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NON-UNIQUENESS OF TOPOLOGY FOR ALGEBRAS OF POLYNOMIALS

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All algebras in this paper are either real or complex. A *topological* (resp. *semitopological*) algebra is a (Hausdorff) topological vector space (t.v.s.) provided with an associative bilinear multiplication that is jointly (resp. separately) continuous. It is called locally convex if the underlying t.v.s. has this property. It was shown in [2] that if an algebra A is not at most countably generated (in the algebraic sense—we shall keep this terminology throughout this paper), then for each fixed p satisfying $0 < p \leq 1$, the topology $\tau_{\max}^p$, i.e. the maximal p-convex topology given by means of all p-homogeneous seminorms, makes A a complete (in the sense of topological vector spaces) semitopological algebra, and for different p the topologies are different. It was also shown there that if A is at most countably generated, then all these topologies coincide with $\tau_{\max}^{\mathrm{LC}}$ $(= \tau_{\max}^1)$, the maximal locally convex topology, so that in this case the problem of uniqueness of a complete topology making A a semitopological algebra remained open. In particular, the question was posed whether for the algebra $P(t)$ of polynomials in one variable, $\tau_{\max}^{\mathrm{LC}}$ is the unique topology making it a complete semitopological algebra (in fact, as shown in [5], this topology makes $P(t)$ and every at most countably generated algebra a topological algebra). This question was answered in the negative in [6] by constructing on $P(t)$ (and also on algebras of polynomials and on free algebras in an arbitrary number of variables) a complete locally convex topology which makes it a semitopological algebra and is different from $\tau_{\max}^{\mathrm{LC}}$ (the latter makes every algebra a complete semitopological algebra). However, this topology does not make $P(t)$ a topological algebra.

In this paper we construct on $P(t)$ a continuum of different complete locally convex topologies making it a topological algebra. Thus we have non-uniqueness of a complete locally convex topology for $P(t)$. We extend these topologies to algebras of polynomials and free algebras in arbitrarily many

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variables and show that the non-uniqueness phenomenon holds there too. In particular, we obtain complete locally convex topologies for polynomial and free algebras in uncountably many variables, which is also a new result. Using our constructions we show that for every infinite set Ω the algebra $C_{00}(\Omega)$ of all finitely supported functions on Ω has a continuum of different topologies making it a complete locally multiplicatively convex algebra. Answering a question posed in [7], we show that each infinite-dimensional algebra has at least two different locally convex topologies making it a semi-topological algebra. We cannot, however, show that there are two such complete topologies, so at the end of the paper we pose this and some other open questions.

Let $c \geq 1$ and denote by $Q(c)$ the set of all increasing sequences $q = (q_i)_{i=1}^{\infty}$ of natural numbers satisfying

$$q_{n+1}/q_n > 2^{n^c} \tag{1}$$

for sufficiently large n, say $n \geq n(q)$. Clearly $Q(c') \subset Q(c)$ for $c' \geq c$. For q in $Q(c)$ we write $\widetilde{q}$ for the sequence $(\max\{1, q_i - 1\})_{i=1}^{\infty}$. For such q and for large n we have $q_{n+1} - 1 > 2^{n^c} q_n - 1 = 2^{n^c}(q_n - 1) + 2^{n^c} - 1 > 2^{n^c}(q_n - 1)$ and so $\widetilde{q}$ is also in $Q(c)$. We can take for $n(\widetilde{q})$ the number $\max\{n(q) - 1, 1\}$.

Put $I_0 = \{1\}$ and $I_n = \{2^{n-1} + 1, \ldots, 2^n\}$ for $n \geq 1$; this gives a partition of the natural numbers onto disjoint segments. For q in $Q(c)$, denote by $R_q(c)$ the family of all sequences $r = (r_i)_{i=0}^{\infty}$ satisfying $r_i \geq 1$, $r_0 = 1$, and $r_i = 1$ except when $i \in I_{q_m}$ for some natural $m \geq n(q)$ or $1 \leq i \leq 2^{q_{n(q)} - 1}$—in this case r_i can be arbitrarily large. Put

$$R(c) = \bigcup_{q \in Q(c)} R_q(c).$$

Clearly $R(c') \subset R(c)$ for $c' > c \geq 1$. Our construction is based upon the following lemma.

