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PROBABILITY MEASURE FUNCTORS PRESERVING INFINITE-DIMENSIONAL SPACES

BY

NGUYEN TO NHU (HANOI AND BLOOMINGTON, INDIANA) AND KATSURO SAKAI (TSUKUBA)

0. Introduction. Let $Q = [-1,1]^\omega$ be the Hilbert cube, $s = (-1,1)^\omega$ the pseudo-interior of Q, $\Sigma = \{(x_i)_{i\in\mathbb{N}} \mid \sup|x_i| < 1\}$ the radial-interior of Q and

$$\sigma = \{(x_i)_{i\in\mathbb{N}} \in s \mid x_i = 0 \text{ except for finitely many } i\}.$$

As is well-known, s is homeomorphic ($\approx$) to the Hilbert space ℓ_2. We put

$$\ell_2^Q = \{(x_i)_{i\in\mathbb{N}} \in \ell_2 \mid \sup|ix_i| < \infty\},$$
$$\ell_2^f = \{(x_i)_{i\in\mathbb{N}} \in \ell_2 \mid x_i = 0 \text{ except for finitely many } i\}.$$

It is shown in [SW] that $(\ell_2, \ell_2^Q, \ell_2^f) \approx (s, \Sigma, \sigma)$, that is, there exists a homeomorphism $h : \ell_2 \to s$ such that $h(\ell_2^Q) = \Sigma$ and $h(\ell_2^f) = \sigma$.

The space of probability measures ([1]) on a metrizable space X is denoted by $P(X)$. By integration, $P(X)$ can be regarded as a subset of the dual $C_{\mathrm{b}}(X)^*$ of the Banach space $C_{\mathrm{b}}(X)$ of all bounded continuous real-valued functions on X with the sup-norm. For details, see [Va$_2$, Part I] and [DS, Introduction]. The topology of $P(X)$ is inherited from the weak*-topology of $C_{\mathrm{b}}(X)^*$. For each $k \in \mathbb{N}$, let $P_k(X) \subset P(X)$ be the subspace of all measures with supports consisting of at most k points, and let $P_{\mathfrak{F}}(X) = \bigcup_{k\in\mathbb{N}} P_k(X)$. It is known that $P_k(Q) \approx Q$ and $P_k(\ell_2) \approx \ell_2$ for each $k \in \mathbb{N}$ ([Fe$_1$] and [NT]). For related topics, see [Fe$_3$]. For a subspace A of a metrizable space X, we can regard $P_k(A)$ as a subspace of $P_k(X)$ by identifying as follows:

$$P_{\mathfrak{F}}(A) = \{\mu \in P_{\mathfrak{F}}(X) \mid \operatorname{supp}\mu \subset A\},$$

where $\operatorname{supp}\mu$ denotes the support of μ. Using the open base in [Va$_2$, Part II, Remark 3 to Theorem 2] (or [NT, Proposition 2.1]), it is easy to see that

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([1]) A non-negative Borel measure μ on X with $\mu(X) = 1$ is called a *probability measure*.

the topology of $P_{\mathfrak{F}}(A)$ is identical with the relative topology inherited from $P_{\mathfrak{F}}(X)$ (cf. [DS, Subspace Lemma]). In the present paper, applying the results of [SW], [CDM], [vM$_2$] and [We], we prove

MAIN THEOREM. *For each* $k \in \mathbb{N}$, *the following hold*:

(a) $(P_k(Q), P_k(s), P_k(\Sigma), P_k(\sigma)) \approx (Q, s, \Sigma, \sigma)$,
(b) $(P_k(Q^\omega), P_k(\Sigma^\omega)) \approx (Q^\omega, \Sigma^\omega)$, *hence* $P_k((\ell_2^f)^\omega) \approx (\ell_2^f)^\omega$ (2), *and*
(c) $(P_k(\ell_2 \times Q), P_k(\ell_2 \times \Sigma)) \approx (\ell_2 \times Q, \ell_2 \times \Sigma)$ (3), *hence* $P_k(\ell_2 \times \ell_2^f) \approx \ell_2 \times \ell_2^f$.

