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m-REDUCTION OF ORDINARY DIFFERENTIAL EQUATIONS

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I. Introduction. Let (F, D) be a differential field. We consider linear differential equations

$$Ly = a_n D^n y + \ldots + a_0 y = 0,$$

where $a_0, \ldots, a_n \in F$, and the solution y is in F or in some extension E of F. We introduce the concept of m-reducibility, i.e. a reduction process to equations of lower order m. This reduction is a generalization of the Liouville property, which is included here in the case $m = 2$ (if inhomogeneous equations are admitted, the equation is Liouville if and only if it is 1-reducible).

Connecting m-reducibility with properties of the Galois group of the equation, we show that the generic equation of order $n \geq 2$ with Galois group $\mathrm{GL}(n, \boldsymbol{C})$ is *not* $(n-1)$-reducible. We call an equation $Ly = 0$ simple if its Galois group G is simple, i.e. if it has no proper infinite normal subgroups. For simple equations we give a lower bound for m-reducibility:

$$m \geq [\dim G]^{1/2}.$$

Combining this with the inverse Galois theorem for the differential field $(\mathbb{C}(z), d/dz)$, we get the existence of simple Fuchsian equations (with polynomial coefficients) which are not m-reducible for any

$$m < [\dim G]^{1/2}. \tag{0}$$

For instance, for the simple group $\mathrm{SL}(n, \mathbb{C})$, $n \geq 2$, there exists a Fuchsian equation $Ly=0$ of order $\operatorname{ord} L = n$ which is not $(n-1)$-reducible. We obtain this result by just checking (0):

$$n - 1 < [\dim \mathrm{SL}(n, \mathbb{C})]^{1/2} = (n^2 - 1)^{1/2} \quad \text{for } n \geq 2.$$

By isomorphy these results are also true for s-equations

$$Ly = p_n(s) D^n y + \ldots + p_0(s) y = 0$$

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in the field of Mikusiński's operators, where $s = \frac{1}{\{1\}}$ $(= \frac{d\cdot}{dt})$, $Dy = \{-ty(t)\}$ and $p_0(s), \ldots, p_n(s)$ are polynomials in s.

M. F. Singer [1]–[3] solved the problem of m-reduction for $m = 2$ and $m = n - 1$. Here we give an independent, simple way to prove the existence of irreducible equations, relying only upon dimensional considerations.

We are using the language of differential algebra and of linear, algebraic groups:

A *derivation* of a ring A is an additive mapping $a \to Da$ of A into itself satisfying

$$D(a \cdot b) = Da \cdot b + a \cdot Db.$$

A *differential field* (F, D) is a commutative field F together with a derivation D. In any differential field (F, D) the elements c with $Dc = 0$ form a subfield $\boldsymbol{C}$, called the *field of constants* (see Kaplansky [1]). We assume—once for all—that the characteristic of the field F is 0, and that the subfield of constants $\boldsymbol{C}$ is algebraically closed.

Let (F, D) be a differential field. We consider monic, linear, differential equations

$$Ly = D^n y + a_{n-1} D^{n-1} y + \ldots + a_0 y = 0, \tag{1}$$

where $a_0, \ldots, a_{n-1} \in F$ and the solution y is in F or in some extension E of F.

There always exists a (minimal, unique) extension E of F where $Ly = 0$ has a full system $u_1, \ldots, u_n$ of linearly independent solutions; it is called the *Picard–Vessiot extension* of F and denoted by

$$\mathrm{PV}\, F = \mathrm{PV}\, F(Ly = 0);$$

for its existence and uniqueness, see Magid [1].

We have

(a) $\mathrm{PV}\, F(Ly = 0) = F\langle u_1, \ldots, u_n \rangle$, where $u_1, \ldots, u_n$ is a full system of linearly independent (over the constants) solutions of $Ly=0$ and $\langle u_1, \ldots, u_n \rangle$ means that we adjoin to F the variables u_j and $D^m u_j$ for all $j = 1, \ldots, n$ and $m \geq 1$, and form polynomials and rational functions in those variables with coefficients from F.

(b) $\mathrm{PV}\, F(Ly = 0)$ has the same field of constants as F.

Looking closely at the existence proof of Magid [1], we see that we can prove a little more:

THEOREM 1. *Let* $(F, D) \overset{\pi}{\leftrightarrow} (\widetilde{F}, \widetilde{D})$ *be two isomorphic differential fields with subfields of constants* $\boldsymbol{C}$ *and* $\widetilde{\boldsymbol{C}}$ *respectively* (*it follows that* $\boldsymbol{C} \overset{\pi}{\leftrightarrow} \widetilde{\boldsymbol{C}}$). *Let L be given by* (1) *and consider its isomorphic image*

$$\pi L y = \widetilde{L} y = \widetilde{D}^n y + \widetilde{a}_{n-1} \widetilde{D}^{n-1} y + \ldots + \widetilde{a}_0 y.$$

Then π extends to a differential isomorphism $\overline{\pi}$ of the Picard–Vessiot extensions

$$\mathrm{PV}\, F(Ly=0) \overset{\overline{\pi}}{\leftrightarrow} \mathrm{PV}\, \widetilde{F}(\widetilde{L}y=0).$$

It follows that the Galois groups G and $\widetilde{G}$ of those extensions are isomorphic:

$$\widetilde{G} = \overline{\pi} \circ G \circ \overline{\pi}^{-1}.$$

Proof. We use the construction of $\mathrm{PV}\, F(Ly=0)$ with the help of the universal solution algebra of L,

$$F[y_{ij} \mid 0 \le i \le n-1,\ 1 \le j \le n][w^{-1}]/P,$$

where $w = \det(y_{ij})$ and P is a maximal differential ideal (see Magid [1]).