Lemma. *For every r in $R(c)$, there are $r', \widetilde{r} \in R(c)$ such that*

$$r_{i+j} \leq r'_i r'_j + \widetilde{r}_i \widetilde{r}_j \tag{2}$$

for all $i, j \geq 0$.

Proof. Put $n_0 = \max\{n(q), n(\widetilde{q}) + 1\}$. We define r'_i so that $r'_0 = 1$ and

$$r_d \leq r'_{d-i} r'_i \tag{3}$$

for $0 \leq i \leq d$ and all d satisfying $0 \leq d \leq 2^{q_{n_0} - 1}$; this can be done by Lemma 3 of [4]. Without loss of generality we can assume $r'_i \geq 1$ for $i \leq d$. Define now $r'_d = 1$ for $d \geq 2^{q_{n_0} - 1} + 1$ and $d \notin I_{q_m}$ for all m. Otherwise, $d \in I_{q_m}$ for some $m \geq n_0$ and in this case we put

$$r'_d = \max\{r_i : i \in I_{q_m}\}.$$

Define now $\widetilde{r}$ in $R_{\tilde{q}}(c)$ setting $\widetilde{r}_d = 1$ for all d satisfying either $d \le 2^{q_{n_0}-2}$ or $d \notin I_{\tilde{q}_m}$ for all $m \ge n_0 - 1$. Otherwise $d \in I_{\tilde{q}_m}$ for some $m \ge n_0 - 1$ and then we put

$$\widetilde{r}_d = \max\{r_i : i \in I_{\tilde{q}_m}\} = \max\{r_i : i \in I_{q_m - 1}\},$$

since clearly $\widetilde{q}_m = q_m - 1$ for $m \ge n_0$.

Suppose now that $d \ge 2^{q_{n_0}-1} + 1$. If $d \notin I_{q_m}$ for all m, we have $r_d = 1$ and relation (2) is satisfied for any choice of r' and $\widetilde{r}$ in R. If $d \in I_{q_m}$ for some $m \ge n_0$, the relation (3) is satisfied if either i or $j = d - i$ is in I_{q_m}, which implies (2) in this case. If neither i nor $d - i$ is in I_{q_m}, then at least one of them must be in $I_{q_m - 1} = I_{\tilde{q}_m}$. In this case we have $r_d \le \widetilde{r}_{d-i}\widetilde{r}_i$, which means that (2) also holds in this case. The conclusion follows.

For x in $P(t)$, $x = \sum_i a_i(x)t^i$, $c \ge 1$, and r in $R(c)$ we put

$$|x|_r = \sum_i |a_i(x)| r_i, \tag{4}$$

which is clearly a norm on $P(t)$. Denote by τ_c the locally convex topology given by all norms of the form (4). Clearly the topology τ_c is stronger than $\tau_{c'}$ for $c' > c \ge 1$ and the linear functionals $x \mapsto a_i(x)$ are continuous in all these topologies.

PROPOSITION 1. *For each $c \ge 1$, $A_c = (P(t), \tau_c)$ is a complete locally convex topological algebra. Moreover, $\tau_c \ne \tau_{c'}$ for $c \ne c'$ and consequently the algebra $P(t)$ has a continuum of different locally convex complete topologies making it a topological algebra.*

Proof. First we show that the multiplication in A_c is jointly continuous. Let $x, y \in P(t)$ and $r \in R(c)$. The formula (2) implies

$$\begin{aligned} |xy|_r &= \sum_i \Big| \sum_j a_{i-j}(x) a_j(y) \Big| r_i \le \sum_{i,j} |a_{i-j}(x)| \cdot |a_j(y)| (r'_{i-j} r'_j + \widetilde{r}_{i-j}\widetilde{r}_j) \\ &= \sum_{i,j} |a_{i-j}(x)| r'_{i-j} |a_j(y)| r'_j + \sum_{i,j} |a_{i-j}(x)| \widetilde{r}_{i-j} |a_j(y)| \widetilde{r}_j \\ &= |x|_{r'} |y|_{r'} + |x|_{\tilde{r}} |y|_{\tilde{r}}, \end{aligned}$$

so that the multiplication in A_c is jointly continuous.