Remark 1. Relating to the above result, one may ask whether $P_k(H) \approx H$ for any infinite-dimensional pre-Hilbert space H or not. This question can be answered negatively. In [vM$_1$], Jan van Mill showed that every separable Banach space (hence ℓ_2) contains a dense linear subspace X which has *restricted domain invariance*, that is, for every continuous injection $g : U \to X$ with domain a non-empty open set in X, there exists a non-empty open set $V \subset U$ such that $g|V$ is an open embedding in X. For such a normed linear space (or a pre-Hilbert space) X, $P_k(X) \not\approx X$ if $k > 1$.

In fact, let $U_1, \ldots, U_k$ be disjoint open sets in X. By $\mathring{\Delta}^{k-1}$, we denote the standard open $(k-1)$-simplex, that is,

$$\mathring{\Delta}^{k-1} = \Big\{(t_1, \ldots, t_k) \in \mathbb{R}^k \Bigm| \sum_{i=1}^{k} t_i = 1 \text{ and } t_i > 0 \text{ for each } i\Big\}.$$

We define $\varphi : U_1 \times \ldots \times U_k \times \mathring{\Delta}^{k-1} \to P_k(X)$ as follows:

$$\varphi(x_1, \ldots, x_k; t_1, \ldots, t_k) = \sum_{i=1}^{k} t_i \delta_{x_i},$$

where $\delta_x \in P(X)$ is the Dirac measure at $x \in X$ (i.e. $\delta_x(\{x\}) = 1$). Then by using the open base in [Va$_2$] (or [NT]), it is easy to see that φ is an open embedding. If $P_k(X) \approx X$ then we have a continuous injection $g : U_1 \to X$ such that $g(U_1)$ has no interior point, which contradicts the restricted domain invariance of X. Therefore $P_k(X) \not\approx X$ for any $k > 1$. ∎

Remark 2. By our results, each $(P_k(X), P_k(M), P_k(N))$ is a (Q, Σ, σ)-manifold (or an $(\ell_2, \ell_2^Q, \ell_2^f)$-manifold) triple if so is (X, M, N) and each functor P_k preserves manifolds modeled on the spaces Q, ℓ_2, ℓ_2^Q, ℓ_2^f, $(\ell_2^f)^\omega$ and $\ell_2 \times \ell_2^f$. However, $P_k(X) \not\approx X$ in general even if X is such a manifold.

In fact, $P_k(X)$ is path-connected for any (disconnected) space X and $k > 1$. To see this, let $x_0 \in X$ and $\mu = \sum_{i=1}^{r} s_i \delta_{x_i} \in P_k(X)$. We define a

(2) It is known that $(\ell_2^f)^\omega \approx \Sigma^\omega$ (cf. the proof of [vM$_2$, Corollary 4.2]).

(3) It is known that $(\ell_2 \times Q, \ell_2 \times \Sigma) \approx (\ell_2 \times Q, \ell_2 \times \sigma)$, hence $\ell_2 \times \Sigma \approx \ell_2 \times \ell_2^f$.

path $\varphi:\mathbf{I}\to P_k(X)$ as follows:

$$\varphi(t)=\begin{cases}\sum_{i=1}^{r}(1-2t)s_i\delta_{x_i}+2t\delta_{x_1} & \text{if } 0\le t\le 1/2,\\ (2-2t)\delta_{x_1}+(2t-1)\delta_{x_0} & \text{if } 1/2\le t\le 1.\end{cases}$$

Then $\varphi(0)=\mu$, $\varphi(1/2)=\delta_{x_1}$ and $\varphi(1)=\delta_{x_0}$. ∎

Remark 3. Let $\mathcal{SM}$ be the category of separable metrizable spaces with (continuous) maps. Then each $P_k:\mathcal{SM}\to\mathcal{SM}$ is a covariant functor. Our Main Theorem holds if P_k is replaced by any covariant functor $F:\mathcal{SM}\to\mathcal{SM}$ satisfying the following conditions:

(1) if A is a subspace of X then $F(A)$ is a subspace $F(X)$;
(2) if A is closed in X then $F(A)$ is also closed in $F(X)$;
(3) for $A\subset X$, any deformation $h:A\times\mathbf{I}\to X$ induces the deformation $h^*:F(A)\times\mathbf{I}\to F(X)$ defined by $h^*_t=F(h_t)$ (hence $h^*_t(F(A))\subset F(h_t(A))$);
(4) $F(\bigcup_{i\in\mathbb{N}}X_i)=\bigcup_{i\in\mathbb{N}}F(X_i)$ for $X_1\subset X_2\subset\ldots$;
(5) $F(X\cap Y)=F(X)\cap F(Y)$;
(6) $F(X\setminus A)\subset F(X)\setminus F(A)$ for $A\subset X$;
(7) $F(\bigcap_{i\in\mathbb{N}}X_i)=\bigcap_{i\in\mathbb{N}}F(X_i)$ for $X_1\supset X_2\supset\ldots$;
(8) if X is a finite-dimensional compactum then so is $F(X)$;
(9) if X is separable completely metrizable then so is $F(X)$;
(10) $F(Q)\approx Q$.

Let $\mathfrak{F}(X)$ be the hyperspace of non-empty finite subsets of X with the Vietoris (or finite) topology (cf. [Na]). For a subspace A of X, we can regard $\mathfrak{F}(A)$ as a subspace of $\mathfrak{F}(X)$ by identifying $\mathfrak{F}(A)=\{F\in\mathfrak{F}(X)\mid F\subset A\}$. From the definition of the Vietoris topology, it follows that the topology of $\mathfrak{F}(A)$ is identical with the relative topology inherited from $\mathfrak{F}(X)$. As easily observed, if A is closed in X then $\mathfrak{F}(A)$ is closed in $\mathfrak{F}(X)$.

For each $k\in\mathbb{N}$, let $\mathfrak{F}_k(X)\subset\mathfrak{F}(X)$ be the subspace of all subsets of X consisting of at most k points. Then the functor $\mathfrak{F}_k:\mathcal{SM}\to\mathcal{SM}$ satisfies (1) and (2). By [Fe_2, Corollary 5], $\mathfrak{F}_k(Q)\approx Q$, that is, $\mathfrak{F}_k$ satisfies (10). We show that $\mathfrak{F}_k$ also satisfies the conditions (3)–(9). Thus the following can be obtained:

THEOREM 2. *For each* $k\in\mathbb{N}$, *the following hold*:

(a) $(\mathfrak{F}_k(Q),\mathfrak{F}_k(s),\mathfrak{F}_k(\Sigma),\mathfrak{F}_k(\sigma))\approx(Q,s,\Sigma,\sigma)$,
(b) $(\mathfrak{F}_k(Q^\omega),\mathfrak{F}_k(\Sigma^\omega))\approx(Q^\omega,\Sigma^\omega)$, *hence* $\mathfrak{F}_k((\ell_2^f)^\omega)\approx(\ell_2^f)^\omega$, *and*
(c) $(\mathfrak{F}_k(\ell_2\times Q),\mathfrak{F}_k(\ell_2\times\Sigma))\approx(\ell_2\times Q,\ell_2\times\Sigma)$, *hence* $\mathfrak{F}_k(\ell_2\times\ell_2^f)\approx\ell_2\times\ell_2^f$.

