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BOUNDS FOR THE SINGULAR VALUES OF SMOOTH KERNELS

BY

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Let $k \in L^2[0,1]^2$. The Hilbert–Schmidt operator K with kernel k is defined on the Hilbert space $L^2[0,1]$ by

$$Kf(x) = \int_0^1 k(x,y)f(y)\,dy.$$

We denote by $\{s_n(K)\}$ the sequence of singular values of K, which are the positive eigenvalues of the positive square root of K^*K. As usual $\{s_n(K)\}$ is arranged in the decreasing order and counted according to multiplicities. As extensions of some classical results of Fredholm [3] and Weyl [7] when K is Hermitian, the asymptotic estimates of $\{s_n(K)\}$ have been obtained in Blyumin–Kotlyar [1], Oehring [5], and Weidmann [6] for the kernel k satisfying some smoothness assumptions analogous to those from the classical Fourier series.

The purpose of this paper is to give upper bounds for $s_n(K)$ in terms of n, which seem more desirable, and improve some of the results cited above. If $k(x,y) \equiv k(x-y)$ for some $k \in L^2[0,1]$ which is periodically extended, then $\{s_n(K)\}$ consists of the moduli of the Fourier coefficients of k. Thus these bounds imply results concerning the absolute convergence of Fourier series under comparable conditions (see Zygmund [8, pp. 240–242]). Our results are based on an interesting inequality of Fan [2], which makes the proofs simple and straightforward.

We first recall that

$$\sum_{n=1}^{\infty} s_n^2(K) = \int_0^1\!\!\int_0^1 |k(x,y)|^2\,dx\,dy. \tag{1}$$

Moreover, it follows from Fan [2, Theorem 1] (see also Gohberg–Krein [4, p. 47]) that for any orthonormal family $\{\phi_j : 1 \le j \le n\}$ in $L^2[0,1]$,

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$$\sum_{j=1}^{n} s_j^2(K) \ge \sum_{j=1}^{n} \int_0^1 |K\phi_j(x)|^2\,dx. \tag{2}$$

Lemma 1. *For any integer* $n \ge 1$,

$$s_{2n}^2(K) \le \frac{1}{2} \sum_{j=1}^{n} \Delta_j(k; n), \tag{3}$$

where for $1 \le j \le n$, $I_j = [(j-1)/n, j/n]$ *and*

$$\Delta_j(k; n) = \int_{I_j} \int_{I_j} \Big[\int_0^1 |k(z,x) - k(z,y)|^2\,dz\Big]dx\,dy.$$

Proof. For $1 \le j \le n$ we define

$$\phi_j(x) = \begin{cases} \sqrt{n} & \text{if } x \in I_j, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\{\phi_j : 1 \le j \le n\}$ is orthonormal. Moreover,

$$\begin{aligned}\int_0^1 |K\phi_j(x)|^2\,dx &= n \int_0^1 \Big| \int_{I_j} k(x,y)\,dy \Big|^2 dx \\ &= n \int_0^1 \Big(\int_{I_j} \overline{k(z,x)}\,dx \Big) \Big(\int_{I_j} k(z,y)\,dy \Big) dz \\ &= n \int_{I_j} \int_{I_j} \Big[\int_0^1 \overline{k(z,x)} k(z,y)\,dz\Big] dx\,dy.\end{aligned}$$

We write

$$\int_0^1\int_0^1 |k(x,y)|^2\,dx\,dy = \frac{n}{2} \sum_{j=1}^{n} \int_{I_j}\int_{I_j} \Big[\int_0^1 (|k(z,x)|^2 + |k(z,y)|^2)\,dz\Big]dx\,dy.$$

By (1), (2),

$$\begin{aligned}\sum_{j=n+1}^{\infty} s_j^2(K) &\le \int_0^1\int_0^1 |k(x,y)|^2\,dx\,dy - \sum_{j=1}^{n} \int_0^1 |K\phi_j(x)|^2\,dx \\ &= \frac{n}{2} \sum_{j=1}^{n} \int_{I_j}\int_{I_j} \Big[\int_0^1 |k(z,x) - k(z,y)|^2\,dz\Big]dx\,dy,\end{aligned}$$

from which (3) follows.

As the inequality of Fan on which the proof of Lemma 1 is based is valid for any compact operator on a Hilbert space, (3) can be easily generalized to the higher dimensional case in which $k \in L^2(D^2)$ and $D \subset \mathbb{R}^d$ is a bounded domain, $d \geq 1$.

We shall keep the notation of Lemma 1 throughout the rest of this paper. Moreover, we set $k^{(0)} \equiv k$ and denote the rth order partial derivative of k with respect to y, if it exists, by

$$k^{(r)}(x,y) \equiv \frac{\partial^r k}{\partial y^r}(x,y).$$

For kernels with the partial derivatives, Lemma 1 can be extended to the following

LEMMA 2. *If $m \geq 1$, $k^{(r)}$ is absolutely continuous in y for almost all $x \in [0,1]$ for $0 \leq r \leq m-1$, and $k^{(m)} \in L^2[0,1]^2$, then for any integer $n \geq 1$,*

$$s^2_{2n+m}(K) \leq \frac{2}{n^{2m}(m-1)!^2} \sum_{j=1}^{n} \Delta_j(k^{(m)}; n). \tag{4}$$

Proof. We choose a point $c_j \in I_j$ for each $1 \leq j \leq n$ such that the second inequality in (6) below holds, and define

$$k_1(x,y) = \begin{cases} \sum_{r=1}^{m} k^{(r)}(x,c_j)(y-c_j)^r/r! & \text{if } x \in [0,1],\ y \in I_j, \\ 0 & \text{otherwise.} \end{cases}$$