Set

$$S = F[y_{ij} \mid 0 \le i \le n-1,\ 1 \le j \le n][w^{-1}], \quad w = \det(y_{ij}),$$

$$\widetilde{S} = \widetilde{F}[\xi_{ij} \mid 0 \le i \le n-1,\ 1 \le j \le n][\mathfrak{w}^{-1}], \quad \mathfrak{w} = \det(\xi_{ij}),$$

$$D(y_{ij}) = y_{i+1,j}, \quad 0 \le i < n-1, \quad \widetilde{D}(\xi_{i,j}) = \xi_{i+1,j},$$

$$Dy_{n-1,j} = -\sum_{i=o}^{n-1} a_i y_{ij}, \quad \widetilde{D}\xi_{n-1,j} = -\sum_{i=0}^{n-1} \widetilde{a}_i \xi_{ij}.$$

It is obvious how to extend the differential isomorphism $\pi : F \leftrightarrow \widetilde{F}$ to a differential isomorphism $\widetilde{\pi} : S \leftrightarrow \widetilde{S}$: we just set $\widetilde{\pi} : y_{ij} \leftrightarrow \xi_{ij}$. Now, if P is a maximal differential ideal of S, it is also prime (Magid [1]), hence $\widetilde{\pi}P = \widetilde{P}$ is also maximal and prime, and $\widetilde{\pi}$ induces a differential isomorphism $\overline{\pi} : S/P \leftrightarrow \widetilde{S}/\widetilde{P}$, which extends to the quotient fields:

$$\begin{tikzcd}
\overline{\pi} : Q(S/P) \arrow[r, leftrightarrow] \arrow[d, equal] & Q(\widetilde{S}/\widetilde{P}) \arrow[d, equal] \\
\overline{\pi} : \mathrm{PV}\, F(Ly=0) \arrow[r, leftrightarrow] & \mathrm{PV}\, \widetilde{F}(\widetilde{L}y=0)
\end{tikzcd}$$

Since Galois groups consist of automorphisms the last statement of the theorem is obvious.

Example 1. Let $\mathbb{C}(z)$ be the field of rational functions in the complex variable $z \in \mathbb{C}$. Then $(\mathbb{C}(z), d/dz)$ is a differential field with $\boldsymbol{C} = \mathbb{C}$. Let $\mathbb{C}(s)$ denote the field of rational functions in the (Mikusiński) operator $s = \frac{1}{\{1\}}$ (see Mikusiński [1]). Defining $D\{f(t)\} = \{-tf(t)\}$ for functions $\{f(t)\}$, and extending D by the quotient rule, we see that $(\mathbb{C}(s), D)$ and $(\mathfrak{M}, D)$ are differential fields; here $\mathfrak{M}$ denotes the Mikusiński field (see Mikusiński [1]). We have (Wloka [1])

$$(\mathbb{C}(z), d/dz) \overset{\pi}{\leftrightarrow} (\mathbb{C}(s), D = d/ds),$$

where the isomorphism π is given by $\pi = \mathrm{id}$ on $\mathbb{C}$ and $\pi(z) = s$. Both fields of constants are $\mathbb{C}$.

The *Galois group* $G(E \mid F)$ of an extension field $E \supseteq F$ consists of all differential automorphisms of E leaving the elements of F fixed. If $E = \mathrm{PV}\, F(Ly = 0)$ is a Picard–Vessiot extension, then the elements $g \in G(E \mid F)$ are $n \times n$ matrices, $n = \mathrm{ord}\, L$, with elements from $\boldsymbol{C}$, the field of constants. G is an algebraic matrix group in the Zariski topology: $G(\mathrm{PV}\, F \mid F) \subseteq \mathrm{GL}(n, \boldsymbol{C})$.

We define the differential field $K \supseteq F$ to be *normal* over F if any element in K but not in F can be moved by a differential automorphism of $G(K \mid F)$.

In several places we shall need the "Fundamental Theorem of differential Galois theory for Picard–Vessiot extensions"; for a proof see Magid [1].

Theorem 2. *There is a lattice inverting bijective correspondence between*

$$\{E \supseteq K \supseteq F \mid K \text{ is an intermediate differential field}\}$$

and

$$\{(e) \subseteq H \subseteq G(E \mid F) \mid H \text{ is a Zariski closed subgroup}\}$$

given by

$$K \to G(E \mid K) \quad \text{and} \quad H \to E^H,$$

where $G(E \mid K)$ is the group of all automorphisms of E leaving the elements of K fixed, and E^H denotes the field of all elements of E which remain fixed under the action of all $h \in H$.

An intermediate field K is normal over F if and only if the subgruop $H = G(E \mid K)$ is normal in $G(E \mid F)$; if it is, then $K = E^H$ and

$$G(K \mid F) = G(E^H \mid F) = G(E \mid F)/G(E \mid K).$$

Also, K is normal over F if and only if it is a Picard–Vessiot extension of F.

We illustrate the situation with the diagram

$$\begin{array}{ccccc} \{e\} & \subseteq & H & \subseteq & G \\ | & & | & & | \\ E & \supseteq & K & \supseteq & F \end{array}$$

Proposition 3. *Let K be an intermediate differential field of a Picard–Vessiot extension E,*

$$F \subseteq K \subseteq E = \mathrm{PV}\, F.$$

If $\sigma K \subseteq K$ for all $\sigma \in G(E \mid F)$, then K is normal over F.

Proof. Let $x \in K$ but $x \notin F$. Since $x \in E$ and E is normal (Theorem 2) there exists a $\sigma \in G(E \mid F)$ which moves x, i.e. $\sigma(x) \neq x$, and the condition $\sigma K \subseteq K$ asserts that the restriction $\sigma|_K$ is an automorphism of K over F.

II. m-Reduction. Let $\overline{E} = \overline{\mathrm{PV}}\, F$ be some PV-extension of F. We define an *m-reduction chain*, $m = 0, 1, 2, \ldots$, as a chain of intermediate fields

$$F = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_l = \overline{E} \tag{2a}$$

such that each F_{i+1} is either a finite algebraic extension of F_i (in which case we put $m = 0$), or we get F_{i+1} from F_i by adjoining some solutions of a homogeneous differential equation of order $\leq m$ with coefficients in F_i.

Definition 1. An equation $Ly = 0$ as in (1) is said to be *m-reducible* if there exists an m-reduction chain (2a) such that the PV-extension of F associated with $Ly = 0$ lies in $\overline{E}$:

$$E = \mathrm{PV}\, F(Ly = 0) \subseteq \overline{E}. \tag{2b}$$

This property is obviously hereditary:

Proposition 4. *Let*

$$\mathrm{PV}\, F(L_1 y = 0) \subseteq \mathrm{PV}\, F(L_2 y = 0)$$

be two PV-*extensions of* F. *If* $\mathrm{PV}\, F(L_2 y = 0)$ *is* m-*reducible*, *then so is* $\mathrm{PV}\, F(L_1 y = 0)$.