We now show that A_c is complete. Let $(x_\alpha)_{\alpha \in \mathfrak{a}}$ be a Cauchy net in A_c. The continuity of the functionals $x \mapsto a_i(x)$ implies that the limits $a_i = \lim_\alpha a_i(x_\alpha)$ exist and are finite for all i.

First we show that only finitely many numbers a_i can be different from zero. If not, there is an increasing sequence (i_k) of natural numbers such that $a_{i_k} \ne 0$ for all k. For each k there is an m_k such that $i_k \in I_{m_k}$. Take a subsequence $q_n = m_{k_n}$ so that $q = (q_i)$ satisfies (1) and thus it is in $Q(c)$. Since $i_{k_n} \in I_{q_n}$, we can define a sequence r in $R_q(c)$ setting $r_i = 1$

if $i \neq i_{k_n}$ for all n and $r_{i_{k_n}} = \max\{2n/a_{i_{k_n}}, 1\}$. Since $|\cdot|_r$ is a continuous norm in A_c, the limit $M = \lim_\alpha |x_\alpha|_r$ exists and is finite. For each n we have $|a_{i_{k_n}}(x_\alpha)| > a_{i_{k_n}}/2$ for sufficiently large α, say $\alpha \succ \alpha(n)$. Thus for $\alpha \succ \alpha(n)$ we have

$$|x_\alpha|_r \geq |a_{i_{k_n}}(x_\alpha)| r_{i_{k_n}} \geq \frac{|a_{i_{k_n}}|}{2} \cdot \frac{2n}{|a_{i_{k_n}}|} = n,$$

so that $M \geq n$ for all n. This is a nonsense proving that only finitely many numbers a_i can be different from 0.

We now put $y = \sum_i a_i t^i$ and $z_\alpha = x_\alpha - y$. Thus (z_α) is a Cauchy net in A_c and $\lim_\alpha a_i(z_\alpha) = 0$ for all i. The completeness of A will be proved if we show that $\lim_\alpha z_\alpha = 0$ in A, because then $\lim_\alpha x_\alpha = y$. If not, there is an r_0 in $R(c)$ such that $\lim_\alpha |z_\alpha|_{r_0} = M_0 > 0$. Define the support of a polynomial x setting $\mathrm{supp}(x) = \{i : a_i(x) \neq 0\}$ so that $\mathrm{supp}(0) = \emptyset$. Observe that if x and y in A_c have disjoint supports, then for each r in $R(c)$,

$$|x + y|_r = |x|_r + |y|_r. \tag{5}$$

Choose α_0 in $\mathfrak{a}$ so that for $\alpha \succeq \alpha_0$ we have

$$|z_\alpha - z_{\alpha_0}|_{r_0} < M_0/2. \tag{6}$$

Define a projection P setting

$$Px = \sum_{i \in \mathrm{supp}(z_{\alpha_0})} a_i(x) t^i,$$

which is a continuous linear operator on A_c. Denoting by I the identity operator on A_c, we have the following relations true for all x in A_c:

$$\mathrm{supp}(Px) \subset \mathrm{supp}(z_{\alpha_0}) \quad \text{and} \quad \mathrm{supp}(Px) \cap \mathrm{supp}((I - P)x) = \emptyset. \tag{7}$$

Thus by (5) we have

$$|z_\alpha - z_{\alpha_0}|_{r_0} = |Pz_\alpha - z_{\alpha_0} + (I - P)z_\alpha|_{r_0} = |Pz_\alpha - z_{\alpha_0}|_{r_0} + |(I - P)z_\alpha|_{r_0},$$

and so by (6) we obtain

$$|(I - P)z_\alpha|_{r_0} < M_0/2 \quad \text{for } \alpha \succeq \alpha_0. \tag{8}$$