Let $h(x,y) = k(x,y) - k_1(x,y)$. Since $h \in L^2[0,1]^2$ and differs from k by a degenerate kernel of rank $\leq m$, it follows that $s_{2n+m}(K) \leq s_{2n}(H)$ (see [4, p. 29]). We define $k_0(x,y) = k_1(x,y) + k(x,c_j)$ so that

$$|h(z,x) - h(z,y)| \leq |k(z,x) - k_0(z,x)| + |k(z,y) - k_0(z,y)|. \tag{5}$$

For a fixed $1 \leq j \leq n$, if $x \in I_j$, then using the Taylor formula with the Cauchy remainder we have

$$|k(z,x) - k_0(z,x)| \leq \frac{1}{n^{m-1}(m-1)!} \int_{I_j} |k^{(m)}(z,t) - k^{(m)}(z,c_j)|\, dt$$

for $z \in [0,1]$ and so by Hölder inequality and the choice of the point $c_j \in I_j$,

$$\begin{aligned} \int_0^1 |k(z,x) - k_0(z,x)|^2\, dz & \\ &\leq \frac{1}{n^{2m-1}(m-1)!^2} \int_0^1 \int_{I_j} |k^{(m)}(z,t) - k^{(m)}(z,c_j)|^2\, dt\, dz \\ &\leq \frac{1}{n^{2m-2}(m-1)!^2} \int_{I_j} \int_{I_j} \Big[\int_0^1 |k^{(m)}(z,t) - k^{(m)}(z,y)|^2\, dz \Big]\, dt\, dy. \end{aligned} \tag{6}$$

Hence

$$\int_{I_j}\int_{I_j}\Big[\int_0^1 |k(z,x)-k_0(z,x)|^2\,dz\Big]\,dx\,dy \le \frac{1}{n^{2m}(m-1)!^2}\Delta(k^{(m)};n).$$

We also have a similar inequality for the second term on the right hand side of (5). Thus (4) follows from (3).

In the following we shall assume that either $m=0$, or $m\ge 1$ and $k^{(r)}$ is absolutely continuous in y for almost all $x\in[0,1]$ for $0\le r\le m-1$.

Theorem 1. *If $k^{(m)}\in L^2[0,1]^2$ satisfies the integrated Lipschitz condition*

$$\int_0^1 |k^{(m)}(z,x)-k^{(m)}(z,y)|^2\,dz \le C|x-y|^{2\alpha}$$

for $0\le x,y\le 1$, where $C>0$, $0<\alpha\le 1$, then for $n\ge 1$,

$$s_{2n+m}(K) \le \frac{\sqrt{2C}}{(m-1)!}\frac{1}{n^{m+1/2+\alpha}}. \tag{7}$$

It is immediate that (7) follows from (4). We refer to an interesting result in [6, Lemma 1], by which together with Lemmas 1, 2 further results along these lines can be obtained.

Theorem 2. *If $k^{(m)}$ satisfies the Lipschitz condition*

$$|k^{(m)}(z,x)-k^{(m)}(z,y)| \le A(z)|x-y|^{\alpha}$$

for $0\le x,y,z\le 1$, where $0<\alpha\le 1$, and for almost all $z\in[0,1]$, $k^{(m)}(z,y)$ is of bounded variation in y with total variation $B(z)$ on $[0,1]$ such that $C=\int_0^1 A(z)B(z)\,dz<\infty$, then for $n\ge 1$,

$$s_{2n+m}(K) \le \frac{\sqrt{2C}}{(m-1)!}\frac{1}{n^{m+1+\alpha/2}}. \tag{8}$$

Proof. For $1\le j\le n$ by definition

$$\int_{I_j}\int_{I_j} |k^{(m)}(z,x)-k^{(m)}(z,y)|\,dx\,dy \le \frac{1}{n^2}B_j(z)$$

for almost all $z\in[0,1]$, where $B_j(z)$ denotes the total variation on I_j of $k^{(m)}(z,y)$ as a function of y, and so

$$\sum_{j=1}^n \Delta_j(k^{(m)};n) \le \frac{1}{n^{2+\alpha}}\sum_{j=1}^n\int_0^1 A(z)B_j(z)\,dz = \frac{C}{n^{2+\alpha}},$$

from which (8) follows.

We refer to [1] for an asymptotic property of $\{s_n(K)\}$ under an assumption similar to that of Theorem 2.

THEOREM 3. *If* $k^{(m)}(x,y)$ *is absolutely continuous in* y *for almost all* $x \in [0,1]$ *and*

$$C = \int_0^1 \Big[\int_0^1 |k^{(m+1)}(x,y)|^p\, dy\Big]^{2/p} dx < \infty,$$

where $1 \le p \le 2$, *then for* $n \ge 1$,

$$s_{2n+m}(K) \le \frac{\sqrt{2C}}{(m-1)!}\frac{1}{n^{m+2-1/p}}. \tag{9}$$

Proof. For $1 \le j \le n$,

$$\begin{aligned}\Delta_j(k^{(m)};n) &\le \frac{1}{n^2}\int_0^1 \Big[\int_{I_j} |k^{(m+1)}(z,t)|\, dt\Big]^2 dz\\ &\le \frac{1}{n^{2+2/q}}\int_0^1 \Big[\int_{I_j} |k^{(m+1)}(z,t)|^p\, dt\Big]^{2/p} dz,\end{aligned}$$

where $q = p/(p-1)$. By assumption $2/p \ge 1$ and so

$$\sum_{j=1}^n \Delta_j(k^{(m)};n) \le \frac{C}{n^{2+2/q}},$$

from which (9) follows.

We refer to [5] for a related result for the case $m = 0$.

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