COLLOQUIUM MATHEMATICUM

VOL. 73 1997 NO. 1

# ON THE NUMBERS OF DISCRETE INDECOMPOSABLE MODULES OVER TAME ALGEBRAS

BY

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**1. Introduction.** Throughout the paper $K$ denotes a fixed algebraically closed field. By an *algebra* we mean an associative finite-dimensional $K$-algebra with identity which we shall assume (without loss of generality) to be basic and connected. For an algebra $A$, by an $A$-*module* we mean a finitely generated right $A$-module. We shall denote by $\operatorname{mod} A$ the category of $A$-modules, by $\operatorname{ind} A$ its full subcategory formed by the indecomposable modules, by $\Gamma_A$ the Auslander–Reiten quiver of $A$, and by $\tau_A$ the Auslander–Reiten translation $D\mathrm{Tr}$ in $\Gamma_A$. We shall identify an indecomposable $A$-module with the vertex of $\Gamma_A$ corresponding to it.

It follows from a well-known result of Yu. Drozd [11] that the class of algebras may be divided into two disjoint classes. One class consists of the wild algebras, whose representation theory is as complicated as the study of finite-dimensional vector spaces together with two non-commuting endomorphisms, for which the classification of indecomposable modules is a known unsolved problem. The second class is formed by the tame algebras, for which the indecomposable modules occur, in each dimension $d$, in a finite number of discrete and a finite number of one-parameter families. Moreover, it has been shown by W. W. Crawley-Boevey [10] that, if $A$ is a tame algebra, then, for any $d \geq 1$, all but finitely many isomorphism classes of indecomposable $A$-modules of dimension $d$ are invariant under the action of $\tau_A = D\mathrm{Tr}$, and hence by a result due to M. Hoshino [13] lie in stable tubes of rank 1 (in $\Gamma_A$). Indecomposable modules over tame algebras which do not lie in stable tubes of rank 1 are said to be *discrete*.

In this article we are interested in the numbers of isomorphism classes of discrete indecomposable modules over tame algebras having the same (simple) composition factors. Recently, tame strongly simply connected algebras are extensively investigated. In particular, in [30] (see also [28]) a criterion for a strongly simply connected algebra to be of polynomial growth has been

1991 *Mathematics Subject Classification*: 16G60, 16G70.

Supported by Polish Scientific Grant KBN No. 2PO3A 020 08.

established. Recall that an algebra $A$ is said to be of *polynomial growth* if there is a positive integer $m$ such that the indecomposable $A$-modules occur, in each dimension $d$, in a finite number of discrete and at most $d^m$ one-parameter families. We shall prove here that a strongly simply connected algebra $A$ is of polynomial growth if and only if there is a common bound on the number of isomorphism classes of discrete indecomposable $A$-modules with any fixed composition factors. In the paper we consider also the following related problem. It follows from the mentioned result by W. W. Crawley-Boevey that any connected component of the Auslander–Reiten quiver $\Gamma_A$ of a tame algebra $A$ has only finitely many indecomposable modules with any fixed composition factors. This is also the case for the connected components of the Auslander–Reiten quivers of wild hereditary algebras [19], [31]. It would be interesting to know when, for a connected component $\mathcal{C}$ of an Auslander–Reiten quiver $\Gamma_A$, there is a common bound on the numbers of indecomposable modules in $\mathcal{C}$ having the same composition factors. We prove that it is true if $\mathcal{C}$ is generalized standard in the sense of [25]. We also show tame algebras (pg-critical algebras of [15]) whose Auslander–Reiten quiver admits a connected component containing arbitrary large numbers of indecomposable modules with the same composition factors.

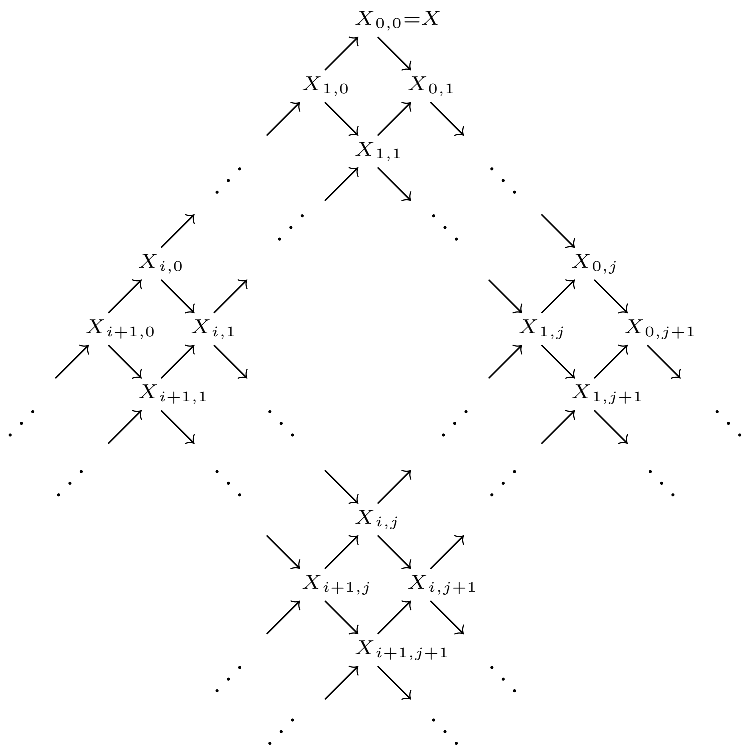
**2. Generalized standard components.** For an algebra $A$, we denote by $\operatorname{rad}(\operatorname{mod} A)$ the Jacobson radical of the category $\operatorname{mod} A$ and by $\operatorname{rad}^\infty(\operatorname{mod} A)$ the intersection of all powers $\operatorname{rad}^i(\operatorname{mod} A)$, $i \geq 1$, of $\operatorname{rad}(\operatorname{mod} A)$. Following [25], a connected component $\mathcal{C}$ of $\Gamma_A$ is said to be *generalized standard* if $\operatorname{rad}^\infty(X, Y) = 0$ for any modules $X$ and $Y$ from $\mathcal{C}$. Moreover, a component $\Gamma$ of $\Gamma_A$ is called *standard* ([9], [21]) if the full subcategory of $\operatorname{ind} A$ given by modules from $\Gamma$ is equivalent to the mesh-category $K(\Gamma)$ of $\Gamma$. It has been proved by S. Liu [14] that any standard component is generalized standard but the converse is not true. For the structure of generalized standard components without oriented cycles we refer to [24]. The structure of arbitrary generalized standard components is not known. It has been proved by the first named author in [25] that if $\mathcal{C}$ is a generalized standard component of $\Gamma_A$ then all but finitely many $\tau_A$-orbits are periodic, and hence $\mathcal{C}$ admits at most finitely many modules of any fixed dimension. We note also that if all components of $\Gamma_A$ are generalized standard then $A$ is tame [29, (2.8)].

Given an algebra $A$ we denote by $K_0(A)$ the Grothendieck group of $A$. It is well known that $K_0(A) \simeq \mathbb{Z}^n$, where $n$ is the number of isomorphism classes of simple $A$-modules. For an $A$-module $M$ we denote by $[M]$ the image of $M$ in $K_0(A)$. Thus $[M] = [N]$ if and only if the modules $M$ and $N$ have the same composition factors including the multiplicities. We may ask when two modules $M$ and $N$ have the same composition factors.

The main aim of this section is to prove the following theorem.

THEOREM 1. *Let $A$ be an algebra and $\mathcal{C}$ be a generalized standard component of $\Gamma_A$. Then there is a positive integer $m$ such that, for each vector $x \in K_0(A)$, the number of modules $X$ in $\mathcal{C}$ with $[X] = x$ is bounded by $m$.*

For the proof of Theorem 1 we need the following concept. By a *proper subtube* of an Auslander–Reiten quiver $\Gamma_A$ we mean a full translation subquiver $\mathcal{T}(X,p,q)$, $p,q \geq 1$, of $\Gamma_A$ obtained from a translation quiver $\mathcal{T}(X)$ of the form



with the set of vertices $X_{r,s}$, $r,s \geq 0$, the set of arrows $X_{r+1,s} \to X_{r,s}$, $X_{r,s} \to X_{r,s+1}$, and the translation $\tau$ defined on $X_{r,s}$, $r \geq 0$, $s \geq 1$, by $\tau(X_{r,s}) = X_{r+1,s-1}$, by identifying the vertices $X_{i+p,j}$ and $X_{i,j+q}$ for all $i,j \geq 0$. Observe that then

$$\{X_{i,j} : i \geq 0,\ 0 \leq j < q\} = \{X_{i,j} :\ 0 \leq i < p,\ j \geq 0\}$$

is a complete set of pairwise different vertices of $\mathcal{T}(X,p,q)$.