Since $\lim_\alpha a_i(z_\alpha) = 0$ for all i and $\mathrm{supp}(z_{\alpha_0})$ is finite, we have $\lim_\alpha P(z_\alpha) = 0$. Thus by (5), (7) and (8) we obtain

$$\begin{aligned} M_0 &= \lim_\alpha |z_\alpha|_{r_0} = \lim_\alpha |Pz_\alpha + (I - P)z_\alpha|_{r_0} \\ &= \lim_\alpha |Pz_\alpha|_{r_0} + \lim_\alpha |(I - P)z_\alpha|_{r_0} = M_0/2, \end{aligned}$$

a contradiction proving the completeness of A_c.

It remains to be shown that the topologies τ_c, $c \geq 1$, are all different. To this end it is sufficient to show that for given $c' > c \geq 1$ there is a τ_c-continuous norm $|\cdot|_0$ on $P(t)$ which is discontinuous in the topology $\tau_{c'}$.

Put $q_1 = 1$ and $q_{n+1} = [2^{n^c} q_n + 1]$, where $[\lambda]$ is the greatest integer less than or equal to λ. Clearly $q = (q_i)$ is in $Q(c)$ and for all natural n we have

$$q_{n+1} \le 2^{n^c} q_n + 1 < 2^{n^c+1} q_n < 2^{(n+1)^c} q_n. \tag{9}$$

This implies

$$q_{n+k} < 2^{k(n+k)^c} q_n \tag{10}$$

for all natural n and k. In fact, for $k = 1$ the formulas (9) and (10) coincide. Assuming (10) for $k = m$, we obtain, by (9) and (10),

$$q_{n+m+1} < 2^{(n+m+1)^c} q_{n+m} < 2^{(n+m+1)^c + m(n+m)^c} q_n < 2^{(m+1)(n+m+1)^c} q_n,$$

and so (10) follows by induction.

Define $r^{(0)} = (r_i)$ setting $r_i = m$ if $i \in I_{q_m}$ for some m and $r_i = 1$ otherwise. We have $r^{(0)} \in R_q(c)$ and put $|x|_0 = |x|_{r^{(0)}}$ for x in $P(t)$. It is a continuous norm in the topology τ_c. We now show that $|\cdot|_0$ is discontinuous in the topology $\tau_{c'}$. If not, there are a finite number of sequences $r^{(1)}, \ldots, r^{(s)}$ in $R(c')$ with $r^{(i)} \in R_{q^{(i)}}(c')$ and $q^{(i)} = (q_j^{(i)})_{j=0}^{\infty} \in Q(c')$, and a positive constant C such that

$$|x|_0 \le C \max\{|x|_{r^{(1)}}, \ldots, |x|_{r^{(s)}}\} \tag{11}$$

for all x in A. Since $c' > c \ge 1$, we have

$$n^{c'} > s(n+s)^c \tag{12}$$

for sufficiently large n. Suppose that n satisfies (12) and also, in addition, $n > C$, and consider the segment $(l, l+1, \ldots, 2^{n^{c'}} l)$ with $l = q_n$. By (1) this segment contains at most s numbers of the form $q_j^{(i)}$, $1 \le i \le s$, $j = 1, 2, \ldots$ On the other hand, by (10) and (12) we obtain

$$q_{n+i} \le 2^{i(n+i)^c} q_n \le 2^{s(n+s)^c} q_n < 2^{n^{c'}} q_n = 2^{n^{c'}} l \quad \text{for } 0 \le i \le s.$$

Thus the considered segment contains all $s+1$ numbers $q_n, q_{n+1}, \ldots, q_{n+s}$, so that at least one of them, say $q_{n'}$, is different from $q_j^{(i)}$ for all i and j. This implies that for $k \in I_{q_{n'}}$ we have $|t^k|_0 = n' \ge n > C$, contrary to (11), since for all i with $1 \le i \le s$ we have $|t^k|_{r^{(i)}} = 1$. This contradiction proves $\tau_c \ne \tau_{c'}$. The conclusion follows.

