mu}A)$ satisfy*

$$(3.11)\quad |Q_\mu(x,u)| \le C_b 2^{d\mu/2}(1 + 2^{\mu/2}|x-u|)^{-b},$$

$$(3.12)\quad \left|\frac{\partial^{|\gamma|}}{\partial x^\gamma}\frac{\partial^{|\gamma'|}}{\partial u^{\gamma'}}Q_\mu(x,u)\right| \le C_{(b,\gamma,\gamma')}2^{(d+|\gamma|+|\gamma'|)\mu/2}(1 + 2^{\mu/2}|x-u|)^{-b}.$$

Let $K_t(x,u)$ be the kernels of the operators $\int_0^\infty \lambda e^{-t\lambda}\,dE_A(\lambda)$. The following proposition is a simple consequence of (3.1), (3.2), (3.5), and the fact that if $\|g\|$ is a Euclidean norm on G then $\|g\| \le C(1+|g|)^{\varepsilon'}$ for some $\varepsilon' > 0$.

PROPOSITION 3.13. *There exist constants $C > 0$ and $\varepsilon > 0$ such that*

$$(3.14)\quad K_t(x,u) = t^{-(2+d)/2}\Xi(t^{-1/2}(u-x), tP(x), \ldots, t^{(|\beta|+2)/2}D^\beta P(x), \ldots),$$

where

$$(3.15)\quad |\Xi(x,\xi)| \le C(1+|x|)^{-d-2},$$

$$(3.16)\quad |\Xi(x,\xi) - \Xi(x,0)| \le C(1+|x|)^{-d-1}|\xi|^\varepsilon.$$

We denote by $T_t(x,u)$ the kernels of the semigroup generated by $-A$.

PROPOSITION 3.17. *For every $b>0$ there exists a constant $C>0$ such that*

$$\begin{aligned}(3.18)\qquad 0 &\le T_t(x,u)\\ &\le Ct^{-d/2}\exp(-|u-x|^2/(5t))\prod_{\beta\le\alpha}(1+|t^{(|\beta|+2)/2}D^\beta P(x)|)^{-b}.\end{aligned}$$

Proof. On $\mathcal{Y}$ we consider the coordinates $y=(y_\beta)_{\beta\le\alpha}=\sum y_\beta Y^{[\beta]}$. Since $\partial/\partial y_\beta = Y^{[\beta]}+\sum_{|\gamma|<|\beta|}c_{\gamma,\beta}(g)Y^{[\gamma]}$, where $c_{\gamma,\beta}$ is a homogeneous polynomial on G of degree $|\beta|-|\gamma|$ (cf. [FS]), we conclude from (3.2) that

$$\begin{aligned}(3.19)\qquad &\left|\left(\frac{\partial}{\partial y_{\beta^{(1)}}}\right)^{k_1}\cdots\left(\frac{\partial}{\partial y_{\beta^{(n)}}}\right)^{k_n}q_1(g)\right|\\ &\qquad\le C(k_1,\ldots,k_n,\beta^{(1)},\ldots,\beta^{(n)})(1+|g|)^{-Q-2}.\end{aligned}$$

This combined with (3.5) and (3.1) gives

$$\begin{aligned}(3.20)\qquad &|T_t(x,u)|\\ &\le C_bt^{-d/2}(1+t^{-1/2}|u-x|)^{-d-2}\prod_{\beta\le\alpha}(1+|t^{(|\beta|+2)/2}D^\beta P(x)|)^{-b}.\end{aligned}$$

On the other hand, the Feynman–Kac formula implies

$$(3.21)\qquad 0\le T_t(x,u)\le Ct^{-d/2}\exp(-|u-x|^2/(4t)).$$

Thus (3.18) follows from (3.20) and (3.21).

4. Applications. In this section we show some applications of the estimates we derived in Section 3. Some results presented here are known (see the remarks following Theorems 4.4 and 4.5) but we believe that the methods can be used in other investigations.

An operator K defined on a dense set $\mathcal{D}$ of $L^2(\mathbb{R}^d)$ by the formula

$$Kf(x)=\int K(x,u)f(u)\,du,$$

where $K(x,u)$ is a continuous function on $\{(x,u)\in\mathbb{R}^d\times\mathbb{R}^d : x\ne u\}$, is a *Calderón–Zygmund operator* if K can be extended to a bounded operator on $L^2(\mathbb{R}^d)$, that is,

$$(4.1)\qquad \|Kf\|_{L^2(\mathbb{R}^d)}\le C\|f\|_{L^2(\mathbb{R}^d)}\quad\text{for } f\in\mathcal{D},$$

and

$$(4.2)\qquad |K(x,u)|\le C|x-u|^{-d},\quad x\ne u,$$

$$(4.3)\qquad |\nabla_xK(x,u)|+|\nabla_uK(x,u)|\le C|x-u|^{-d-1},\quad x\ne u.$$

The smallest constant C such that (4.1)–(4.3) hold is called the *bound* of the Calderón–Zygmund operator.