PROPOSITION 2. *Let $A$ be an algebra and $\mathcal{T} = \mathcal{T}(X,p,q)$ a proper subtube of $\Gamma_A$. Then, for each vector $z \in K_0(A)$, the number of modules $Z$ in $\mathcal{T}$ with $[Z] = z$ is bounded by $pq$.*

Proof. We use the notation introduced above. We shall prove that $[X_{p,0}] - [X_{0,0}] > 0$ and for $i = mp + r$, $m \geq 0$, $0 \leq r < p$, $0 \leq j < q$, the following equality holds:

$$[X_{i,j}] = [X_{r,j}] + m([X_{p,0}] - [X_{0,0}]).$$

Then, since any module in $\mathcal{T}$ is of the form $X_{i,j}$, $i \geq 0$, $0 \leq j < q$, for any given $z \in K_0(A)$ the number of indecomposable modules $Z$ in $\mathcal{T}$ with $[Z] = z$ is bounded by $pq$. Observe that for any mesh

$$\begin{tikzcd} & X_{s,t} \arrow[dr] & \\ X_{s+1,t} \arrow[ur] \arrow[dr] & & X_{s,t+1} \\ & X_{s+1,t+1} \arrow[ur] & \end{tikzcd}$$

in $\mathcal{T}$, $s, t \geq 0$, we have an Auslander–Reiten sequence

$$0 \to X_{s+1,t} \to X_{s,t} \oplus X_{s+1,t+1} \to X_{s,t+1} \to 0,$$

and hence $[X_{s+1,t+1}] + [X_{s,t}] = [X_{s+1,t}] + [X_{s,t+1}]$. Let $l \geq 1$. From the rectangle in $\mathcal{T}$ given by the modules $X_{lp,0}$, $X_{lp,q}$, $X_{0,0}$ and $X_{0,q}$ we get $[X_{lp,0}] + [X_{0,q}] = [X_{lp,q}] + [X_{0,0}]$. Since $X_{0,q} = X_{p,0}$ and $X_{lp,q} = X_{(l+1)p,0}$ this gives $[X_{(l+1)p,0}] - [X_{lp,0}] = [X_{p,0}] - [X_{0,0}]$. By induction we infer that

$$[X_{mp,0}] = m([X_{p,0}] - [X_{0,0}]) + [X_{0,0}]$$

for any $m \geq 0$. In particular, $[X_{p,0}] - [X_{0,0}] \geq 0$. Suppose that $[X_{p,0}] - [X_{0,0}] = 0$. Then $[X_{mp,0}] = [X_{0,0}]$ for any $m \geq 1$. Choose now irreducible maps $f_i : X_{i+1,0} \to X_{i,0}$, $i \geq 0$, and put $g_m = f_{mp-1} \circ \ldots \circ f_{(m-1)p}$ for any $m \geq 1$. Thus we get the family of maps

$$\cdots \longrightarrow X_{mp,0} \xrightarrow{g_m} X_{(m-1)p,0} \longrightarrow \cdots \longrightarrow X_{2p,0} \xrightarrow{g_2} X_{p,0} \xrightarrow{g_1} X_{0,0}.$$

Since the morphisms $f_i$, $i \geq 0$, form a sectional path in $\Gamma_A$, we conclude (see [7, VII.2.4]) that, for each $m \geq 1$, the composition $g_m \ldots g_1$ is non-zero. This contradicts the lemma of Harada and Sai [12] (see also [20]), because the modules $X_{mp,0}$, $m \geq 0$, have the same dimension $d = \dim_K X_{0,0}$. Therefore, $[X_{p,0}]-[X_{0,0}] > 0$. Finally, take arbitrary $i \geq 0$, $0 \leq j < q$, and let $i = mp+r$ for $m \geq 0$, $0 \leq r < p$. From the rectangle in $\mathcal{T}$ given by $X_{i,0}$, $X_{r,0}$, $X_{i,j}$ and $X_{r,j}$ we have $[X_{i,0}]+[X_{r,j}] = [X_{i,j}]+[X_{r,0}]$. Further, from the rectangle given by $X_{r,0}$, $X_{0,0}$, $X_{r,mq}$ and $X_{0,mq}$ we get $[X_{r,0}] + [X_{0,mq}] = [X_{r,mq}] + [X_{0,0}]$. Hence, we obtain the equalities

$$\begin{aligned} [X_{i,j}] &= [X_{i,0}] + [X_{r,j}] - [X_{r,0}] = [X_{r,j}] + [X_{mp+r,0}] - [X_{r,0}] \\ &= [X_{r,j}] + [X_{r,mq}] - [X_{r,0}] = [X_{r,j}] + [X_{0,mq}] - [X_{0,0}] \\ &= [X_{r,j}] + [X_{mp,0}] - [X_{0,0}] = [X_{r,j}] + m([X_{p,0}] - [X_{0,0}]). \end{aligned}$$

This finishes the proof.

P r o o f  o f  T h e o r e m 1. Assume that $M$ is an indecomposable module in $\mathcal{C}$ which does not lie on an oriented cycle (in $\mathcal{C}$). We claim that then $M$ is uniquely determined by $[M]$. We shall use arguments similar to that in [18, (2.1)]. Let $N$ be an indecomposable module in $\mathcal{C}$ such that $[M] = [N]$. Let

$$P_1 \to P_0 \to M \to 0$$

be a minimal projective presentation of $M$ and

$$0 \to M \to I_0 \to I_1$$

a minimal injective copresentation of $M$ in $\operatorname{mod} A$. Then for any $A$-module $X$ we have, by [6, (1.4)], the following equalities:

$$[M, X] - [X, \tau_A M] = [P_0, X] - [P_1, X],$$
$$[X, M] - [\tau_A^- M, X] = [X, I_0] - [X, I_1],$$

where we abbreviate $\dim_K \operatorname{Hom}_A(X, Y)$ by $[X, Y]$. Since $[M] = [N]$, we have $[P_0, M] = [P_0, N]$, $[P_1, M] = [P_1, N]$, $[M, I_0] = [N, I_0]$, and $[M, I_1] = [N, I_1]$. Letting $X = M$ and $X = N$ we get the equalities

$$[M, M] - [M, \tau_A M] = [M, N] - [N, \tau_A M],$$
$$[M, M] - [\tau_A^- M, M] = [N, M] - [\tau_A^- M, N].$$

Since $\mathcal{C}$ is generalized standard and $M$ does not lie on an oriented cycle in $\mathcal{C}$ we get $[M, \tau_A M] = 0$ and $[\tau_A^- M, M] = 0$. Hence $[M, N] - [N, \tau_A M] = [M, M] > 0$ and $[N, M] - [\tau_A^- M, N] = [M, M] > 0$, and consequently $[M, N] \neq 0$ and $[N, M] \neq 0$. By our assumption on $\mathcal{C}$ we have $\operatorname{rad}^\infty(M, N) = 0$ and $\operatorname{rad}^\infty(N, M) = 0$. Now, if $M \not\simeq N$ then $\operatorname{rad}(M, N) \neq 0$, $\operatorname{rad}(N, M) \neq 0$ and we infer that $\mathcal{C}$ contains an oriented cycle passing through $M$ and $N$, a contradiction. Therefore, $M \simeq N$.

Now since $\mathcal{C}$ is generalized standard, we know from [25, (2.3)] that $\mathcal{C}$ admits at most finitely many nonperiodic $\tau_A$-orbits. Then there is a finite family $\mathcal{T}_1 = \mathcal{T}(X_1, p_1, q_1), \ldots, \mathcal{T}_r = \mathcal{T}(X_r, p_r, q_r)$ of pairwise disjoint subtubes of $\mathcal{C}$ such that all but finitely many modules lying on oriented cycles in $\mathcal{C}$ belong to the sum $\mathcal{T}_1 \cup \ldots \cup \mathcal{T}_r$ (see [32, (3.6)] for a detailed proof). Then, by Proposition 2, there is a positive integer $m$ such that for each $x \in K_0(A)$, the number of modules $X$ in $\mathcal{C}$ with $[X] = x$ is bounded by $m$.

**3. Components with unbounded numbers of discrete modules.** In this section we shall exhibit a class of components which occur in the Auslander–Reiten quivers of tame algebras and have arbitrary large numbers of indecomposable modules with the same composition factors.

The *one-point* extension of an algebra $B$ by a $B$-module $M$ is the algebra

$$B[M] = \begin{bmatrix} K & M \\ 0 & B \end{bmatrix}$$

with the usual addition and multiplication of matrices. The quiver of $B[M]$ contains the quiver of $B$ as a full convex subquiver and there is an additional (extension) vertex which is a source. The $B[M]$-modules are usually identified with the triples $(V, X, \varphi)$, where $V$ is a $K$-vector space, $X$ is a $B$-module and $\varphi : V \to \mathrm{Hom}_B(M, X)$ is a $K$-linear map. A $B[M]$-linear map $(V, X, \varphi) \to (V', X', \varphi')$ is thus a pair $(f, g)$, where $f : V \to V'$ is $K$-linear and $g : X \to X'$ is $B$-linear such that $\varphi' f = \mathrm{Hom}_B(M, g)\varphi$. One defines dually the *one-point coextension* $[M]B$ of $B$ by $M$.

Let $B$ be an algebra and $\Gamma$ a generalized standard component of $\Gamma_B$ and $X$ a $B$-module from $\Gamma$. Denote by $\mathcal{H}_X$ the full subcategory of $\mathrm{ind}\, B$ formed by the indecomposable modules $Z$ in $\Gamma$ such that $\mathrm{Hom}_B(X, Z) \neq 0$, and by $\mathcal{I}_X$ the ideal of $\mathcal{H}_X$ consisting of morphisms $f : Y \to Z$ such that $\mathrm{Hom}_B(X, f) = 0$. Then the quotient category $\mathcal{S}(X) = \mathcal{H}_X/\mathcal{I}_X$ is said to be the *support* of the functor $\mathrm{Hom}_B(X, -)|_\Gamma$. We usually identify the $K$-linear category $\mathcal{S}(X)$ with its quiver.