The above proof is given in full detail. For the following Propositions 2 and 3, though formally more complicated, we omit some elements of the proofs since they are performed according to the same pattern as the proof of Proposition 1.

First, we consider the case of polynomials in arbitrarily many variables. Denote by $\mathfrak{I}$ a non-void index set of an arbitrary cardinality and let $\mathbf{t} = (t_\alpha)_{\alpha\in\mathfrak{I}}$ be a family of (commuting) variables. Denote by Φ the family of all finitely supported functions ϕ on $\mathfrak{I}$ with non-negative integral values. The support of such a function will be denoted by $\mathrm{Supp}(\phi)$ (to distinguish it from the support of a polynomial introduced in the proof of the previous proposition). For two such functions ϕ and ψ write $\phi \le \psi$ for the pointwise inequality; in this case the function $\psi - \phi$ is also in Φ. Put $\mathbb{W} = \mathbb{N}^{\mathfrak{I}}$, the set of all functions on $\mathfrak{I}$ with natural values, whose elements will be called *weights*. For a weight $w \in \mathbb{W}$ write $S_w(\phi) = \sum_{\alpha \in \mathfrak{I}} \phi(\alpha) w(\alpha)$; for each ϕ in Φ this is a non-negative integer, and clearly $\phi \le \psi$ implies $S_w(\phi) \le S_w(\psi)$ for each w in $\mathbb{W}$. For every $\phi \in \Phi$ put $\mathbf{t}^\phi = \prod_\alpha t_\alpha^{\phi(\alpha)}$, where we put 1 for t_α^0, so that the product is, in fact, well defined and finite. With this notation we can write an arbitrary polynomial in $P(\mathbf{t})$ in the form

$$x(\mathbf{t}) = \sum_\phi a_\phi(x) \mathbf{t}^\phi,$$

where only finitely many scalar coefficients $a_\phi(x)$, $\phi \in \Phi$, are different from zero. The product of two polynomials x and y is given by the formula

$$xy = \sum_\psi \Big(\sum_{\phi \le \psi} a_{\psi-\phi}(x) a_\phi(y) \Big) \mathbf{t}^\psi.$$

Fix a $c \ge 1$. For $r \in R(c)$ and $w \in \mathbb{W}$ write

$$|x|_{r,w} = \sum_\phi |a_\phi(x)| r_{S_w(\phi)}.$$

The topology given on $P(\mathbf{t})$ by all these norms will also be denoted by τ_c. Similarly to Proposition 1 we now prove

PROPOSITION 2. *For every set* $\mathbf{t}$ *of variables the algebra* $P(\mathbf{t})$ *of all polynomials in these variables has a continuum of different complete locally convex topologies making it a topological algebra.*

Proof. Let r, r' and $\widetilde{r}$ be elements in $R(c)$ satisfying the lemma and let $w \in \mathbb{W}$. As in the proof of Proposition 1, we obtain

$$|xy|_{r,w} \le |x|_{r',w} |y|_{r',w} + |x|_{\tilde{r},w} |y|_{\tilde{r},w},$$

so that the multiplication in A_c, i.e. in $P(\mathbf{t})$ equipped with the topology τ_c, is jointly continuous.

We now prove that A_c is complete. Suppose that (x_μ) is a Cauchy net in A_c. As in Proposition 1, we show that only finitely many numbers $a_\phi = \lim_\mu a_\phi(x_\mu)$ can be different from zero. If not, there is a sequence $(\phi_i) \subset \Phi$ with $a_i = a_{\phi_i} \ne 0$. Consider now two cases: either the set $\bigcup_i \mathrm{Supp}(\phi_i)$ is finite or not. In the first case the sequence (ϕ_i) is unbounded and taking $w = w_0$, the weight identically equal to 1, we have

$$\lim_i S_w(\phi_i) = \infty. \tag{13}$$

In the second case, passing to a subsequence if necessary, we can assume that for each natural i there is $\alpha_i \in \mathfrak{I}$ such that

$$\alpha_i \in \mathrm{Supp}(\phi_i) \setminus \bigcup_{j<i} \mathrm{Supp}(\phi_j) \tag{14}$$