THEOREM 4.4. *For every $\gamma, \gamma' \in \mathbb{Z}_+^d$ the operator*

$$R = D^\gamma A^{-(|\gamma|+|\gamma'|)/2} D^{\gamma'}$$

is a Calderón–Zygmund operator with bound that depends only on γ, γ', and the degree of the polynomial P.

REMARK. For $|\gamma| + |\gamma'| \le 2$ Theorem 4.4 was proved by Zhong in [Z].

Proof. Fix γ and γ'. Without loss of generality, by taking α large if necessary, we can assume that $Q > |\gamma| + |\gamma'|$, where Q is the homogeneous dimension of the group G. Therefore the operator $H^{-(|\gamma|+|\gamma'|)/2}$ has a convolution kernel, which is a regular kernel of order $-|\gamma| - |\gamma'|$ on the group G. Let $\zeta \in C_c^\infty(1/2, 2)$ be such that $\sum_{\mu\in\mathbb{Z}} \zeta(2^{-\mu}\lambda) = 1$ for $\lambda > 0$.

Set

$$\Theta_\mu = \int_0^\infty \lambda^{-(|\gamma|+|\gamma'|)/2} \zeta(2^{-\mu}\lambda)\, dE_H(\lambda).$$

Clearly the convolution kernel $\Theta_\mu(g)$ of the operator Θ_μ is given by

$$\Theta_\mu(g) = 2^{-\mu(|\gamma|+|\gamma'|)/2} 2^{\mu Q/2} \Theta_0(\delta_{2^{\mu/2}} g),$$

where $\Theta_0 \in \mathcal{S}(G)$. Thus

$$(X^\gamma * \Theta_\mu * X^{\gamma'})(g) = 2^{\mu Q/2} (X^\gamma * \Theta_0 * X^{\gamma'})(\delta_{2^{\mu/2}} g)$$

and $\int_G X^\gamma * \Theta_0 * X^{\gamma'}\, dg = 0$.

Therefore, by the almost orthogonality principle, $\sum_{\mu\in\mathbb{Z}} \Pi^P_{X^\gamma * \Theta_\mu * X^{\gamma'}} f$ converges in the norm $L^2(\mathbb{R}^d)$ for every $f \in C_c^\infty(\mathbb{R}^d)$. Moreover, since the spectrum of the operator A is strictly positive, the operator $A^{-(|\gamma|+|\gamma'|)/2}$ is bounded on $L^2(\mathbb{R}^d)$, and the series converges in $\mathcal{S}'(\mathbb{R}^d)$ to Rf. Thus R is bounded on $L^2(\mathbb{R}^d)$ and

$$Rf = \sum_{\mu\in\mathbb{Z}} \Pi^P_{X^\gamma * \Theta_\mu * X^{\gamma'}} f.$$

It follows from (3.5) that the kernel $R_\mu(x, u)$ of $\Pi^P_{X^\gamma * \Theta_\mu * X^{\gamma'}}$ is

$$R_\mu(x, u) = 2^{\mu d/2} \Xi(2^{\mu/2}(u - x), 2^{-\mu} P(x), \ldots, 2^{-\mu(|\beta|+2)/2} D^\beta P(x), \ldots),$$

where $\Xi \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^D)$. It is now not difficult to check that the kernel of R satisfies (4.2) and (4.3).

THEOREM 4.5. *For every $q > 0$ and every $\gamma, \gamma' \in \mathbb{Z}_+^d$ the operator*

$$R = P^q(x) D^\gamma A^{-k} D^{\gamma'}, \tag{4.6}$$

where $k = q + (|\gamma| + |\gamma'|)/2$, can be extended to a bounded operator on L^p for $1 \le p < \infty$, that is,

$$\|Rf\|_{L^p} \le C_p \|f\|_{L^p} \quad \text{for } f \in C_c^\infty(\mathbb{R}^d).$$

REMARK. In the case where P satisfies a reverse Hölder inequality, and $k = \frac{1}{2}, 1$, the boundedness of R on L^p spaces (for a certain range of p) was shown by Shen (see [Sh] for details).