PROPOSITION 3. *Let $B$ be an algebra, $\Gamma$ a generalized standard component of $\Gamma_B$, and $\mathcal{T}$ a proper subtube of $\Gamma$. Assume that $X$ is an indecomposable module in $\Gamma$ satisfying the following conditions*:

(i) *$\mathcal{S}(X)$ is given by two parallel infinite sectional paths*

$$\begin{tikzcd}
 & Y_1 \arrow[r] & Y_2 \arrow[r] & \cdots \arrow[r] & Y_{i+1} \arrow[r] & Y_{i+2} \arrow[r] & \cdots \\
X = X_0 \arrow[ur] \arrow[r] & X_1 \arrow[ur] \arrow[r] & \cdots \arrow[r] & X_i \arrow[ur] \arrow[r] & X_{i+1} \arrow[ur] \arrow[r] & \cdots &
\end{tikzcd}$$

*formed by pairwise different modules.*

(ii) *$\mathcal{T}$ contains all but finitely many modules of $\mathcal{S}(X)$.*

*Let $A = B[X]$ and $\mathcal{C}$ be the component of $\Gamma_A$ containing $X$. Then, for any positive integer $r$, there exists a vector $x \in K_0(A)$ such that $\mathcal{C}$ admits $r$ pairwise different modules $M_1, \ldots, M_r$ with $[M_i] = x$ and $M_i \not\cong \tau_A M_i$ for all $1 \leq i \leq r$.*

Proof. We may choose irreducible maps $f_i : X_i \to X_{i+1}$, $g_i : X_i \to Y_{i+1}$, $h_{i+1} : Y_{i+1} \to Y_{i+2}$, $i \geq 0$, such that $h_{i+1}g_i = g_{i+1}f_i$ for any $i \geq 0$. Hence, $\mathrm{Hom}_B(X, X_i)$, $i \geq 0$, and $\mathrm{Hom}_B(X, Y_j)$, $j \geq 1$, are one-dimensional $K$-vector spaces generated by $u_0 = 1_X$, $u_i = f_{i-1} \ldots f_0$, for $i \geq 1$, and $v_1 = g_0$, $v_j = h_{j-1} \ldots h_1 g_0$, $j \geq 2$, respectively. Moreover, $\mathrm{Hom}_B(X, \varphi) = 0$ for any $\varphi : Y_j \to X_i$, $j \geq 1$, $i \geq 0$, and $\mathrm{Hom}_B(X, \psi) = 0$ for any $\psi : X_i \to Y_j$

with $1 \le j \le i$. Then we get the following indecomposable $A$-modules:

$$Z_{i,j} = (K, X_i \oplus Y_j, \Delta_{i,j}), \qquad 1 \le j \le i,$$

where

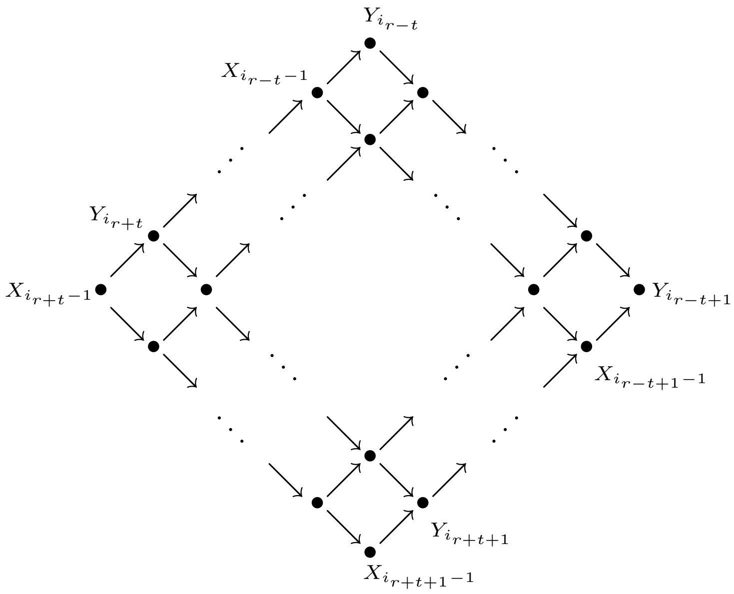
$$\Delta_{i,j} : K \to \mathrm{Hom}_B(X, X_i \oplus Y_j) = \mathrm{Hom}_B(X, X_i) \oplus \mathrm{Hom}_B(X, Y_j)$$

assigns to $1 \in K$ the pair $(u_i, v_j)$. Consider also the indecomposable $A$-modules $X'_i = (K, X_i, \eta_i)$, $i \ge 0$, where $\eta_i(1) = u_i$ for each $i \ge 0$. Observe that $X'_0$ is the (new) indecomposable projective $A$-module whose radical is $X$. Applying now [21, (2.5)] and calculating the corresponding cokernels, we infer that $\mathcal{C}$ admits a full translation subquiver $\mathcal{D}$ of the form

$$\begin{tikzcd}[column sep=small, row sep=small]
 & Y_1 \arrow[dr] & & X'_1 \arrow[dr] & & Y_3 \arrow[dr] & & \cdots & \\
X=X_0 \arrow[ur] \arrow[r] \arrow[dr] & X'_0 \arrow[r] & Z_{1,1} \arrow[ur] \arrow[r] \arrow[dr] & Y_2 \arrow[r] & Z_{2,2} \arrow[ur] \arrow[r] \arrow[dr] & X'_2 \arrow[r] & Z_{3,3} \arrow[ur] \arrow[r] \arrow[dr] & \cdots & \\
 & X_1 \arrow[ur] \arrow[dr] & & Z_{2,1} \arrow[ur] \arrow[dr] & & Z_{3,2} \arrow[ur] \arrow[dr] & & & \ddots \\
 & & X_2 \arrow[ur] \arrow[dr] & & Z_{3,1} \arrow[ur] \arrow[dr] & & & \ddots & \\
 & & & X_3 \arrow[ur] \arrow[dr] & & & \ddots & & \\
 & & & & \ddots & & & &
\end{tikzcd}$$

formed by the modules $X_i$, $X'_i$, $i \ge 0$, $Y_j$, $j \ge 1$, and $Z_{i,j}$, $1 \le j \le i$, which is closed under successors in $\Gamma_A$.

We shall find the required modules $M_1, \ldots, M_r$ (with the same composition factors) among the modules $Z_{i,j}$ in $\mathcal{D}$. Denote by $\Sigma$ the infinite sectional path formed by the modules $X_i$, $i \ge 0$, and by $\Omega$ the infinite sectional path (in $\Gamma$) consisting of the modules $Y_j$, $j \ge 1$. Let $m \ge 1$ be such that all modules $X_{j-1}$, $Y_j$, $j \ge m$, belong to the subtube $\mathcal{T}$. Without loss of generality, we may assume that $\mathcal{T} = \mathcal{T}(Y, p, q)$ for $Y = Y_m$ and some $p \ge 2$, $q \ge 1$. Denote by $\Theta$ the infinite sectional path in $\mathcal{T}$ with target $Y_m$. Then there exists a sequence $m = i_1 < i_2 < \ldots$ such that $\Omega \cap \Theta$ consists of the modules $Y_{i_1}, Y_{i_2}, \ldots,$ and $\Sigma \cap \Theta$ consists of the modules $X_{i_1-1}, X_{i_2-1}, \ldots$ Finally, for any fixed $r \ge 1$, consider the indecomposable $A$-modules $M_t = Z_{i_{r+t}-1, i_{r-t+1}}$, $1 \le t \le r$. Clearly, the modules $M_1, \ldots, M_r$ are pairwise nonisomorphic. We shall show that they have the same composition factors. It is enough to prove that the $B$-modules $X_{i_{r+t}-1} \oplus Y_{i_{r-t+1}}$, $1 \le t \le r$, have the same composition factors. We may assume $r \ge 2$. Take $1 \le t < r$. Observe that we have in $\mathcal{T}$ a rectangle




Hence, we get

$$\begin{aligned}[X_{i_{r+(t+1)}-1}\oplus Y_{i_{r-(t+1)+1}}] &= [X_{i_{r+t+1}-1}]+[Y_{i_{r-t}}]\\ &= [X_{i_{r+t}-1}]+[Y_{i_{r-t+1}}]=[X_{i_{r+t}-1}\oplus Y_{i_{r-t+1}}].\end{aligned}$$

This shows that the modules $X_{i_{r+t}-1}\oplus Y_{i_{r-t+1}}$, and hence the modules $M_t$, $1\le t\le r$, have the same composition factors. This finishes the proof.