and define a weight w inductively: having already the values $w(\alpha)$ for $\alpha \in \bigcup_{j<i} \mathrm{Supp}(\phi_j)$ we put $w(\beta) = 1$ if β is in the set (14) and $\beta \neq \alpha_i$ and take $w(\alpha_i)$ so large that $S_w(\phi_i) > i$. Finally, we put $w(\alpha) = 1$ for all $\alpha \notin \bigcup_i \mathrm{Supp}(\phi_i)$. With this choice of w the formula (13) holds true also in this case. We now proceed as in the proof of Proposition 1 to obtain a contradiction proving that only finitely many numbers a_ϕ are different from zero. The proof of completeness of A_c is now performed exactly as in Proposition 1 (the formula (5) obviously holds true if we replace $|\cdot|_r$ by $|\cdot|_{r,w}$).

We shall be done if we prove that all topologies τ_c are different. To this end it is sufficient to show that the present topology τ_c equals on each algebra $P(t_\alpha)$, $\alpha \in \mathfrak{I}$, the topology τ_c', the latter being the topology τ_c in the sense of Proposition 1. Since every norm $|\cdot|_r$ giving on $P(t_\alpha)$ the topology τ_c' is a restriction of the norm $|\cdot|_{r,w_0}$ to this subalgebra, and so it is continuous in the topology τ_c, we only have to show that every norm $|\cdot|_{r,w}$, $r \in R(c)$, $w \in \mathbb{W}$, restricted to $P(t_\alpha)$ is continuous in the topology τ_c'.

So fix $r \in R(c)$, $w \in \mathbb{W}$ and $\alpha \in \mathfrak{I}$. Put $m = w(\alpha)$. For any $x \in P(t_\alpha)$ with $x = \sum_i a_i t_\alpha^i$, we have $|x|_{r,w} = \sum_i |a_i| r_{mi}$. If $m = 1$ this is already a τ_c'-continuous norm. So consider only the case $m > 1$ and define a natural s so that $2^{s-1} < m \leq 2^s$. If $mi \in I_p$, so that $2^{p-1} < mi \leq 2^p$, then $2^{p-s-1} < i < 2^{p-s+1}$. It follows that either $i \in I_{p-s+1}$ or $i \in I_{p-s}$. We have $r \in R_q(c)$ for some q in $Q(c)$. Iterating l times the process $q \mapsto \widetilde{q}$ and denoting the result by $\widetilde{q}^{(l)}$, observe that for large i (say $i > i_0$) we can have $r_{mi} > 1$ only if $mi \in I_{q_k}$ for some k, or, equivalently, if either $i \in I_{\tilde{q}_k^{(s-1)}}$ or $i \in I_{\tilde{q}_k^{(s)}}$. Thus setting $\varrho_i = r_{mi}$ for $i \leq i_0$ or for $i > i_0$ and $i \in I_{\tilde{q}_k^s}$ for some k, and $\varrho_i = 1$ otherwise, and setting $\varrho_i' = r_{mi}$ for $i \in I_{\tilde{q}_k^{(s-1)}}$ for some k, $i > i_0$, and $\varrho_i' = 1$ otherwise, we see that $|x|_{r,w} \leq |x|_\varrho + |x|_{\varrho'}$ for all x in $P(t_\alpha)$. Since the norms $|\cdot|_\varrho$ and $|\cdot|_{\varrho'}$ are τ_c'-continuous, both topologies τ_c' and τ_c coincide on $P(t_\alpha)$ (this justifies our previous notation). The conclusion follows.