It is obvious how to define the m-reducibility of some solutions $y_1, \ldots, y_r$ of (1):

Definition 1′. The solutions $y_1, \ldots, y_r$ of the equation (1) are said to be *m-reducible* if there exists an m-reduction chain (2a) such that

$$\{y_1, \ldots, y_r\} \subset \overline{E}. \tag{2b$'$}$$

Here are some special cases of m-reductions: 0-chains are finite algebraic extensions $\overline{E}|F$. For $m = 1$ we get the so-called "special Liouville extensions", where as building blocks are allowed finite algebraic extensions and adjoining exponentials of integrals.

Since integrals $\int a$ satisfy a second order homogeneous equation

$$D^2 y - \frac{Da}{a} Dy = 0$$

(in general they do not satisfy a first order *homogeneous* equation!), we see that general Liouville extensions are a special case of 2-reduction chains.

Remark 1. Taking inhomogeneous differential equations instead of homogeneous ones, we get different definitions only for $m = 1$ but not for $m \geq 2$.

m-reducibility is invariant under isomorphism:

Proposition 5. *Suppose that the equation* (1) *is* m-*reducible*, *and let* $\pi : F \leftrightarrow \widetilde{F}$ *be a differential isomorphism. Then the isomorphic equation* $\pi L y = \widetilde{L} y = 0$ (see Theorem 1) *is also* m-*reducible.*

Proof. Take $\overline{E} = \overline{\mathrm{PV}}\, F$ from the m-chain (2a). Using Theorem 1 we extend the isomorphism $\pi : F \leftrightarrow \widetilde{F}$ to $\overline{\pi} : \overline{E} \leftrightarrow \overline{\pi}\overline{E}$ and obtain isomorphically (all other $\updownarrow$ are restrictions)

$$\begin{array}{ccccccc} F \subseteq & F_1 & \subseteq \ldots \subseteq & \overline{E} & \supseteq \mathrm{PV}\, F(Ly=0) \\ \updownarrow{\scriptstyle \pi} & \updownarrow{\scriptstyle \overline{\pi}} & & \updownarrow{\scriptstyle \overline{\pi}} & \updownarrow{\scriptstyle \overline{\pi}} \\ \widetilde{F} \subseteq & \overline{\pi}F_1 & \subseteq \ldots \subseteq & \overline{\pi}\overline{E} & \supseteq \mathrm{PV}\, \widetilde{F}(\widetilde{L}y=0) \end{array}$$

EXAMPLE 2. Applying Proposition 5 to the equations

$$D^2y + sy = 0, \quad D^2y + s^2y = 0,$$

and

(B) $$s^2D^2y + sDy + (s^2 - n^2)y = 0, \quad \text{where } n - 1/2 \notin \mathbb{Z},$$

we see that they are not Liouville over $\mathbb{C}(s)$ in the Mikusiński field, because the isomorphic equations in the complex domain $z \in \mathbb{C}$:

$$y'' + zy = 0, \quad y'' + z^2y = 0,$$

and

$$z^2y'' + zy' + (z^2 - n^2)y = 0 \quad \text{for } n - 1/2 \notin \mathbb{Z},$$

are not Liouville (see Kaplansky [1] and Skórnik–Wloka [1]).

REMARK 2. Equation (B) is the operator form of an equation (which?) which has nothing to do with Bessel functions.

Sometimes it is advantageous to simplify the Galois group of the equation to a subgroup of the special linear group

$$\mathrm{SL}(n, \boldsymbol{C}) = \{g \in \mathrm{GL}(n, \boldsymbol{C}) \mid \det g = 1\}.$$

Let us substitute $y = z \cdot \exp \frac{-1}{n} \int a_{n-1}$ into (1). After cancelling the factor $\exp \frac{-1}{n} \int a_{n-1}$, we obtain an equation for z (in F!)

$$\widehat{L}z = D^nz + \widehat{a}_{n-2}D^{n-2}z + \ldots + \widehat{a}_0z = 0, \quad \widehat{a}_0, \ldots, \widehat{a}_{n-2} \in F,$$

without second term: $\widehat{a}_{n-1} = 0$. Computing the Wrońskian W for $\widehat{L}z = 0$ we have

$$DW = -\widehat{a}_{n-1}W = 0,$$

which implies $W = \mathrm{const}$, and we see that the Galois group of $\widehat{L}z = 0$ consists only of unimodular matrices, i.e. it is a subgroup of $\mathrm{SL}(n, \boldsymbol{C})$. Now we have

PROPOSITION 6. *For $m \ge 1$, the m-reducibility of* (1) *is equivalent to the m-reducibility of $\widehat{L}z = 0$.*

Proof. Let (1) be m-reducible, i.e. we have

$$F \subseteq F_1 \subseteq \ldots \subseteq F_{l-1} \subseteq \overline{E} = \mathrm{PV}\, F(\overline{L}y = 0) \supseteq \mathrm{PV}\, F(Ly = 0).$$

Let us add to F_{l-1}, $\overline{E}$ and $\mathrm{PV}\, F(Ly = 0)$ the element $\exp \frac{-1}{n} \int a_{n-1}$, which is a solution of the first order ($m \geq 1$!) equation in F,

$$Dy + \frac{a_{n-1}}{n} y = 0, \qquad a_{n-1} \in F.$$

We obtain the m-chain

$$\begin{aligned} F \subseteq F_1 \subseteq \ldots \subseteq F_{l-1} \subseteq F_l &= \left\langle F_{l-1}, \exp \frac{-1}{n} \int a_{n-1} \right\rangle \\ &\subseteq \mathrm{PV}\, F\left(\overline{L}y = 0,\ Dy + \frac{a_{n-1}}{n} y = 0\right) \\ &\supseteq \mathrm{PV}\, F\left(Ly = 0,\ Dy + \frac{a_{n-1}}{n} y = 0\right); \end{aligned}$$

here $\langle A, b\rangle$ denotes the differential field spanned by A and b. Now by differential algebra (Magid [1]), $\mathrm{PV}\, F(\overline{L}y = 0, Dy + (a_{n-1}/n)y = 0)$ is the Picard–Vessiot extension (of F) of some other equation, $\mathrm{PV}\, F(\widetilde{L}y = 0)$, and $\mathrm{PV}\, F(Ly = 0, Dy + (a_{n-1}/n)y = 0)$ contains $\mathrm{PV}\, F(\widehat{L}z = 0)$ (look at the substitution!). Finally, we obtain the m-chain (2a) and (2b):

$$F \subseteq F_1 \subseteq \ldots \subseteq F_{l-1} \subseteq F_l \subseteq \mathrm{PV}\, F(\widetilde{L}y = 0) \supseteq \mathrm{PV}\, F(\widehat{L}z = 0),$$

hence the m-reducibility of $Ly = 0$ implies the m-reducibility of $\widehat{L}z = 0$.