In the representation theory of tame simply connected algebras an important role is played by the polynomial growth critical algebras introduced and investigated by R. Nörenberg and A. Skowroński in [15]. By a *polynomial growth critical algebra* (*briefly pg-critical algebra*) we mean an algebra $A$ satisfying the following conditions:

(i) $A$ is one of the matrix algebras

$$B[X]=\begin{bmatrix} K & X\\ 0 & B\end{bmatrix},\quad B[Y,t]=\begin{bmatrix} K & K & \dots & K & K & K & Y\\ & K & \dots & K & K & K & 0\\ & & \ddots & \vdots & \vdots & \vdots & \vdots\\ & & & K & K & K & 0\\ & & & & K & 0 & 0\\ & \mathbf{0} & & & & K & 0\\ & & & & & & B\end{bmatrix},$$

where $B$ is a representation-infinite tilted algebra of Euclidean type $\widetilde{\mathbb{D}}_n$, $n\ge 4$, with a complete slice in the preinjective component of $\Gamma_B$, $X$ (respectively, $Y$) is an indecomposable regular $B$-module of regular length 2 (respectively, regular length 1) lying in a tube of $\Gamma_B$ with $n-2$ rays, $t+1$ ($t\ge 2$) is the number of isoclasses of simple $B[Y,t]$-modules which are not $B$-modules.

(ii) Every proper convex subcategory of $A$ is of polynomial growth.

The pg-critical algebras have been classified by quivers and relations in [15]. There are 31 frames of such algebras. In particular, it is known that if $A$ is a pg-critical algebra then: (1) $A$ is tame minimal of non-polynomial growth, (2) $\mathrm{gl.dim}\, A = 2$, (3) $A$ is simply connected (in the sense of [1]), (4) the opposite algebra $A^{\mathrm{op}}$ is also pg-critical.

We shall prove now the following properties of pg-critical algebras.

PROPOSITION 4. *Let $A$ be a polynomial growth critical algebra. Then $\Gamma_A$ admits a component $\mathcal{C}$ such that, for any positive integer $r$, $\mathcal{C}$ contains pairwise different modules $M_1, \ldots, M_r$ having the following properties*:

(i) $[M_i] = [M_j]$ *for any* $1 \le i, j \le r$.

(ii) $M_i \not\simeq \tau_A M_i$ *for any* $1 \le i \le r$.

(iii) $\mathrm{pd}_A M_i = 1$ *for any* $1 \le i \le r$.

(iv) $\dim_K \mathrm{End}_A(M_i) > \dim_K \mathrm{Ext}^1_A(M_i, M_i)$ *for any* $1 \le i \le r$.

Proof. It is well known that all components of the Auslander–Reiten quiver of a tilted algebra of Euclidean type are standard [21, (4.9)]. Assume first that $A$ is of the form $B[X]$. Then $\mathcal{S}(X)$ is given by two parallel infinite sectional paths

$$\begin{array}{ccccccccc} & Y_1 & \to & Y_2 & \to \cdots \to & Y_{i+1} & \to & Y_{i+2} & \to \cdots \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ X = & X_0 & \to & X_1 & \to \cdots \to & X_i & \to & X_{i+1} & \to \cdots \end{array}$$

Let $\mathcal{C}$ be the connected component of $\Gamma_A$ containing the module $X$. Applying Proposition 3 we infer that, for any positive integer $r$, there are pairwise different modules $M_1, \ldots, M_r$ in $\mathcal{C}$ satisfying the conditions (i) and (ii). Moreover, we may choose the modules $M_t$, $1 \le t \le r$, of the form $M_t = Z_{i_t, j_t} = (K, X_{i_t} \oplus Y_{j_t}, \Delta_{i_t, j_t})$ for the corresponding pairs $(i_t, j_t)$ with $2 \le j_t \le i_t$. Hence, in order to show that the modules $M_1, \ldots, M_r$ satisfy the conditions (iii) and (iv), it is enough to prove that $\mathrm{pd}_A Z_{i,j} = 1$ and $\dim_K \mathrm{End}_A(Z_{i,j}) > \dim_K \mathrm{Ext}^1_A(Z_{i,j}, Z_{i,j})$ for any $2 \le j \le i$. Fix a pair $i, j$ with $2 \le j \le i$. Then $\tau_A Z_{i,j} = Z_{i-1,j-1}$. Since $\mathrm{pd}_B U \le 1$ for any $B$-module $U$ which is not in the preinjective component of $\Gamma_B$, we get $\mathrm{Hom}_B(D(B), X_i) = 0 = \mathrm{Hom}_B(D(B), Y_j)$ for all $i \ge 0$, $j \ge 1$. Then $\mathrm{Hom}_A(D(A), \tau_A Z_{i,j}) = \mathrm{Hom}_A(D(A), Z_{i-1,j-1}) = 0$, and consequently $\mathrm{pd}_A Z_{i,j} \le 1$. Further, observe that the image of any map $Z_{i,j} \to Z_{i-1,j-1}$ is contained in the submodule $X_{i-1} \oplus Y_{j-1}$ of $Z_{i-1,j-1}$. Hence the canonical embedding $X_{i-1} \oplus Y_{j-1} \to Z_{i-1,j-1}$ induces an isomorphism of $K$-vector spaces $\mathrm{Hom}_A(Z_{i,j}, X_{i-1} \oplus Y_{j-1}) \xrightarrow{\sim} \mathrm{Hom}_A(Z_{i,j}, Z_{i-1,j-1})$. Choose now the irreducible morphisms $f : X_{i-1} \to X_i$ and $g : Y_{j-1} \to Y_j$. Since the arrows

$X_{i-1} \to X_i$ and $Y_{j-1} \to Y_j$ belong to rays of a standard ray tube of $\Gamma_B$, $f$ and $g$ are monomorphisms. Thus we get a monomorphism

$$\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} : X_{i-1} \oplus Y_{j-1} \to X_i \oplus Y_j.$$

Therefore, we have a chain of monomorphisms of $K$-vector spaces

$$\begin{aligned} \operatorname{Hom}_A(Z_{i,j}, X_{i-1} \oplus Y_{j-1}) &\to \operatorname{Hom}_A(Z_{i,j}, X_i \oplus Y_j) \\ &\to \operatorname{rad}(Z_{i,j}, Z_{i,j}) \to \operatorname{End}_A(Z_{i,j}). \end{aligned}$$

Together with the Auslander–Reiten formula [7; (IV.4.6)]

$$\operatorname{Ext}^1_A(Z_{i,j}, Z_{i,j}) \simeq D\overline{\operatorname{Hom}}_A(Z_{i,j}, \tau_A Z_{i,j}) \simeq D\overline{\operatorname{Hom}}_A(Z_{i,j}, Z_{i-1,j-1})$$

this gives the inequalities

$$\begin{aligned} \dim_K \operatorname{End}_A(Z_{i,j}) &- \dim_K \operatorname{Ext}^1_A(Z_{i,j}, Z_{i,j}) \\ &\geq \dim_K \operatorname{End}_A(Z_{i,j}) - \dim_K \operatorname{Hom}_A(Z_{i,j}, Z_{i-1,j-1}) \\ &\geq \dim_K \operatorname{End}_A(Z_{i,j}) - \dim_K \operatorname{Hom}_A(Z_{i,j}, X_{i-1} \oplus Y_{j-1}) \\ &\geq \dim_K \operatorname{End}_A(Z_{i,j}) - \dim_K \operatorname{rad}(Z_{i,j}, Z_{i,j}) > 0, \end{aligned}$$

and we are done.

Consider now the case when $A = B[Y, t]$. Observe that $A$ is obtained from the one-point extension $B[Y]$ by glueing the extension vertex of $B[Y]$ with the vertex $a_0$ of the following (free) quiver

$$\begin{tikzcd}[row sep=small] & & & & \bullet\, a_{t-1} \\ a_0\, \bullet \arrow[r] & \bullet \arrow[r, dash] & \cdots \arrow[r] & \bullet\, a_{t-2} \arrow[ur] \arrow[dr] & \\ & & & & \bullet\, a_t \end{tikzcd}$$

Denote by $A'$ the algebra obtained from $A$ by reversing the arrow $a_{t-2} \to a_t$ on $a_{t-2} \leftarrow a_t$. Then $A'$ is a $pg$-critical algebra of the form $B'[X']$, where $B'$ is the full subcategory of $A'$ (and $A$) given by all vertices except $a_t$ and $X'$ is the indecomposable projective $A'$-module given by the vertex $a_{t-2}$. Consider now the APR-tilting $A$-module $T = \tau_A^- S_A(a_t) \oplus P$ associated to $a_t$, where $S_A(a_t)$ is the simple $A$-module given by $a_t$ and $P$ is the direct sum of indecomposable projective $A$-modules given by all objects of $A$ except $a_t$. Then $A' = \operatorname{End}_A(T)$. Further, by [5], the functor $F = \operatorname{Hom}_A(T, -)$ induces an equivalence between the full subcategory $\mathcal{G}(T)$ of $\operatorname{mod} A$ formed by all modules having no $S_A(a_t)$ as a direct summand and the full subcategory $\mathcal{Y}(T)$ of $\operatorname{mod} A'$ formed by all modules having no $S_{A'}(a_t)$ as a direct summand. Moreover, there is an isomorphism $\sigma_T : K_0(A) \to K_0(A')$ of groups (see [21, (4.1)]) such that $\sigma_T([Z]) = [F(Z)]$ for any module $Z$ from