Consider now the case of a free algebra (algebra of polynomials in non-commuting variables). Let $\mathbf{t} = \{t_\alpha\}_{\alpha \in \mathfrak{I}}$ be an arbitrary family of non-commuting variables and denote the corresponding free algebra by $F(\mathbf{t})$. Put $\mathfrak{I}^{(\infty)} = \bigcup_{n=0}^\infty \mathfrak{I}^n$, where $\mathfrak{I}^0 = \{0\}$ and 0 is not an element in $\mathfrak{I}$. We put $\mathbf{t}^\mathbf{i} = t_{\alpha_1} \ldots t_{\alpha_k}$ for $\mathbf{i} = (\alpha_1, \ldots, \alpha_k)$ in $\mathfrak{I}$, and $\mathbf{t}^0 = e$, the unit element of $F(\mathbf{t})$. With this notation every element of $F(\mathbf{t})$ can be written in the form

$$x = \sum_{\mathbf{i} \in \mathfrak{I}^{(\infty)}} a_{\mathbf{i}}(x) \mathbf{t}^{\mathbf{i}},$$

where only finitely many scalar coefficients $a_{\mathbf{i}}(x)$ are different from zero. Writing $\mathbf{ij} = (\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_n)$ if $\mathbf{i} = (\alpha_1, \ldots, \alpha_k)$ and $\mathbf{j} = (\beta_1, \ldots, \beta_n)$, and $\mathbf{i}0 = 0\mathbf{i} = \mathbf{i}$ for $\mathbf{i}, \mathbf{j} \in \mathfrak{I}^{\infty}$, we define the multiplication in $F(\mathbf{t})$ by the formula

$$xy = \sum_{\mathbf{k} \in \mathfrak{I}^{(\infty)}} \Big(\sum_{\mathbf{ij} = \mathbf{k}} a_{\mathbf{i}}(x) a_{\mathbf{j}}(y) \Big) \mathbf{t}^{\mathbf{k}}.$$

We define the topology τ_c on $F(\mathbf{t})$ by means of all norms of the form

$$|x|_{r,w} = \sum_{\mathbf{i}} |a_{\mathbf{i}}(x)| r_{|\mathbf{i}|_w}$$

for $r \in R(c)$ and $w \in \mathbb{W}$, where $|\mathbf{i}|_w = \sum_i w(\alpha_i)$ or 0 according as $\mathbf{i}$ equals $(\alpha_1, \ldots, \alpha_k)$ or 0.

Exactly in the same way as Proposition 2 we obtain

PROPOSITION 3. *Let $F(\mathbf{t})$ be a free algebra in arbitrarily many variables. Then there is a continuum of different complete locally convex topologies making it a topological algebra.*

A locally convex algebra A is said to be *locally multiplicatively-convex* (briefly m-convex) if its topology can be given by means of a family of submultiplicative seminorms, i.e. seminorms satisfying $|xy| \le |x| \cdot |y|$ for all $x, y \in A$ (see [3]). Clearly any m-convex algebra is topological. A commutative algebra is said to be *semisimple* if the intersection of all of its maximal regular ideals is the zero ideal. In [1] Carpenter has shown that a semisimple complex algebra A has at most one topology making it a completely metrizable m-convex algebra. In contrast, we now show that for some semisimple algebra there may exist a continuum of different topologies making it a complete m-convex algebra, of course in a non-metrizable way. Let Ω be an infinite set. Denote by $C_{00}(\Omega)$ the algebra of all (real of complex) finitely supported functions on Ω. It is clearly a semisimple algebra, since the set M_ω of all elements of $C_{00}(\Omega)$ vanishing at a single point $\omega \in \Omega$ is a maximal ideal in this algebra, and the intersection of all these ideals is the zero ideal. Using the construction of Propositions 1 and 2 we obtain the following

PROPOSITION 4. *Let Ω be an infinite set. Then the algebra $C_{00}(\Omega)$ has at least a continuum of different topologies making it a complete m-convex algebra.*

Proof. We use the notation of Proposition 2. For every infinite set $\mathfrak{I}$ the set Φ has the same cardinality as $\mathfrak{I}$, so that any infinite set Ω can be identified with some set of the form Φ. The elements of $P(\mathbf{t})$ can be treated as finitely supported functions $\phi \mapsto a_\phi = a(\omega)$ for $\phi = \omega \in \Phi = \Omega$.

Thus $C_{00}(\Omega)$ has a continuum of different topologies τ_c, $c \geq 1$, making it a complete locally convex t.v.s.