Proof (of Theorem 4.5). Let $R(x,u)$ be the integral kernel of the operator R. It suffices to show that there exists a constant C such that

$$\sup_{x\in\mathbb{R}^d}\int |R(x,u)|\,du + \sup_{u\in\mathbb{R}^d}\int |R(x,u)|\,dx \le C. \tag{4.7}$$

Let ζ be as in the proof of Theorem 4.4. Since the infimum of the spectrum of A is strictly positive, there exists a constant B_1 such that

$$R(x,u) = \sum_{\mu > B_1} R_\mu(x,u), \tag{4.8}$$

where $R_\mu(x,u)$ is the kernel of the operator

$$R_\mu = P^q(x)D^\gamma\Big(\int_0^\infty \lambda^{-k}\zeta(2^{-\mu}\lambda)\,dE_A(\lambda)\Big)D^{\gamma'}.$$

As in the proof of Theorem 4.4, we obtain

$$\begin{aligned}R_\mu(x,u) &= 2^{-\mu q}2^{d\mu/2}P^q(x)\\ &\quad\times \Xi(2^{\mu/2}(u-x), 2^{-\mu}P(x), \ldots, 2^{-\mu(|\beta|+2)/2}D^\beta P(x), \ldots),\end{aligned}$$

where $\Xi \in \mathcal{S}(\mathbb{R}^d\times\mathbb{R}^D)$.

It is easy to check that $\sum_{\mu>B_1}\int |R_\mu(x,u)|\,du \le C$ with C independent of x.

It remains to prove that $\sup_{u\in\mathbb{R}^d}\int |R(x,u)|\,dx \le C$.

For a positive integer m we set

$$\begin{aligned}R_\mu^{[m]}(x,u) &= 2^{d\mu/2}2^{-\mu q}P^q(x)\chi_{[-m,m]}(2^{\mu/2}|u-x|)\\ &\quad\times\prod_{\beta\le\alpha}\chi_{[-m,m]}(2^{-\mu(|\beta|+2)/2}D^\beta P(x)).\end{aligned}$$

We see that

$$|R_\mu(x,u)| \le \sum_{m\ge 2} b_m R_\mu^{[m]}(x,u), \tag{4.9}$$

where $b_m \le C_l m^{-l}$ for every $l > 0$.

For fixed $u\in\mathbb{R}^d$ let n be an integer such that

$$2^{n/2} \le \sum_{\beta\le\alpha}|D^\beta P(u)|^{1/(|\beta|+2)} < 2^{(n+1)/2}. \tag{4.10}$$

Since P is a nonzero polynomial, there exists a constant B_2 such that $n > B_2$ for every u. If $R_\mu^{[m]}(x,u) \ne 0$ then $|x-u| \le 2^{-\mu/2}m$ and $|D^\beta P(x)| \le 2^{\mu(|\beta|+2)/2}m$ for every $\beta\le\alpha$. Applying the Taylor formula we obtain

$$|D^\beta P(u)| \le C2^{\mu(|\beta|+2)/2}m^{|\alpha|+1}$$

for every $\beta \le \alpha$. It follows from (4.10) that there exists $\beta \le \alpha$ such that $2^{(|\beta|+2)n/2} \le C|D^\beta P(u)|$. Therefore

$$2^{n/2} \le C2^{\mu/2} m^{|\alpha|+1}. \tag{4.11}$$

On the other hand, (4.10) implies

$$|P(x)| \le C \sum_{\beta\le\alpha} |D^\beta P(u)| \cdot |x-u|^{|\beta|} \le C_0 m^{|\alpha|} 2^n \sum_{\beta\le\alpha} 2^{|\beta|(n-\mu)/2}. \tag{4.12}$$

Finally, by (4.9), (4.11) and (4.12), we obtain

$$\begin{aligned}\int |R(x,u)|\,dx &\le \sum_{\mu>B_1} \int |R_\mu(x,u)|\,dx \\ &\le \sum_{\mu>B_1}\sum_{m\ge2} b_m \int |R_\mu^{[m]}(x,u)|\,dx \\ &\le \sum_{m\ge 2}\ \sum_{\mu>n-C\log_2 m} C_1 b_m 2^{d\mu/2} 2^{-\mu q} m^{cq} 2^{nq} \\ &\quad\times \sum_{\beta\le\alpha} 2^{q|\beta|(n-\mu)/2} \int \chi_{B(0,m)}(2^{\mu/2}(u-x))\,dx \le C'.\end{aligned}$$