To prove the converse, we use the (converse) substitution

$$z = y \cdot \exp \frac{1}{n} \int a_{n-1}.$$

PROPOSITION 7. *Let $m \geq 2$ and let Ly be factorizable into factors of order $\leq m$ in F:*

$$Ly = L_r \circ \ldots \circ L_1 y, \qquad \mathrm{ord}\, L_i \leq m,\ i = 1, \ldots, r. \tag{3}$$

Then the equation $Ly = 0$ is m-reducible.

Proof. We show how to obtain a fundamental system of solutions of $Ly = 0$ by an m-chain, i.e. by solving equations of order $\leq m$. First we take the solutions of

$$L_1 y_1 = 0, \qquad \mathrm{ord}\, L_1 \leq m,$$

into the fundamental system. Next we consider the solutions of

$$L_2 x_2 = 0, \qquad \mathrm{ord}\, L_2 \leq m,$$

and solve the inhomogeneous equations

$$L_1 y_2 = x_2 \tag{4}$$

for y_2. By a classical formula (see Ince [1], "variation of constants"), we get the solutions of (4) from the fundamental system of L_1 and x_2 by integrations, i.e. by solving equations of order $\leq m$ and of order 2; here we need the assumption $m \geq 2$. From (3) and (4) we obtain

$$Ly_2 = L_r \circ \ldots \circ L_2 \circ L_1 y_2 = L_r \circ \ldots \circ L_2 x_2 = 0,$$

and we take the y_2's as further elements into the fundamental system of $Ly = 0$. The check of linear independence is straightforward. Now it is obvious how to proceed: consider

$$L_3 x_3 = 0, \qquad \operatorname{ord} L_3 \leq m,$$

solve for y_3's

$$L_2 \circ L_1 y_3 = x_3.$$

Because the y_1's, y_2's also constitute a fundamental system for $L_2 \circ L_1 y = 0$, once more the classical formula does the job of getting the y_3's (integration!).

We have not covered the case $m = 1$, but there is a known result: An equation (1) which factors into order-1 factors is general Liouville (see Skórnik–Wloka [1]).

There is a Galois group criterion for the factorization (3) of an operator L (Kolchin [1]).

PROPOSITION 7′. *L factors into* (3) *if and only if the Galois group G of the equation* (1) *is block-reducible*, *i.e. if G can be represented by matrices of the form*

$$\begin{pmatrix} G_r & \ldots & * \\ \vdots & \ddots & \vdots \\ 0 & \ldots & G_1 \end{pmatrix}.$$

EXAMPLE 3. We consider the equation

$$D^3 y + sDy = 0.$$

Since it factors into

$$(D^2 + s) \circ Dy = 0,$$

it is 2-reducible (Proposition 7), but it is not (general) Liouville (see Example 2).

If (1) is *not* m-reducible, it may still happen that some solutions can be found by m-reduction; but we have

PROPOSITION 8. *Let* $\operatorname{ord} L = n \geq 3$, *and suppose that* $Ly = 0$ *is not* $(n-1)$*-reducible. Then no solutions* $\neq 0$ *of* $Ly = 0$ *are* $(n-1)$*-reducible* (Definition 1′).

P r o o f. Suppose that a solution $y_1 \neq 0$ of (1) is $(n-1)$-reducible, i.e. we have an $(n-1)$-chain

$$F \subseteq F_1 \subseteq \ldots \subseteq F_l = \overline{E} = \mathrm{PV}\, F(\overline{L}y = 0) \ni y_1.$$

Since in differential fields there exists a euclidean algorithm (Ince [1], Skórnik–Wloka [1]), the operator L factors in $\overline{E}$ into

$$Ly = L_{n-1} \circ L_1 y, \quad \text{where} \quad L_1 y = Dy - \frac{Dy_1}{y_1} y. \tag{5}$$

Now, adjoining to $\overline{E}$ the solutions $y_1, \ldots, y_n$ of (1), we obtain

$$E_2 = \overline{E}\langle y_1, \ldots, y_n \rangle = \mathrm{PV}\, F(\overline{L}z = 0, Ly = 0),$$

which is another Picard–Vessiot extension of F for some equation $L_2 y = 0$ (Magid [1]).

Applying to (5) the reasoning of Proposition 7, we see that E_2 is connected to $\overline{E}$ by an $(n-1)$-chain. Because we also have to integrate, we need $n-1 \geq 2$.

Taking all threads together, we get a long $(n-1)$-chain

$$F \subseteq F_1 \subseteq \ldots \subseteq \overline{E} \subseteq \ldots \subseteq E_2 = \mathrm{PV}\, F(\overline{L}z = 0, Ly = 0) \supseteq \mathrm{PV}\, F(Ly = 0)$$

for $Ly = 0$, which is a contradiction.

Proposition 9. *Consider the m-chain* (2a) *and Definition* 1. *We get an equivalent definition* (*that is, a new m-chain $F \subseteq E_1 \subseteq \ldots \subseteq E_l = \overline{E}$*) *demanding that for $i = 0, \ldots, l-1$ the building block E_{i+1} is a* PV-*extension of E_i belonging to an equation $L_{i+1} y = 0$ with* $\operatorname{ord} L_{i+1} \leq m$ *and with coefficients from E_i*,

$$E_{i+1} = \mathrm{PV}\, E_i(L_{i+1} y = 0, \operatorname{ord} L_{i+1} \leq m), \quad i = 1, \ldots, l-1$$

(*or that E_{i+1} is a finite*, *normal*, *algebraic extension of E_i*). *The final field $E_l = \overline{E}$ remains unchanged. For the first link E_1 in the new chain we have*

$$\dim G(E_1 \mid F) \leq m^2.$$

P r o o f. If $F = \overline{E} = \mathrm{PV}\, F(Ly = 0)$, there is nothing to prove.