We show that each of these topologies makes $C_{00}(\Omega)$, provided with pointwise algebra operations, an m-convex algebra. To this end put $|a|_\infty = \max\{|a(\omega)| : \omega \in \Omega\}$. This is clearly a τ_c-continuous norm for every $c \geq 1$ since $|a|_\infty \leq |a|_{r,w}$ for all $r \in R(c)$, $c \geq 1$, and $w \in \mathbb{W}$. Using the last relation we obtain for all $a, b \in C_{00}(\Omega)$, $r \in R(c)$ and $w \in \mathbb{W}$,

$$\begin{aligned} |ab|_{r,w} &= \sum_{\omega \in \Omega} |a(\omega)b(\omega)| r_{S_w(\omega)} \leq |a|_\infty \sum_{\omega} |b(\omega)| r_{S_w(\omega)} \\ &= |a|_\infty |b|_{r,w} \leq |a|_{r,w} |b|_{r,w}, \end{aligned}$$

and so each topology τ_c makes $C_{00}(\Omega)$ a complete m-convex algebra. The conclusion follows.

Remark. The fact that non-metrizable semisimple algebras can have different m-convex topologies is already known. In [3] there is given a non-metrizable complete m-convex topology on the algebra $C[0,1]$.

We now pass to semitopological algebras. Our next observation solves a problem posed in [7].

Proposition 5. *Every infinite-dimensional algebra A has at least two different locally convex topologies making it a semitopological algebra.*

Proof. For one topology we can take the topology $\tau_{\max}^{\mathrm{LC}}$ and for the second the maximal weak topology $\tau_{\max}^{\mathrm{w}}$ given by means of all seminorms of the form $x \mapsto |f(x)|$, where f is an arbitrary linear functional on A. Clearly both topologies make the multiplication separately continuous and they are different. The conclusion follows.

It is not hard to observe that for any infinite-dimensional vector space the topology $\tau_{\max}^{\mathrm{w}}$ is never complete. In fact, we do not know the answer to the following question.

Problem 1. Let A be a real or complex algebra. Can it be made into a complete locally convex semitopological algebra in two different ways?

We cannot even answer the following weaker question.

Problem 2. Let A be a real or complex algebra. Can it be made into a complete semitopological algebra in two different ways?

As mentioned in the introduction, this question is open only for at most countably generated algebras. Perhaps the problem should be first attacked for algebras whose elements are all algebraic (cf. [6]). We do not know the answer for Problem 1 for arbitrarily generated algebras. Of course, for more general classes of algebras the construction of counterexamples is easier, so perhaps solving (in the affirmative) the following more restrictive questions 3–6 still will not be too hard.

PROBLEM 3. Let A be an infinite-dimensional topological algebra. Can it be made into a topological algebra by means of a different topology?

PROBLEM 4. Let A be an infinite-dimensional complete topological algebra. Can it be made into a complete topological algebra with a different topology?

PROBLEM 5. Let A be an infinite-dimensional locally convex topological algebra. Can it be made into such an algebra with a different topology?

PROBLEM 6. This is Problem 5 but with "locally convex" replaced by "complete locally convex".

In case when the answers to Problems 1, 5 or 6 are negative we can ask the following question.

PROBLEM 7. Suppose that an algebra A has a unique locally convex topology making it a topological (resp. complete topological or complete semitopological) algebra. Does it follow that it has a unique topology making it a topological (resp. complete topological or complete semitopological) algebra?

Finally, we ask two questions concerning cardinalities of topologies on an algebra.

PROBLEM 8. Consider an algebra of all polynomials in a given number of variables. How many different topologies make it a complete topological algebra?

The answer should provide a function leading from the cardinality of the set of variables to the cardinality of the set of topologies. A similar question can be asked about the cardinality of locally convex topologies. One can also consider the case of incomplete topologies and of topologies making the algebra in question a semitopological algebra.

The following question extends Problem 7.

PROBLEM 9. Does there exist an algebra for which the cardinality of the set of all topologies making it a complete topological algebra (resp. a topological algebra, a complete semitopological algebra, a semitopological algebra) is larger than the cardinality of the set of all such locally convex topologies?

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