5. Triebel–Lizorkin spaces associated with A. For a smooth function ϕ such that

$$\operatorname{supp}\phi \subset [1/2,2], \quad |\phi(\lambda)| \ge c > 0 \quad \text{for } \lambda\in[3/4,7/4], \tag{5.1}$$

and for $\kappa\in\mathbb{R}$, $0<p,q<\infty$, we define a Triebel–Lizorkin norm $\|\ \|_{A_p^{\kappa,q}(\phi)}$ associated with $A=-\Delta+P$ by

$$\|f\|_{A_p^{\kappa,q}(\phi)} = \Big\| \Big[\sum_{\mu\in\mathbb{Z}} (2^{\mu\kappa}|Q_\mu f|)^q\Big]^{1/q}\Big\|_{L^p(\mathbb{R}^d)}, \tag{5.2}$$

where

$$Q_\mu f = \phi(2^{-\mu}A) f = \int_0^\infty \phi(2^{-\mu}\lambda)\,dE_A(\lambda) f. \tag{5.3}$$

Observe that if $P\equiv 0$ then the Triebel–Lizorkin norm $\|\ \|_{(-\Delta)_p^{\kappa,q}}$ is equivalent to the classical homogeneous norm $\|\ \|_{F_p^{2\kappa,q}}$.

In the present section we shall show that different ϕ's give equivalent Triebel–Lizorkin norms, that is,

THEOREM 5.4. *Let $\kappa\in\mathbb{R}$, $0<p<\infty$, $0<q<\infty$. If $\phi^{(1)}$ and $\phi^{(2)}$ are two C^∞ functions satisfying* (5.1), *then there is a constant $C>0$ such that*

$$C^{-1}\|f\|_{A_p^{\kappa,q}(\phi^{(1)})} \le \|f\|_{A_p^{\kappa,q}(\phi^{(2)})} \le C\|f\|_{A_p^{\kappa,q}(\phi^{(1)})}. \tag{5.5}$$

The proof uses ideas of Peetre [P] (see also Epperson [E]).

For $a > 0$ and a fixed C^∞ function ϕ for which (5.1) holds define the maximal function A_μ by

$$A_\mu f(x) = \sup_{y\in\mathbb{R}^d} \frac{|Q_\mu f(y)|}{(1+2^{\mu/2}|x-y|)^a}, \tag{5.6}$$

where $Q_\mu = \phi(2^{-\mu}A)$. We also consider

$$B_\mu f(x) = \sup_{y\in\mathbb{R}^d} \frac{|\nabla Q_\mu f(y)|}{(1+2^{\mu/2}|x-y|)^a}. \tag{5.7}$$

LEMMA 5.8. *For every $a > 0$ there is a constant $C > 0$ such that for all $\mu \in \mathbb{Z}$, $x \in \mathbb{R}^d$, $f \in L^2(\mathbb{R}^d)$,*

$$B_\mu f(x) \le C 2^{\mu/2} A_\mu f(x). \tag{5.9}$$

Proof. The proof is essentially the same as that of Lemma 2.1 in [E]. Let ψ be a C_c^∞ function on $\mathbb{R}$ such that $\operatorname{supp}\psi \subset [1/2, 2]$, $\sum_{j\in\mathbb{Z}} \psi(2^j\lambda)\phi(2^j\lambda) = 1$. Set $\zeta(\lambda) = \sum_{j=-1}^{1} \psi(2^j\lambda)\phi(2^j\lambda)$ Obviously, (3.11) and (3.12) hold for the kernels $M_{2^{-\mu}}(x,s)$ of the operators $\zeta(2^{-\mu}A)$. Therefore

$$\begin{aligned}\left|\frac{\partial}{\partial x_j} Q_\mu f(x)\right| &= \left|\frac{\partial}{\partial x_j}\zeta(2^{-\mu}A)Q_\mu f(x)\right| = \left|\int \frac{\partial}{\partial x_j} M_{2^{-\mu}}(x,s) Q_\mu f(s)\,ds\right| \\ &\le C_b \int 2^{(d+1)\mu/2}(1+2^{\mu/2}|x-s|)^{-b} \\ &\qquad \times (1+2^{\mu/2}|s-u|)^a (1+2^{\mu/2}|s-u|)^{-a} |Q_\mu f(s)|\,ds \\ &\le C 2^{\mu/2} A_\mu f(u)(1+2^{\mu/2}|x-u|)^a,\end{aligned}$$

which gives (5.9).

LEMMA 5.10. *There is a constant C such that for all $\mu \in \mathbb{Z}$, $x \in \mathbb{R}^d$, $f \in L^2(\mathbb{R}^d)$,*

$$A_\mu f(x) \le C[\mathcal{M}(|Q_\mu f|)^r(x)]^{1/r}, \tag{5.11}$$

where $a = d/r$ and $\mathcal{M}$ is the Hardy–Littlewood maximal operator.