$\mathcal{G}(T)$. Let $\mathcal{C}$ be the components of $\Gamma_A$ containing $S_A(a_t)$ and $\mathcal{C}'$ the component of $\Gamma_{A'}$ containing $X'$, or equivalently, the indecomposable projective $A'$-module given by $a_t$. Take an arbitrary positive integer $r$. From the first part of the proof there exist pairwise different modules $M'_1, \dots, M'_r$ in $\mathcal{C}'$ satisfying the conditions (i)–(iv). Clearly, the modules $M'_1, \dots, M'_r$ belong to $\mathcal{Y}(T)$, and hence there exist pairwise different modules $M_1, \dots, M_r$ in $\mathcal{C}$ such that $M'_i = F(M_i)$ for $1 \le i \le r$. Since $M_1, \dots, M_r$ belong to $\mathcal{G}(T)$, for $1 \le i, j \le r$, we have $\sigma_T([M_i]) = [M'_i] = [M'_j] = \sigma_T([M_j])$, and so $[M_i] = [M_j]$. Moreover, for each $1 \le i \le r$, we obtain

$$\begin{aligned}\dim_K \operatorname{End}_A(M_i) &= \dim_K \operatorname{End}_{A'}(M'_i)\\ &> \dim_K \operatorname{Ext}^1_{A'}(M'_i, M'_i) = \dim_K \operatorname{Ext}^1_A(M_i, M_i).\end{aligned}$$

Further, since the indecomposable $A$-modules nonisomorphic to $S_A(a_t)$ belong to $\mathcal{G}(T)$, the indecomposable $A'$-modules nonisomorphic to $S_{A'}(a_t)$ belong to $\mathcal{Y}(T)$, and $F$ induces an equivalence $\mathcal{G}(T) \simeq \mathcal{Y}(T)$, we conclude that $M_i \not\simeq \tau_A M_i$ for $1 \le i \le r$. Moreover, $\operatorname{Hom}_A(D(A), \tau_A M_i) = 0$, and so $\operatorname{pd}_A M_i \le 1$ for any $1 \le i \le r$. Consequently, the modules $M_1, \dots, M_r$ satisfy the required conditions (i)–(iv). This completes the proof.

**4. Polynomial growth strongly simply connected algebras.** Let $A$ be an algebra. Then there exists an isomorphism $A \simeq KQ/I$, where $KQ$ is the path algebra of the ordinary (Gabriel) quiver $Q = Q_A$ of $A$ and $I$ is an admissible ideal in $KQ$. Equivalently, $A = KQ/I$ may be considered as a $K$-category whose object class is the set $Q_0$ of vertices of $Q$, and the set of morphisms $A(x, y)$ from $x$ to $y$ is the quotient of the $K$-vector space $KQ(x, y)$, formed by the $K$-linear combinations of the paths in $Q$ from $x$ to $y$, by the subspace $I(x, y) = KQ(x, y) \cap I$. An algebra $A$ with $Q_A$ having no oriented cycle is said to be *triangular*. A full subcategory $C$ of $A$ is said to be *convex* if any path in $Q_A$ with source and target in $Q_C$ lies entirely in $Q_C$. A triangular algebra $A$ is called *simply connected* [1] if, for any presentation $A \simeq KQ/I$ of $A$ as a bound quiver algebra, the fundamental group $\pi_1(Q, I)$ of $(Q, I)$ is trivial. Following [23], an algebra $A$ is said to be *strongly simply connected* if every convex subcategory $C$ of $A$ is simply connected. It is shown in [23, (4.1)] that a triangular algebra $A$ is strongly simply connected if and only if, for any convex subcategory $C$ of $A$, the first Hochschild cohomology group $H^1(C, C)$ vanishes.

The Tits form $q_A$ of a triangular algebra $A = KQ/I$ with the quiver $Q = (Q_0, Q_1)$ is the integral quadratic form $q : \mathbb{Z}^n \to \mathbb{Z}$, $n = |Q_0|$, defined by

$$q_A(x) = \sum_{i \in Q_0} x_i^2 - \sum_{(i \to j) \in Q_1} x_i x_j + \sum_{i, j \in Q_0} r(i, j) x_i x_j,$$

where $r(i,j)$ is the cardinality of $\mathcal{R} \cap I(i,j)$ for a minimal set of generators $\mathcal{R} \subset \bigcup_{i,j \in Q_0} I(i,j)$ of the ideal $I$ [8]. The Euler form $\chi_A$ of $A$ is the integral quadratic form

$$\chi_A(x) = \sum_{i \in Q_0} x_i^2 - \sum_{i,j \in Q_0} x_i x_j \dim_K \mathrm{Ext}_A^1(S_i, S_j) + \sum_{i,j \in Q_0} x_i x_j \dim_K \mathrm{Ext}_A^2(S_i, S_j),$$

where $S_i$ are the simple $A$-modules associated with $i \in Q_0$. It is known that for any $A$-module $X$ we have (see [21])

$$\chi_A([X]) = \sum_{i=0}^{\infty} (-1)^i \dim_K \mathrm{Ext}_A^i(X, X).$$

If $\mathrm{gl.dim}\, A \leq 2$ then $q_A$ and $\chi_A$ coincide [8].

Following [11], an algebra $A$ is said to be *tame* if, for any dimension $d$, there exists a finite number of $K[X]$–$A$-bimodules $M_i$, $1 \leq i \leq n_d$, which are finitely generated and free as left $K[X]$-modules and all but finitely many isomorphism classes of indecomposable $A$-modules of dimension $d$ are of the form $K[X]/(X - \lambda) \otimes_{K[X]} M_i$ for some $\lambda \in K$ and some $i$. Let $\mu_A(d)$ be the least number of $K[X]$-$A$-bimodules satisfying the above condition for $d$. Then $A$ is said to be of *polynomial growth* [22] if there is a positive integer $m$ such that $\mu_A(d) \leq d^m$ for all $d \geq 1$. From the validity of the second Brauer-Thrall conjecture, we know that $A$ is representation-finite if and only if $\mu_A(d) = 0$ for all $d \geq 1$. Examples of polynomial growth algebras are provided by all tilted algebras of Euclidean type, tubular algebras and tame coil enlargements of such algebras (see [21], [4]). The polynomial growth critical algebras are tame but not of polynomial growth. It is known that if $A$ is triangular and tame then the Tits form $q_A$ is weakly nonnegative (see [16]). Recently it was shown in [30] that a strongly simply connected algebra $A$ is of polynomial growth if and only if $q_A$ is weakly nonnegative and $A$ does not contain a *pg*-critical convex subcategory. This gives a handy criterion for a strongly simply connected algebra to be of polynomial growth. We note that among the 31 frames of *pg*-critical algebra described in [15] we have only 16 frames which are strongly simply connected. Finally, we also mention that, by [17], if $A$ is a strongly simply connected algebra of polynomial growth, $X$ an indecomposable $A$-module and $[X] = x$, then $\mathrm{Ext}_A^i(X, X) = 0$ for $i \geq 2$ and

$$q_A(x) \geq \chi_A(x) = \dim_K \mathrm{End}_A(X) - \dim_K \mathrm{Ext}_A^1(X, X) \geq 0.$$

Here, we shall prove the following characterizations of polynomial growth strongly simply connected algebras.

THEOREM 5. *Let $A$ be a strongly simply connected algebra and $n$ be the rank of $K_0(A)$. The following conditions are equivalent*:

(i) *$A$ is of polynomial growth.*

(ii) *$A$ is tame and there exists $m \in \mathbb{N}$ such that for each $x \in \mathbb{N}^n$, the number of isomorphism classes of indecomposable $A$-modules $X$ with $[X] = x$ and $X \not\simeq \tau_A X$ is bounded by $m$.*

(iii) *$q_A$ is weakly nonnegative and there exists $m \in \mathbb{N}$ such that*, *for each $x \in \mathbb{N}^n$, there are at most $m$ isomorphism classes of indecomposable $A$-modules $X$ with $[X] = x$ and $q_A(x) \neq 0$.*

(iv) *$A$ is tame and there exists $m \in \mathbb{N}$ such that for each $x \in \mathbb{N}^n$, there are at most $m$ isomorphism classes of indecomposable $A$-modules $X$ with $[X] = x$ and $\chi_A(x) \neq 0$.*

In the representation theory of polynomial growth strongly simply connected algebras developed in [30] a fundamental role is played by tame coil enlargements of tame concealed algebras (coil algebras). In our proof of Theorem 5 we need information on the numbers of isomorphism classes of discrete indecomposable modules lying in the Auslander–Reiten components (multicoils) of such algebras. We recall first briefly the notion of admissible operations [2, 3]. Let $A$ be an algebra and $\Gamma$ be a standard component of $\Gamma_A$. For an indecomposable module $X$ in $\Gamma$, called the *pivot*, three admissible operations (ad 1), (ad 2), (ad 3) (and their duals) are defined, depending on the shape of the support $\mathcal{S}(X)$ of $\mathrm{Hom}_B(X, -)|_\Gamma$. These admissible operations yield in each case a modified algebra $A'$ of $A$, and a modified component $\Gamma'$ of $\Gamma$:

(ad 1) If $\mathcal{S}(X)$ is of the form

$$X = X_0 \to X_1 \to X_2 \to \cdots$$

then $X$ is called an (ad 1)-*pivot*, we set $A' = (A \times D)[X \oplus Y_1]$, where $D$ is the full $t{\times}t$ lower triangular matrix algebra, and $Y_1$ is the unique indecomposable projective-injective $D$-module. In this case, $\Gamma'$ is obtained from $\Gamma$ and $\Gamma_D$ by inserting a rectangle consisting of the modules $Z_{i,j} = \big(K, X_i \oplus Y_j, \binom{1}{1}\big)$ for $i \geq 0$, $1 \leq j \leq t$, and $X_i' = (K, X_i, 1)$ for $i \geq 0$, where $Y_j$, $1 \leq j \leq t$, denote the indecomposable injective $D$-modules. The translation $\tau' = \tau_{A'}$ in $\Gamma'$ is defined as follows: $\tau' Z_{i,j} = Z_{i-1,j-1}$ if $i \geq 1$, $j \geq 2$, $\tau' Z_{i,1} = X_{i-1}$ if $i \geq 1$, $\tau' Z_{0,j} = Y_{j-1}$ if $j \geq 2$, $Z_{0,1} = P$ is projective, $\tau' X_0' = Y_t$, $\tau' X_i' = Z_{i-1,t}$ if $i \geq 1$, $\tau'(\tau_A^- X_i) = X_i'$ provided $X_i$ is not an injective $A$-module, otherwise $X_i'$ is injective in $\Gamma'$. For the remaining points of $\Gamma$ and $\Gamma_D$, the translation $\tau'$ coincides with $\tau_A$ and $\tau_D$, respectively. If $t = 0$, we set $A' = A[X]$, and the rectangle reduces to the ray formed by the modules of the form $X_i'$, $i \geq 0$.

(ad 2) If $\mathcal{S}(X)$ is of the form

$$Y_t \leftarrow \cdots \leftarrow Y_1 \leftarrow X = X_0 \to X_1 \to X_2 \to \cdots$$

with $t \geq 1$ (so that $X$ is injective), then $X$ is called an (ad 2)-*pivot*, we set $A' = A[X]$. In this case, $\Gamma'$ is obtained by inserting in $\Gamma$ a rectangle consisting of the modules $Z_{i,j} = \big(K, X_i \oplus Y_j, \binom{1}{1}\big)$, $i \geq 1$, $1 \leq j \leq t$, and $X'_i = (K, X_i, 1)$, $i \geq 0$. The translation $\tau' = \tau_{A'}$ in $\Gamma'$ is defined as follows: $P = X'_0$ is projective-injective, $\tau' Z_{i,j} = Z_{i-1,j-1}$ if $i \geq 2$, $j \geq 2$, $\tau' Z_{i,1} = X_{i-1}$ if $i \geq 1$, $\tau' Z_{1,j} = Y_{j-1}$ if $j \geq 2$, $\tau' X'_i = Z_{i-1,t}$ if $i \geq 2$, $\tau' X'_1 = Y_t$, $\tau'(\tau_A^- X_i) = X'_i$ if $i \geq 1$, provided $X_i$ is not an injective $A$-module, otherwise $X'_i$ is injective in $\Gamma'$. For the remaining points of $\Gamma'$, the translation $\tau'$ coincides with the translation $\tau_A$.

(ad 3) If $\mathcal{S}(X)$ is of the form

$$
\begin{array}{ccccccccc}
 & & Y_1 & \to & Y_2 & \to \cdots \to & Y_t & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
X = X_0 & & \to & X_1 & & \to \cdots \to & X_{t-1} & \to X_t & \to \cdots
\end{array}
$$

with $t \geq 2$ (so that $X_{t-1}$ is injective), then $X$ is called an (ad 3)-*pivot*, we set $A' = A[X]$. In this case $\Gamma'$ is obtained by inserting in $\Gamma$ a rectangle consisting of the modules $Z_{i,j} = \big(K, X_i \oplus Y_j, \binom{1}{1}\big)$ for $1 \leq j \leq i \leq t$ and $i > t$, $1 \leq j \leq t$, and $X'_i = (K, X_i, 1)$ for $i \geq 0$. The translation $\tau' = \tau_{A'}$ in $\Gamma'$ is defined as follows: $P = X'_0$ is projective, $\tau' Z_{i,j} = Z_{i-1,j-1}$ if $i \geq 2$, $j \geq 2$, $\tau' Z_{i,1} = X_{i-1}$ if $i \geq 1$, $\tau' X'_i = Z_{i-1,t}$ if $i > t$, $\tau' X'_i = Y_i$ if $1 \leq i \leq t$, $\tau' Y_j = X'_{j-2}$ if $2 \leq j \leq t$, $\tau'(\tau_A^- X_i) = X_i$ if $i \geq t$, provided $X_i$ is not an injective $A$-module, otherwise $X'_i$ is injective in $\Gamma'$. For the remaining points of $\Gamma'$, the translation $\tau'$ coincides with $\tau_A$. We note that $X'_{t-1}$ is injective.

Finally, together with each of the admissible operations (ad 1), (ad 2) and (ad 3), we consider its dual, denoted by (ad 1*), (ad 2*) and (ad 3*), respectively. These six operations are called the *admissible operations*. Clearly, the admissible operations can be defined as operations on translation quivers rather on Auslander–Reiten components. The definitions are done in the obvious manner (see [2] or [27] for details). A translation quiver $\Gamma$ is called a *coil* if there exists a sequence of translation quivers $\Gamma_0, \Gamma_1, \ldots, \Gamma_m = \Gamma$ such that $\Gamma_0$ is a stable tube and, for each $i$, $0 \leq i < m$, $\Gamma_{i+1}$ is obtained from $\Gamma_i$ by an admissible operation. For an axiomatic description of the coils we refer to [3].

Let $C$ be an algebra, and $\mathcal{T} = (\mathcal{T}_i)_{i \in I}$ a family of pairwise orthogonal (generalized) standard stable tubes of $\Gamma_C$. Following [4], an algebra $B$ is called a *coil enlargement of $C$ using modules from $\mathcal{T}$* if there exists a finite sequence of algebras $C = C_0, C_1, \ldots, C_m = B$ such that, for each $0 \leq j < m$, $C_{j+1}$ is obtained from $C_j$ by an admissible operation with pivot either on a stable tube of $\mathcal{T}$ or on a coil of $\Gamma_{C_j}$ obtained from a stable tube of $\mathcal{T}$ by means of the sequence of admissible operations done so far. The sequence $C = C_0, C_1, \ldots, C_m = B$ is then called an *admissible sequence*. In this process the family $\mathcal{T}_i$, $i \in I$, of stable tubes is transformed into a family $\mathcal{C}_i$, $i \in I$, of pairwise orthogonal standard coils of $\Gamma_B$. A tame coil enlargement $B$ of a tame concealed algebra $C$ using modules from its unique $\mathbb{P}_1(K)$-family $\mathcal{T} = (\mathcal{T}_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ of stable tubes is said to be a *coil algebra*.

We shall use the following lemma:

LEMMA 6. *Let $B$ be an algebra and $\Gamma$ a standard coil of $\Gamma_B$. Then for any sectional path $\Sigma$ in $\Gamma$ and $x \in K_0(B)$, $\Sigma$ admits at most one module $X$ with $[X] = x$.*

Proof. We divide our proof into three steps. Without loss of generality we may assume that $B$ is the support algebra of $\Gamma$.

(1) Assume first that $\Gamma$ is a standard stable tube. In this case, if $M$ and $N$ are two nonisomorphic indecomposable modules in $\Gamma$, then $[M] = [N]$ if and only if $\mathrm{ql}(M) = \mathrm{ql}(N) = cr$ for some $c \geq 1$, where $r$ is the rank of $\Gamma$ and $\mathrm{ql}(Z)$ denotes the quasi-length of a module $Z$ in $\Gamma$ [26, (4.3)]. Clearly then our claim follows.

(2) Assume that $\Gamma$ is a standard ray tube, containing at least one projective module. Then there exists a convex subcategory $C$ of $B$ and a standard stable tube $\mathcal{T}$ of $\Gamma_C$ such that $B$ is obtained from $C$ (respectively, $\Gamma$ is obtained from $\mathcal{T}$) by a sequence of admissible operations of type (ad 1) (see [3, (5.9)]). Let $C = C_0, C_1, \ldots, C_m = B$ be an admissible sequence of algebras such that each $C_k$, $1 \leq k \leq n$, is obtained from $C_{k-1}$ by an admissible operation of type (ad 1). We than get also a sequence $\mathcal{T} = \Gamma_0, \Gamma_1, \ldots, \Gamma_m = \Gamma$ of ray tubes such that, for each $1 \leq k \leq m$, $\Gamma_k$ is a standard ray tube of $\Gamma_{C_k}$ obtained from the standard ray tube $\Gamma_{k-1}$ of $\Gamma_{C_{k-1}}$ by the corresponding admissible operation of type (ad 1). We shall prove our claim by induction on $k$. Let $1 \leq k \leq m$ and assume that $M$ and $N$ are two indecomposable $C_k$-modules with $[M] = [N]$ and lying on a sectional path $\Sigma$ in $\Gamma_k$. If $M$ and $N$ are $C_{k-1}$-modules, then they lie on a sectional path $\Omega$ of $\Gamma_{k-1}$, and by our inductive assumption we get $M \simeq N$. Hence, we may assume that both $M$ and $N$ are not $C_{k-1}$-modules. For the new indecomposable modules in $\Gamma_k = \Gamma'_{k-1}$ we use the notation introduced above. Thus $M = Z_{i,j}$ or $M = X'_i$, and $N = Z_{r,s}$ or $N = X'_r$ for some $i, r \geq 0$, $1 \leq j, s \leq t$. In our case, the equality $[M] = [N]$ implies $[X_i] = [X_r]$. Moreover, $X_i$ and $X_r$ lie on a sectional path $\Theta$ of $\Gamma_{k-1}$. Hence, by our inductive assumption, we have $i = r$. It remains now to consider the case when $M = Z_{i,j}$ and $N = Z_{i,s}$. But then $[M] = [N]$ implies $[Y_j] = [Y_s]$. Since $Y_j$ and $Y_s$ are indecomposable directing $C_{k-1}$-modules we obtain $j = s$. Therefore $M \simeq N$, and we are done.