Supposing

$$F \subsetneqq F_1 \subseteq \overline{E},$$

we first consider the case when F_1 is an algebraic extension. Let $\xi_1 \in F_1$ ($\xi_1 \notin F$) be algebraic over F and let

$$P(\xi) = \xi^n + a_{n-1}\xi^{n-1} + \ldots + a_0, \quad a_0, \ldots, a_{n-1} \in F, \tag{6}$$

be the irreducible polynomial for ξ_1. We take all those solutions $\xi_1, \ldots, \xi_r$ of (6) which are contained in $\overline{E}$ and consider $F\langle \xi_1, \ldots, \xi_r \rangle$. We have

$$F \subsetneqq F_1 \subseteq F\langle \xi_1, \ldots, \xi_r \rangle \subseteq \overline{E},$$

and since the mappings σ in $G(\overline{E}\,|\,F)$ preserve the differential field $F\langle\xi_1,\dots$ $\dots,\xi_r\rangle$: $\sigma(\xi_i)\in\overline{E}$ and it is once more a solution of (6), Proposition 3 implies that $F\langle\xi_1,\dots,\xi_r\rangle$ is normal, hence a PV-extension of F (Theorem 2). We have

$$F\langle\xi_1,\dots,\xi_r\rangle = F(\xi_1,\dots,\xi_r) = F[\xi_1,\dots,\xi_r],$$

where the first equality holds because of the formula

$$D\xi_i = \frac{-[Da_{n-1}\xi_i^{n-1}+\dots+Da_0]}{\left(\frac{d}{d\xi}P\right)\xi_i}, \qquad i=1,\dots,r,$$

and the second because it is a finite algebraic extension. In those equalities, we denote by ( ) the rational functions and by [ ] the polynomials in $\xi_1,\dots,\xi_r$. Hence the PV-extension is a normal, finite algebraic extension of F and we have

$$m = \dim G(F\langle\xi_1,\dots,\xi_r\rangle\,|\,F) = 0. \tag{6a}$$

Now, let F_1 be a nonalgebraic extension of F. There exists an element $\varsigma_1\in F_1, \varsigma_1\notin F$, such that ς_1 is a solution of an equation

$$\widetilde{L}y = a_mD^my+\dots+a_0y=0, \qquad a_0,\dots,a_m\in F, \tag{7}$$

the order of $\widetilde{L}$ being $\le m$, by our m-reducibility assumption. As before we take all those solutions $\varsigma_1,\dots,\varsigma_r$ of (7) (linearly independent) which are contained in $\overline{E}$. We have

$$F \underset{\neq}{\subseteq} F_1 \subseteq F\langle\varsigma_1,\dots,\varsigma_r\rangle \subseteq \overline{E},$$

and since each $\sigma\in G(\overline{E}\,|\,F)$ maps $F\langle\varsigma_1,\dots,\varsigma_r\rangle$ into itself: $\sigma(\varsigma_i)\in\overline{E}$ and it is again a solution of (7), Proposition 3 implies that $F\langle\varsigma_1,\dots,\varsigma_r\rangle$ is normal, hence a PV-extension of F (Theorem 2). $\varsigma_1,\dots,\varsigma_r$ are linearly independent solutions of the equation

$$L_1y = \frac{W(y,\varsigma_1,\dots,\varsigma_r)}{W(\varsigma_1,\dots,\varsigma_r)} = 0, \qquad r\le m \text{ (see Magid [1])},$$

where W denotes the Wroński determinant, and we have

$$F\langle\varsigma_1,\dots,\varsigma_r\rangle = \mathrm{PV}\,F(L_1y=0).$$

Obviously the coefficients of L_1y belong to F: $G(\overline{E}\,|\,F)$ leaves invariant the solution space $V=$ linear span$[\xi_1,\dots,\xi_r]$ of $L_1y=0$, thus the determinant formula for the coefficients of W (see Magid [1]) shows that the coefficients of L_1y are left fixed by $G(\overline{E}\,|\,F)$, hence belong to F.

$G(F\langle\varsigma_1,\dots,\varsigma_r\rangle\,|\,F)$ is a subgroup of $\mathrm{GL}(r,\boldsymbol{C})$ and also of $\mathrm{GL}(m,\boldsymbol{C})$:

$$G(F\langle\varsigma_1,\dots,\varsigma_r\rangle\,|\,F)\subseteq \mathrm{GL}(r,\boldsymbol{C})\subseteq\mathrm{GL}(m,\boldsymbol{C}), \qquad r\le m,$$

hence

$$\dim G(F\langle\varsigma_1,\dots,\varsigma_r\rangle\,|\,F)\le\dim\mathrm{GL}(m,\boldsymbol{C})=m^2. \tag{6b}$$

Denoting $F\langle \xi_1, \ldots, \xi_r\rangle$ or $F\langle \varsigma_1, \ldots, \varsigma_r\rangle$ by E_1 we have

$$F \subseteq E_1 \subseteq \overline{E}, \qquad \dim G(E_1 \,|\, F) \leq m^2,$$

and with E_1 a PV, we have proved the first step of Proposition 9. We remark that since $\overline{E}$ is a PV over F it is also a PV-extension of E_1.

Consider the composition $\widetilde{F}_2 = \langle E_1, F_2\rangle$. We have

$$E_1 \subseteq \widetilde{F}_2 \subseteq \overline{E},$$

and $\widetilde{F}_2$ is either finite algebraic over E_1 or it is obtained from E_1 by adjoining some solutions $\varsigma_1, \ldots$ of an m-differential equation. Reasoning as before, that is, adjoining all solutions in $\overline{E}$ to $\widetilde{F}_2$ we get $E_1\langle \xi_1, \ldots, \xi_{r_1}\rangle$ (algebraic case) or $E_1\langle \varsigma_1, \ldots, \varsigma_{r_2}\rangle$, $r_2 \leq m$. Using now the Galois group $G(\overline{E} \,|\, E_1)$, we see that both fields are PV-extensions of E_1. Calling them E_2, we have

$$F \subseteq E_1 \subseteq E_2 \subseteq \overline{E}.$$

For the second field we have, as before,

$$E_2 = E_1\langle \varsigma_1, \ldots, \varsigma_{r_2}\rangle = \mathrm{PV}\, E_1(L_2 y = 0), \qquad \operatorname{ord} L_2 = r_2 \leq m,$$

and the coefficients of $L_2 y$ belong to E_1.