Proof (cf. [P]). For $\delta_1 > 0$ set $\delta = 2^{-\mu/2}\delta_1$. By the mean value theorem there is a constant $C > 0$ such that for every $\delta > 0$,

$$\begin{aligned}|Q_\mu f(x-u)| &\le C\delta^{-d/r}\Big(\int_{|x-u-y|<\delta} |Q_\mu f(y)|^r\,dy\Big)^{1/r} \\ &\qquad + C\delta \sup_{|x-u-y|<\delta} |\nabla Q_\mu f(y)| \\ &\le C\delta^{-d/r}(\delta+|u|)^{d/r}[\mathcal{M}(|Q_\mu f|^r)(x)]^{1/r} \\ &\qquad + C\delta B_\mu f(x)(1+2^{\mu/2}\delta+2^{\mu/2}|u|)^a.\end{aligned}$$

Applying (5.9), we obtain

$$\begin{aligned}|Q_\mu f(x-u)| &\le C2^{\mu d/(2r)}\delta_1^{-d/r}(2^{-\mu/2}\delta_1+|u|)^{d/r}[\mathcal{M}(|Q_\mu f|^r)(x)]^{1/r}\\ &\quad + C\delta_1(A_\mu f)(x)(1+\delta_1+2^{\mu/2}|u|)^a\\ &\le C\delta_1^{-d/r}(1+\delta_1+2^{\mu/2}|u|)^{d/r}[\mathcal{M}(|Q_\mu f|^r)(x))]^{1/r}\\ &\quad + C\delta_1(A_\mu f)(x)(1+\delta_1+2^{\mu/2}|u|)^a.\end{aligned}$$

By the above, there exists a constant $C>0$ such that for every $0<\delta_1<1$,

$$|Q_\mu f(x-u)|(1+2^{\mu/2}|u|)^{-a} \le C\delta_1^{-d/r}[\mathcal{M}(|Q_\mu f|^r)(x)]^{1/r} + C\delta_1 A_\mu f(x).$$

Taking δ_1 small enough, we get (5.11).

Proof of Theorem 5.4. Let $0<r<\min\{p,q\}$ and $a=d/r$, and let $\psi^{(2)}$ be a smooth function satisfying (5.1) such that

$$\sum_{\mu\in\mathbb{Z}}\psi^{(2)}(2^{-\mu}\lambda)\phi^{(2)}(2^{-\mu}\lambda)=1 \quad \text{for } \lambda>0. \tag{5.12}$$

If $R_\nu^{(2)}=\psi^{(2)}(2^{-\nu}A)$, then

$$Q_\mu^{(1)}=\phi^{(1)}(2^{-\mu}A)=\sum_{\nu=\mu-1}^{\mu+1}Q_\mu^{(1)}R_\nu^{(2)}Q_\nu^{(2)}. \tag{5.13}$$

By Proposition 3.10,

$$\begin{aligned}|Q_\mu^{(1)}f(x)| &\le C_b\sum_{\nu=\mu-1}^{\mu+1}\int_{\mathbb{R}^d}2^{d\nu/2}(1+2^{\nu/2}|x-y|)^{-b}|Q_\nu^{(2)}f(y)|\,dy\\ &\le C_b\sum_{\nu=\mu-1}^{\mu+1}\int_{\mathbb{R}^d}2^{d\nu/2}(1+2^{\nu/2}|x-y|)^{a-b}A_\nu^{(2)}f(x)\,dy\\ &\le C_b\sum_{\nu=\mu-1}^{\mu+1}A_\nu^{(2)}f(x).\end{aligned}$$

From Lemma 5.10, we conclude

$$|Q_\mu^{(1)}f(x)|\le C\sum_{\nu=\mu-1}^{\mu+1}[\mathcal{M}(|Q_\nu^{(2)}f|^r)(x)]^{1/r}.$$

Finally, using the Fefferman–Stein vector-valued maximal inequality [FeS], we have

$$\|f\|_{A_p^{\kappa,q}(\phi^{(1)})}\le C\Big\|\Big(\sum_{\mu=-\infty}^{\infty}(2^{\mu\kappa}[\mathcal{M}(|Q_\mu^{(2)}f|^r)(x)]^{1/r})^q\Big)^{1/q}\Big\|_{L^p}$$

$$\leq C\Big\|\Big(\sum_{\mu=-\infty}^{\infty}\big(2^{\mu\kappa r}[|Q_\mu^{(2)}f|^r(x)]\big)^{q/r}\Big)^{r/q}\Big\|_{L^{p/r}}^{1/r}$$
$$\leq C\|f\|_{A_p^{\kappa,q}(\phi^{(2)})}.$$

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