(3) Let $\Gamma$ be an arbitrary standard coil of $\Gamma_B$, and $M$, $N$ indecomposable $B$-modules with $[M] = [N]$ and lying on a sectional path $\Sigma$ in $\Gamma$. If one of the modules $M$ and $N$ is directing, then $M \simeq N$. Hence, we may assume

that $M$ and $N$ lie on oriented cycles in $\Gamma$. In this case, $\Sigma$ can be extended to an infinite sectional path. By symmetry, we may assume that $M$ and $N$ lie on an infinite sectional path of the form $U_1 \to U_2 \to U_3 \to \cdots$. It follows from [3, (5.9)] and [4, (3.5)] that there is a convex subcategory $B^*$ of $B$ and a standard coray tube $\Gamma^*$ in $\Gamma_{B^*}$ such that $B$ is obtained from $B^*$ (respectively, $\Gamma$ is obtained from $\Gamma^*$) by a sequence of admissible operations of type (ad 1), (ad 2) and (ad 3). Let $B^* = B_0, B_1, \ldots, B_m = B$, $m \geq 0$, be an admissible sequence of algebras such that each $B_k$, $1 \leq k \leq m$, is obtained from $B_{k-1}$ by an admissible operation of type (ad 1), (ad 2), or (ad 3). We then get also a sequence $\Gamma^* = \Gamma_0, \Gamma_1, \ldots, \Gamma_m = \Gamma$ of coils such that, for each $1 \leq k \leq m$, $\Gamma_k$ is a standard coil of $\Gamma_{B_k}$ obtained from the standard coil $\Gamma_{k-1}$ of $\Gamma_{B_{k-1}}$ by the corresponding admissible operation of type (ad 1), (ad 2), or (ad 3). We shall prove our claim by induction on $0 \leq k \leq m$. The case $k = 0$ is dual to (1) and (2). Assume now that $k \geq 1$ and $M$, $N$ are indecomposable $B_k$-modules with $[M] = [N]$ and lying on an infinite sectional path $U_1 \to U_2 \to U_3 \to \cdots$ of $\Gamma_k$. Again, if $M$ and $N$ are $B_{k-1}$-modules, then they lie on a sectional path of $\Gamma_{k-1}$, and hence $M \simeq N$ by our inductive assumption. Assume now that $M$ and $N$ are not $B_{k-1}$-modules. For the new indecomposable modules in $\Gamma_k = \Gamma'_{k-1}$ we use the notation introduced above. Since $M$ and $N$ lie on an infinite sectional path in $\Gamma_k$ consisting of arrows pointing to infinity, we have two possibilities for $M$ and $N$: $M = X'_i$ and $N = X'_r$, or $M = Z_{i,j}$ and $N = Z_{r,j}$. In both cases, $[M] = [N]$ implies $[X_i] = [X_r]$, and hence, by our inductive assumption, we get $i = r$. Therefore, $M \simeq N$, and this finishes our proof.

Recall that a short cycle $M \xrightarrow{f} N \xrightarrow{g} M$ of nonzero nonisomorphisms in $\operatorname{ind} A$ is called *infinite* [27] if $f$ or $g$ belongs to $\operatorname{rad}^{\infty}(\operatorname{mod} A)$. We have the following consequence of the above lemma and results proved in [25] and [4].

PROPOSITION 7. *Let $B$ be an algebra, $n$ the rank of $K_0(B)$, $x$ a vector of $K_0(B)$, and $\Gamma$ a standard coil of $\Gamma_B$. Then the number of indecomposable modules $X$ in $\Gamma$ with $[X] = x$ is bounded by $n$. Moreover, if $\Gamma$ consists of modules which do not lie on infinite short cycles then the number of indecomposable modules $X$ in $\Gamma$ with $[X] = x$ is bounded by $n - 1$.*

Proof. We may assume that $B$ is the support algebra of $\Gamma$. Let $C$ be a convex subcategory of $B$ and $\mathcal{T}$ a standard stable tube of $\Gamma_C$ such that $B$ (respectively, $\Gamma$) is obtained from $C$ (respectively, $\mathcal{T}$) by a sequence of admissible operations. It follows from [4, (3.5)] that the admissible sequence leading from $C$ to $B$ can be replaced by another one consisting of a block of operations of type (ad $1^*$) followed by a block of operations of types (ad 1), (ad 2), (ad 3). The block of operations of type (ad $1^*$) creates a tubular coextension $B^*$ of $C$ and a standard coray tube $\Gamma^*$ in $\Gamma_{B^*}$ such that $B$ is obtained from $B^*$ and $\Gamma$ is obtained from $\Gamma^*$ by the block of operations of types (ad 1), (ad 2) and (ad 3). Denote by $m$ the rank of $K_0(C)$, by $r$ the rank of $\mathcal{T}$, and by $p$ and $q$ the numbers of rays and corays in $\Gamma$. Then $q$ coincides with the number of corays in $\Gamma^*$, and is the sum of $r$ and the number of corays inserted by application of the operations of type (ad $1^*$). Clearly, $r$ is the number of rays in $\Gamma^*$. Further, $p$ is the sum of $r$ and the number of rays inserted by application of the operations of types (ad 1), (ad 2) and (ad 3). It is also known that the indecomposable modules in $\Gamma^*$ which do not lie on an oriented cycle in $\Gamma^*$ are uniquely determined by their composition factors. In particular, the modules $Y_j$ which occur in the description of the operations (ad 1), (ad 2) and (ad 3) have this property. Finally, observe that if two rays in $\Gamma$ have nonempty intersection, then one of the rays consists of a finite number of directing $B^*$-modules from $\Gamma^*$ followed by infinitely many modules which belong to the second ray. From Lemma 6 we know that each ray of $\Gamma$ contains at most one module $X$ with $[X] = x$. We know also that $p - r$ is the number of objects of $B$ which are not objects of $B^*$. Then we conclude that $m + (p - r) \leq n$. Since $\mathcal{T}$ is generalized standard, it follows from [25, (5.11)] that $r \leq m$, and then $p \leq n$. Moreover, if $\Gamma$ consists of indecomposable modules which do not lie on infinite short cycles in $\operatorname{ind} B$, then $\mathcal{T}$ consists of indecomposable modules which do not lie on infinite short cycles in $\operatorname{ind} C$. In this case, by [25, (5.14)], we get $r \leq m - 1$, and hence $p \leq n - 1$. Therefore, the statements of the proposition follow.

We shall need also the following concepts. A component $\mathcal{C}$ of an Auslander–Reiten quiver is said to be a *multicoil* [3] if it contains a full translation subquiver $\Gamma$ such that: (i) $\Gamma$ is a disjoint union of coils; (ii) no vertex of $\mathcal{C} \setminus \Gamma$ lies on an oriented cycle in $\mathcal{C}$. The component quiver $\Sigma_A$ of an algebra $A$ [27] is the quiver whose vertices are the (connected) components of $\Gamma_A$, and two components $\mathcal{C}$ and $\mathcal{D}$ are connected in $\Sigma_A$ by an arrow $\mathcal{C} \to \mathcal{D}$ if $\operatorname{rad}^\infty(X, Y) \neq 0$ for some modules $X$ from $\mathcal{C}$ and $Y$ from $\mathcal{D}$.