It is now obvious how to finish the proof.

III. Simple equations. We use the language of algebraic groups (Humphreys [1], Merzlyakov [1]), and call an (algebraic) matrix group G *simple* if it has no proper *infinite* closed, normal subgroups.

If $\boldsymbol{C}$ has characteristic 0 and is algebraically closed, then all simple algebraic groups $G \subset \mathrm{GL}(n, \boldsymbol{C})$ are known (Zalesskij [1], Humphreys [1]). Let us take one of them, the special linear group $\mathrm{SL}(n, \boldsymbol{C})$.

Example 4. We have

$$\mathrm{SL}(n, \boldsymbol{C}) := \{g \in \mathrm{GL}(n, \boldsymbol{C}) \mid \det g = 1\}, \quad \dim \mathrm{SL}(n, \boldsymbol{C}) = n^2 - 1, \quad n \geq 1;$$

$\mathrm{SL}(n, \boldsymbol{C})$ is simple. The center $Z(\mathrm{SL})$ of $\mathrm{SL}(n, \boldsymbol{C})$ consists of the n-roots of unity:

$$Z(\mathrm{SL}) = \mathrm{Cycl}(n);$$

it is a finite, normal subgroup, and all other closed normal subgroups are subgroups of the center, hence finite.

The quotient group

$$\mathrm{PSL}(n, \boldsymbol{C}) := \mathrm{SL}(n, \boldsymbol{C})/Z(\mathrm{SL})$$

is once more a simple, linear group (Humphreys [1], Lang [1]), and it is also simple as an abstract group. Since $\boldsymbol{C}$ is algebraically closed, we have (Humphreys [1])

$$\mathrm{PSL}(n, \boldsymbol{C}) \cong \mathrm{GL}(n, \boldsymbol{C})/\boldsymbol{C}^* = \mathrm{PGL}(n, \boldsymbol{C}),$$

which is the projective group. From the quotient theorem (Humphreys [1], Merzlyakov [1]), we obtain the dimensions:

$$\dim \mathrm{PGL}(n, \boldsymbol{C}) = \dim \mathrm{PSL}(n, \boldsymbol{C}) = n^2 - 1.$$

Let $E = \mathrm{PV}\, F(Ly = 0)$ be a PV-extension of F associated with the equation $Ly = 0$.

Definition 2. We call the equation $Ly = 0$ *simple* if its Galois group $G(E \mid F)$ is simple.

We now give a lower bound for m-reducibility.

Theorem 10. *Let $Ly = 0$ be a simple equation, let $E = \mathrm{PV}\, F(Ly = 0)$ with Galois group $G(E \mid F)$, and let*

$$\dim G(E|F) > 0. \tag{8}$$

Then the equation $Ly = 0$ is not m-reducible for

$$m < \sqrt{\dim G(E \mid F)}. \tag{9}$$

Remark 3. The assumption $\dim G(E \mid F) > 0$ is natural, because $\dim G(E \mid F) = 0$ implies that E is a finite algebraic extension of F, which is 0-reducible (see Definition 1).

For the proof of Theorem 10 we need some new concepts (Chevalley [1], Merzlyakov [1]).

A sequence $\{e\} \subseteq A_1 \subseteq \ldots \subset A_m \subseteq G$ of subgroups is called *normal* if the subgroups A_i are normal in G. If $\{e\} \subseteq A_1 \subseteq \ldots \subseteq A_n \subseteq G$ is a subsequence of $\{e\} \subseteq A'_1 \subseteq \ldots \subseteq A'_l \subseteq G$, we say that the latter is a *refinement* of the former. By *factor groups* of a normal sequence we understand the groups $A_1, A_{i+1}/A_i$ $(i = 1, \ldots, m-1)$ and G/A_m.

Two groups G_1, G_2 are *isogenic* if there exist a third group G_3 and finite, normal subgroups H_1, H_2 of G_3 such that $G_3/H_1 \cong G_1$ and $G_3/H_2 \cong G_2$. Since finite groups have dimension 0, we then have $\dim G_1 = \dim G_3 = \dim G_2$.

We have

Chevalley's Theorem [1]. *Any two given normal sequences*

$$\{e\} \subseteq A_1 \subseteq \ldots \subseteq A_n \subseteq G \quad \textit{and} \quad \{e\} \subseteq B_1 \subseteq \ldots \subseteq B_m \subseteq G$$

have respective normal refinements

$$\{e\} \subseteq A'_1 \subseteq \ldots \subseteq A'_l = G \quad \textit{and} \quad \{e\} \subseteq B'_1 \subseteq \ldots \subseteq B'_l = G,$$

both of the same length and such that the factor groups A'_{i+1}/A'_i and B'_{i+1}/B'_i $(i = 1, \ldots, l-1)$ are isogenic.

For the proof, see also Merzlyakov [1].

Proof of Theorem 10. We proceed in five steps:

STEP 1. Let $\overline{G} = G(\overline{E}\,|\,F)$ be the Galois group of $\overline{E}$ in (2a). Supposing m-reducibility, by Proposition 9 there exist a PV-extension E_1 of F and a normal subgroup H_1 of $\overline{G}$ such that

$$\begin{array}{ccccc} F & \subsetneqq & E_1 & \subseteq & \overline{E} \\ | & & | & & | \\ \overline{G} & \supsetneqq & H_1 & \supseteq & \{e\} \end{array} \qquad (\text{“}\neq\text{” because of (8)})$$

(see Theorem 2). Proposition 9, (6a,b) and Theorem 2 imply

$$\dim \overline{G}/H_1 = \dim G(E_1\,|\,F) \leq m^2.$$

STEP 2. If H_2 is another normal subgroup with

$$\{e\} \subseteq H_1 \subseteq H_2 \subsetneqq \overline{G} \tag{10}$$

we still have

$$\dim \overline{G}/H_2 \leq m^2 \tag{11}$$

because (see Humphreys [1] or Merzlyakov [1])

$$\begin{aligned} \dim \overline{G}/H_2 &= \dim \overline{G} - \dim H_2 \\ &\leq \dim \overline{G} - \dim H_1 = \dim \overline{G}/H_1 \leq m^2. \end{aligned}$$

STEP 3. By (2b) we have $F \subseteq E \subseteq \overline{E}$, and there exists a normal subgroup K_1 with

$$\begin{array}{ccccc} F & \subsetneqq & E & \subseteq & \overline{E} \\ | & & | & & | \\ \overline{G} & \supsetneqq & K_1 & \supseteq & \{e\} \end{array} \qquad (\text{“}\neq\text{” because of (8)})$$

and we have (Theorem 2) $\overline{G}/K_1 = G(E\,|\,F)$.

STEP 4. If we refine with another normal subgroup K_2 with

$$\{e\} \subseteq K_1 \subseteq K_2 \subsetneqq \overline{G}, \tag{12}$$

then simplicity of $G(E\,|\,F)$ implies

$$\dim \overline{G}/K_2 = \dim \overline{G}/K_1 = \dim G(E|F). \tag{13}$$

Indeed, applying Theorem 2 to (12) we get the existence of E_2 (a PV-extension of F) such that

$$\begin{array}{ccccccc} \{e\} & \subseteq & K_1 & \subseteq & K_2 & \subsetneqq & \overline{G} \\ | & & | & & | & & | \\ \overline{E} & \supseteq & E & \supseteq & E_2 & \supsetneqq & F \end{array} \tag{14}$$

Considering the shorter sequence $E \supseteq E_2 \supsetneqq F$ we have (Theorem 2)

$$\begin{array}{ccccc} E & \supseteq E_2 & \supsetneqq & & F \\ | & | & & & | \\ \{e\} & \subset \widetilde{K} & \subsetneqq & & G(E\,|\,F) \end{array}$$

where $\widetilde{K}$ is normal in $G(E\,|\,F)$. Because the group $G(E\,|\,F)$ is simple by assumption, $\widetilde{K}$ can only be a finite normal subgroup, which means that the extension $E \supseteq E_2$ is (finite) algebraic. Returning to diagram (14) we have

$$\dim K_2/K_1 = \dim G(E|E_2) = 0$$

i.e. $\dim K_2 = \dim K_1$; the last equation in (13) comes from step 3.

Step 5. By Chevalley's Theorem, there exist normal refinements of (10) and (12) which are of the same length and have isogenic factor groups:

$$\{e\} \subseteq \widetilde{H}_1 \subseteq \ldots \subseteq \widetilde{H}_l \subsetneqq \overline{G}, \qquad \{e\} \subseteq \widetilde{K}_1 \subseteq \ldots \subseteq \widetilde{K}_l \subsetneqq \overline{G}.$$

We are interested in the last factor groups, $\overline{G}/\widetilde{H}_l$ isogenic to $\overline{G}/\widetilde{K}_l$. Since isogenic groups have the same dimension, we obtain from (13) and (11),

$$\dim G(E\,|\,F) = \dim \overline{G}/\widetilde{K}_l = \dim \overline{G}/\widetilde{H}_l \leq m^2,$$

hence if (9) holds then equation $Ly = 0$ is not m-reducible.

Theorem 11. *Let $b_0, \ldots, b_{n-1}$ be indeterminates over a differential field F. The generic equation* ("general equation" in Magid [1])

$$L_{\mathrm{GL}}y = D^n y + b_{n-1}D^{n-1}y + \ldots + b_0 y = 0, \qquad n \geq 2, \tag{15}$$

is not $(n-1)$-reducible, hence it cannot be reduced to equations of lower order.

Using Proposition 8 and supposing that $\operatorname{ord} L_{\mathrm{GL}} = n \geq 3$, we obtain a stronger result: *no solution $y \neq 0$ of (15) is $(n-1)$-reducible.*

Proof. We start with $f = F\langle b_0, \ldots, b_{n-1}\rangle$, construct $\mathrm{PV}\,f(L_{\mathrm{GL}}y = 0)$ and obtain (see Magid [1])

$$G(\mathrm{PV}\,f(L_{\mathrm{GL}}y = 0)\,|\,f) = \mathrm{GL}(n, \boldsymbol{C}).$$

The scalar subgroup

$$\boldsymbol{C}^* = \left\{ \begin{pmatrix} c & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & c \end{pmatrix}_{n\times n} \middle|\, 0 \neq c \in \boldsymbol{C} \right\}$$

is normal in $\mathrm{GL}(n, \boldsymbol{C})$ and we have (see Example 4)

$$\mathrm{GL}(n, \boldsymbol{C})/\boldsymbol{C}^* \cong \mathrm{SL}(n, \boldsymbol{C})/Z(\mathrm{SL}) = \mathrm{PSL}(n, \boldsymbol{C}).$$

By Galois theory (Theorem 2 applied to $\boldsymbol{C}^*$) we get the existence of a PV-extension PV $f(L_{\mathrm{PSL}}y=0)$, such that

$$\begin{array}{ccccc} \mathrm{GL}(n,\boldsymbol{C}) \supseteq & & \boldsymbol{C}^* & \supseteq & \{e\} \\ | & & | & & | \\ f & \subseteq & \mathrm{PV}\, f(L_{\mathrm{PSL}}y=0) & \subseteq & \mathrm{PV}\, f(L_{\mathrm{GL}}y=0) \end{array} \tag{16}$$

We have $G(\mathrm{PV}\, f(L_{\mathrm{PSL}}y=0)\,|\,f) = \mathrm{GL}(n,\boldsymbol{C})/\boldsymbol{C}^* = \mathrm{PSL}(n,\boldsymbol{C})$, the Galois group $\mathrm{PSL}(n,\boldsymbol{C})$ of $L_{\mathrm{PSL}}y=0$ is simple (see Example 4) and

$$n-1 < (n^2-1)^{1/2} = [\dim \mathrm{PSL}(n,\boldsymbol{C})]^{1/2} \quad \text{for } n \geq 2.$$

So Theorem 10 tells us that the equation $L_{\mathrm{PSL}}y=0$ is not $(n-1)$-reducible, hence by Proposition 4 (applied to (16)) $L_{\mathrm{GL}}y=0$ is not either.