Proof of Theorem 5. We shall prove first that (i) implies (ii), (iii) and (iv). Assume that $A$ is of polynomial growth. It is shown in [30, (4.1)] that then $\Sigma_A$ is directed and every component of $\Gamma_A$ is a standard multicoil. In particular, every cycle $M = M_0 \to M_1 \to \cdots \to M_s \to M_{s+1} = M$, $s \geq 0$, of nonzero nonisomorphisms in $\operatorname{ind} A$ is finite, that is, the morphisms forming it do not belong to $\operatorname{rad}^\infty(\operatorname{mod} A)$, and consequently, the modules $M_0, \ldots, M_s$ belong to a coil of a multicoil of $\Gamma_A$. We also know that if an indecomposable $A$-module $M$ does not lie on such a cycle ($M$ is directing) then $M$ is uniquely determined by $[M]$, by [21, (2.4)] or [18, (2.2)]. Moreover, if $\mathcal{T}$ is a stable tube in $\Gamma_A$ then the support of $\mathcal{T}$ is a tame concealed or tubular convex subcategory of $A$ [30, (4.6)]. Hence, for any indecomposable $A$-module $X$ lying in a stable tube of rank 1, we have

$q_A([X]) = \chi_A([X]) = 0$. Let $x$ be a vector in $K_0(A)$ such that there exists a nondirecting indecomposable $A$-module $X$ with $[X] = x$ and $X \not\simeq \tau_A X$. Then $X$ belongs to a proper coil $\Gamma$ of a standard multicoil $\mathcal{C}$ of $\Gamma_A$. Recall that a coil $\Gamma$ is called *proper* if any vertex of $\Gamma$ lies on an oriented cycle of $\Gamma$ (see [3, (3.3)]). Furthermore, by [30, (4.8)], $\Gamma$ is the full translation subquiver of $\Gamma_A$ consisting of all nondirecting modules of a standard coil $\Gamma'$ of the Auslander–Reiten quiver $\Gamma_B$ of a convex coil subcategory $B$ of $A$. Assume first that $\mathrm{Hom}_A(P, X) \neq 0$ for some indecomposable projective module in $\Gamma'$. Then it follows from the inductive proof of [30, (4.1)] that any indecomposable $A$-module $Y$ with $[Y] = x$ also lies in $\Gamma$, and hence in $\Gamma'$. Applying now Proposition 7 we conclude that the number of isomorphism classes of indecomposable $A$-modules $Z$ with $[Z] = [X] = x$ is bounded by $n - 1$. We get the same statement in the case when $\mathrm{Hom}_A(X, I) \neq 0$ for an indecomposable injective module $I$ in $\Gamma'$. Hence, it remains to consider the case when the support of $X$ is contained in a convex subcategory, say $C$, which is tame concealed or tubular. Then $\Gamma$ belongs to a $\mathbb{P}_1(K)$-family $\mathcal{T} = (\mathcal{T}_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ of standard stable tubes of $\Gamma_C$. Moreover, if $Z$ is an indecomposable $A$-module with $[Z] = [X] = x$ then $Z$ is a $C$-module and lies in one of the tubes $\mathcal{T}_\lambda$ (see [21] or [26]). Denote by $m$ the rank of $K_0(C)$, and by $r_\lambda$ the rank of the tube $\mathcal{T}_\lambda$, $\lambda \in \mathbb{P}_1(K)$. Then the following equality holds:

$$\sum_{\lambda \in \mathbb{P}_1(K)} (r_\lambda - 1) = m - 2$$

(see [21]). Further, if $Y \in \mathcal{T}_\lambda$ and $Z \in \mathcal{T}_\mu$ are two nonisomorphic modules in $\mathcal{T}$ with $[Y] = [Z]$ then the quasi-length of $Y$ is divisible by $r_\lambda$ and the quasi-length of $Z$ is divisible by $r_\mu$. We note that then $q_A([Y]) = q_C([Y]) = \chi_A([Y]) = 0$ and $q_A([Z]) = q_C([Z]) = \chi_A([Z]) = 0$ (see [26, (3.6)]), since $\mathrm{gl.dim}\, C \leq 2$. Now a simple inspection of tubular types of tame concealed and tubular algebras shows that, if $\lambda_1, \ldots, \lambda_t$ are all indices $\lambda \in \mathbb{P}_1(K)$ with $r_\lambda \neq 1$, then $r_{\lambda_1} + \ldots + r_{\lambda_t} \leq m + 2 \leq n + 2$. Therefore, the number of isomorphism classes of indecomposable $A$-modules $Z$ with $[Z] = [X] = x$ is bounded by $n + 2$. Thus we proved that (i) implies the conditions (ii), (iii) and (iv).

Assume now that $q_A$ is weakly nonnegative but $A$ is not of polynomial growth. Then, by [30, (4.2)], $A$ admits a convex subcategory $\Lambda$ which is pg-critical. Since $\mathrm{gl.dim}\, \Lambda = 2$, we have $q_\Lambda = \chi_\Lambda$. Let $r$ be an arbitrary positive integer. Then, by Proposition 4, there exist pairwise nonisomorphic indecomposable $\Lambda$-modules $M_1, \ldots, M_r$ such that

(a) $[M_1] = \ldots = [M_r]$.
(b) $\mathrm{pd}_\Lambda M_1 = \ldots = \mathrm{pd}_\Lambda M_r = 1$.
(c) $\dim_K \mathrm{End}_\Lambda(M_i) > \dim_K \mathrm{Ext}^1_\Lambda(M_i, M_i)$ for any $1 \leq i \leq r$.

(d) $M_i \not\simeq \tau_\Lambda M_i$ for any $1 \le i \le r$.

We may clearly consider $M_1, \ldots, M_r$ as indecomposable $A$-modules. Observe that, if $M_i \simeq \tau_A M_i$, then we have an Auslander–Reiten sequence $0 \to M_i \to E \to M_i \to 0$ in $\operatorname{mod} A$. Then $[E] = 2[M_i]$, which implies that $E$ is a $\Lambda$-module, and so it is an Auslander–Reiten sequence in $\operatorname{mod}\Lambda$, a contradiction. Therefore, $M_i \not\simeq \tau_A M_i$ for any $1 \le i \le r$. Finally, since $\mathrm{pd}_\Lambda M_i = 1$ we have

$$q_\Lambda([M_i]) = \chi_\Lambda([M_i]) = \dim_K \operatorname{End}_\Lambda(M_i) - \dim_K \operatorname{Ext}^1_\Lambda(M_i, M_i) > 0,$$

and hence

$$q_A([M_i]) = q_\Lambda([M_i]) > 0 \quad \text{and} \quad \chi_A([M_i]) = \chi_\Lambda([M_i]) > 0.$$

This proves that each of the conditions (ii), (iii) and (iv) implies (i).

It is well known that if $A$ is a representation-finite (strongly) simply connected algebra then any indecomposable $A$-module $X$ is directing, hence uniquely determined by $[X]$, and $q_A([X]) = \chi_A([X]) = 1$. As a direct consequence of our proof of Theorem 5 we get the following

Corollary 8. *Let $A$ be a representation-infinite strongly simply connected algebra of polynomial growth, $n$ be the rank of $K_0(A)$, and $x$ be a vector of $K_0(A)$. Then*

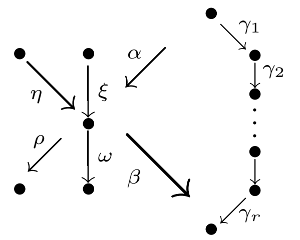
(i) *The number of isomorphism classes of indecomposable $A$-modules $X$ with $[X] = x$ and $X \not\simeq \tau_A X$ is bounded by $n + 2$.*

(ii) *The number of isomorphism classes of indecomposable $A$-modules $X$ with $[X] = x$ and $q_A(x) \neq 0$ is bounded by $n - 1$.*

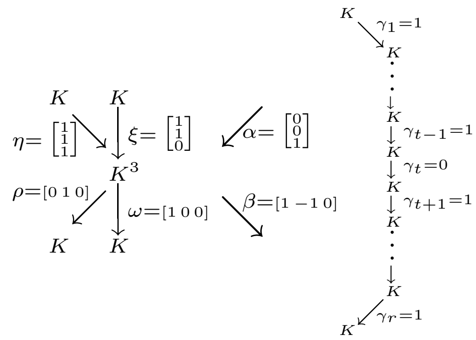
(iii) *The number of isomorphism classes of indecomposable $A$-modules $X$ with $[X] = x$ and $\chi_A(x) \neq 0$ is bounded by $n - 1$.*

We note that for a tubular algebra $C$ of type $(2, 2, 2, 2)$ the rank of $K_0(C)$ is 6 and we have $8 = 6 + 2$ pairwise nonisomorphic indecomposable modules with the same composition factors, and of $\tau_A$-period 2. Hence, the bound $n+2$ in (i) of the above corollary is optimal. Possibly $n-1$ is not the optimal bound in the statements (ii) and (iii). We end the paper with examples of polynomial growth strongly simply connected algebras for which there exist large numbers of pairwise nonisomorphic indecomposable modules $X$ with $X \not\simeq \tau_A X$, $q_A([X]) \neq 0$, $\chi_A([X]) \neq 0$ and having the same composition factors.

Example 9. Let $r \ge 2$. Denote by $A$ the algebra $KQ/I$ given by the quiver $Q$ of the form




and the ideal $I$ in $KQ$ generated by $\omega\alpha$, $\rho\alpha$, $\beta\xi$, $\beta\eta$, $\beta\alpha - \gamma_r \dots \gamma_2\gamma_1$. Then $A$ is a strongly simply connected (coil) algebra of polynomial growth and gl.dim $A = 2$. In fact, $\mu_A(d) \le 1$ for any $d \ge 1$, and the one-parameter families of indecomposable $A$-modules are those given by the unique convex hereditary subcategory $H$ of $A$ of type $\widetilde{\mathbb{D}}_4$. For each $1 \le t \le r$, consider the indecomposable $A$-module $M_t$ given by




Then the modules $M_1, \dots, M_r$ are pairwise nonisomorphic with the same composition factors given by the vector

$$x = \begin{array}{cccc} & & 1 & \\ & & & 1 \\ 1 & 1 & & \\ & 3 & & \vdots \\ 1 & 1 & & \\ & & & 1 \\ & & 1 & \end{array}$$

and $\chi_A(x) = q_A(x) = (r + 14) - (18 + r) + 5 = 1$. Note also that the rank of $K_0(A)$ is equal to $r + 6$.

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*Received 27 June 1996;*
*revised 14 August 1996*