We give a lower bound for the order of the equation $L_{\mathrm{PSL}}y=0$ occurring in (16). First some definitions. It may happen that different equations $L_1y=0$ and $L_2y=0$ generate the same PV-extension of F, i.e.

$$\mathrm{PV}\, F(L_1y=0) = \mathrm{PV}\, F(L_2y=0),$$

therefore, we call the minimal order of all equations $Ly=0$ generating the same PV-extension $E = \mathrm{PV}\, F(Ly=0)$ of F the *order* $\widetilde{n}$ of E:

$$\widetilde{n} = \mathrm{ord}\, E = \min\{\mathrm{ord}\, L \mid E = \mathrm{PV}\, F(Ly=0)\}.$$

To avoid complications with algebraic extensions, we suppose that $\dim G(E\,|\,F) > 0$. Since for the Galois group $G(E\,|\,F)$ of the extension $E = \mathrm{PV}\, F(Ly=0)$ we have $G(E\,|\,F) \subseteq \mathrm{GL}(n,\boldsymbol{C})$, where $n = \mathrm{ord}\, L$, we define the *rank* r of $G(E\,|\,F)$ to be the minimal n such that we have the minimal (rational) faithful representation

$$G(E\,|\,F) \subseteq \mathrm{GL}(r,\boldsymbol{C}).$$

Obviously we have $\widetilde{n} = \mathrm{ord}\, E \geq r = \mathrm{rank}\, G(E\,|\,F)$.

PROPOSITION 12. *Let m be the order of the equation $L_{\mathrm{PSL}}y=0$ in* (16). *For $n \geq 2$ we have the estimate*

$$m \geq \mathrm{ord}\,\mathrm{PV}\, f(L_{\mathrm{PSL}}y=0) \geq r = \mathrm{rank}\,\mathrm{PSL}(n,\boldsymbol{C}) \geq n+1. \tag{17}$$

Proof. We have to show that $r = \mathrm{rank}\,\mathrm{PSL}(n,\boldsymbol{C}) \geq n+1$. Comparing dimensions we see at once that $r \leq n-1$ is not possible, since then

$$\mathrm{PSL}(n,\boldsymbol{C}) \subseteq \mathrm{GL}(r,\boldsymbol{C}) \subseteq \mathrm{GL}(n-1,\boldsymbol{C})$$

and so (see Example 4) $\dim \mathrm{PSL}(n,\boldsymbol{C}) = n^2-1 \leq (n-1)^2$ for $n \geq 2$, which is false.

Now we rule out the possibility $r=n$. The inclusion

$$\mathrm{PSL}(n,\boldsymbol{C}) \subseteq \mathrm{GL}(n,\boldsymbol{C}), \quad r=n,$$

would mean that there exists a homomorphism

$$\varphi : \mathrm{SL}(n, \boldsymbol{C}) \to \mathrm{GL}(n, \boldsymbol{C}) \tag{18}$$

with $\ker \varphi = Z(\mathrm{SL})$ (see Humphreys [1]). The composition

$$\mathrm{SL}(n) \underset{\varphi}{\to} \mathrm{GL}(n) \underset{\det}{\longrightarrow} \mathbb{C} \setminus \{0\}$$

would be a character of $\mathrm{SL}(n)$; but $\mathrm{SL}(n)$ has only trivial ($= 1$) characters (Humphreys [1]), so $\varphi(\mathrm{SL}(n)) \subseteq \mathrm{SL}(n)$. Hence

$$\mathrm{PSL}(n) = \mathrm{SL}(n)/Z(\mathrm{SL}) \cong \varphi(\mathrm{SL}(n)) \subseteq \mathrm{SL}(n). \tag{19}$$

The irreducible component of unity of the group $\mathrm{SL}(n)$ is the whole group (Humphreys [1]) and so the group $\mathrm{SL}(n)$ is irreducible as an algebraic manifold; the same is true for the group $\mathrm{PSL}(n)$, since it is (abstractly) simple. In (19) we have an inclusion of two irreducible, closed, algebraic manifolds, hence by a fundamental theorem about dimensions (Humphreys [1]) we get

$$\dim \mathrm{PSL}(n) = \dim \varphi(\mathrm{SL}(n)) \le \dim \mathrm{SL}(n),$$

where equality holds if and only if equality holds in (19). But both dimensions are equal ($= n^2 - 1$, see Example 4) hence $\mathrm{PSL}(n, \boldsymbol{C}) \cong \mathrm{SL}(n, \boldsymbol{C})$, which is a contradiction for $n \ge 2$, because one group is abstractly simple and the other not.

This establishes (17).

IV. Simple Fuchsian equations. C. Tretkoff and M. Tretkoffi [1] solved the inverse Galois problem for the differential field $(\mathbb{C}(z), d/dz)$; by isomorphy (Theorem 1) it is also solved for the field $(\mathbb{C}(s), d/ds = D)$ of Mikusiński operators, i.e. for every closed algebraic matrix group $G \subseteq \mathrm{GL}(n, \mathbb{C})$, there exists an ordinary, linear, Fuchsian differential equation $L_G y = 0$ of order n (with polynomial coefficients in z or s)

$$L_G y = p_n(z) y^{(n)} + p_{n-1}(z) y^{(n-1)} + \ldots + p_0(z) y = 0, \tag{20}$$

such that the PV-extension $E = \mathrm{PV}\, F(L_G y = 0)$ over $F = \mathbb{C}(z)$ (or $\mathbb{C}(s)$) has Galois group G:

$$G(E \mid F) = G.$$

The inverse Galois theorem and Theorem 10 imply

THEOREM 13. *Let F be $(\mathbb{C}(z), d/dz)$ or $(\mathbb{C}(s), d/ds)$. For each simple group $G \subset \mathrm{GL}(n, \mathbb{C})$ there exists a Fuchsian equation* (20) *of order $L_G = \operatorname{rank} G = n$, which is not m-reducible for any $m < \sqrt{\dim G}$.*

Combining Theorem 13 and Example 4 with the inverse Galois theorem, we get a "best" result for $F = \mathbb{C}(z)$ or $\mathbb{C}(s)$.

THEOREM 14. *For each group* $\mathrm{SL}(n,\mathbb{C})$, $n \geq 2$, *there exists a Fuchsian equation*

$$L_{\mathrm{SL}} y = 0, \qquad \operatorname{ord} L_{\mathrm{SL}} = n,$$

which is not $(n-1)$*-reducible.*

REMARK 4. For Fuchsian equations $L_f y = 0$ over $F = (\mathbb{C}(s), d/ds)$ we have

$$\mathrm{PV}\, F(L_f y = 0) \subseteq \mathfrak{M} \qquad \text{(Mikusiński operators)}$$

(see Wloka [1]), thus we need not go outside $\mathfrak{M}$ with our PV-extensions.

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