Let G be a subgroup of the kth symmetric group $\mathfrak{S}_k$. Then G acts on X^k as a permutation group of the coordinates. The orbit space of this action is denoted by $\mathrm{SP}^k_G(X)$ and called the *G-symmetric power* of X, where $\mathrm{SP}^k_G(X)$ is the quotient space of X^k. We put $\mathrm{SP}^k(X)=\mathrm{SP}^k_{\mathfrak{S}_k}(X)$, which is called the *symmetric power* of X. For a subspace A of X, we can regard $\mathrm{SP}^k_G(A)$ as a subspace of $\mathrm{SP}^k_G(X)$ by identifying $\mathrm{SP}^k_G(A) = q(A^k)$, where $q : X^k \to \mathrm{SP}^k_G(X)$ is the quotient map. In fact, since q is an open map, it is easy to see that the topology of $\mathrm{SP}^k_G(A)$ is identical with the relative topology inherited from $\mathrm{SP}^k_G(X)$. Since $\mathrm{SP}^k_G(X) \setminus \mathrm{SP}^k_G(A) = q(X^k \setminus A^k)$, if A is closed in X then $\mathrm{SP}^k_G(A)$ is closed in $\mathrm{SP}^k_G(X)$. Thus the functor $\mathrm{SP}^k_G : \mathcal{SM} \to \mathcal{SM}$ satisfies (1) and (2). By [Fe$_2$, Corollary 5], $\mathrm{SP}^k_G(Q) \approx Q$, that is, SP^k_G satisfies (10). We show that SP^k_G also satisfies the conditions (3)–(9). Thus we can obtain the following:

THEOREM 3. *For any subgroup G of the kth symmetric group*, *the following hold*:

(a) $(\mathrm{SP}^k_G(Q), \mathrm{SP}^k_G(s), \mathrm{SP}^k_G(\Sigma), \mathrm{SP}^k_G(\sigma)) \approx (Q, s, \Sigma, \sigma)$,
(b) $(\mathrm{SP}^k_G(Q^\omega), \mathrm{SP}^k_G(\Sigma^\omega)) \approx (Q^\omega, \Sigma^\omega)$, *hence* $\mathrm{SP}^k_G((\ell^f_2)^\omega) \approx (\ell^f_2)^\omega$, *and*
(c) $(\mathrm{SP}^k_G(\ell_2 \times Q), \mathrm{SP}^k_G(\ell_2 \times \Sigma)) \approx (\ell_2 \times Q, \ell_2 \times \Sigma)$, *hence* $\mathrm{SP}^k_G(\ell_2 \times \ell^f_2) \approx \ell_2 \times \ell^f_2$.

It should be remarked that Theorems 2(a) and 3(a) refine the results in [Ng$_2$] and [Ng$_1$], respectively.

1. Preliminaries. Let $X_1 \subset X_2 \subset \ldots$ be a tower of closed sets in X. We say that $(X_n)_{n\in\mathbb{N}}$ is *expansive* (or *finitely expansive*) [Cu] if for each $n \in \mathbb{N}$, there is an embedding $h : X_n \times Q \to X_{n+1}$ (or $h : X_n \times \mathbf{I} \to X_{n+1}$) such that $h(x, 0) = x$ for every $x \in X_n$ ([4]). It is said that $(X_n)_{n\in\mathbb{N}}$ has the *mapping absorption property for compacta* in X provided for any compactum $A \subset X$ and for any $\varepsilon > 0$ and $n \in \mathbb{N}$ there exists a map $f : A \to X_m$ for some $m \ge n$ such that $f|A \cap X_n = \mathrm{id}$ and f is ε-close to id (cf. [Cu, Definition 4.5]). It is said that $(X_n)_{n\in\mathbb{N}}$ has the *compact absorption property* (abbrev. cap) (or the *finite-dimensional compact absorption property* (abbrev. fd-cap)) in X and $M = \bigcup_{n\in\mathbb{N}} X_n$ is called a *cap set* (or an *fd-cap set*) for X [Ch] if each X_n is a (finite-dimensional) compact Z-set in X and for each (finite-dimensional) compact Z-set A in X, $\varepsilon > 0$ and $n \in \mathbb{N}$, there is an embedding $g : A \to X_m$ for some $m \ge n$ such that g is ε-close to id and $g|A \cap X_n = \mathrm{id}$, where a closed set A in X is a *Z-set* if each map $f : Q \to X$ can be approximated by maps $g : Q \to X \setminus A$. In case X is an ANR, a closed set A in X is Z-set if and only if there are maps $f : X \to X \setminus A$ arbitrarily close to id [vM$_3$, 7.2.5], or, more strongly, there is a deformation $h : X \times \mathbf{I} \to X$ such that $h_0 = \mathrm{id}$ and $h_t(X) \subset X \setminus A$ if $0 < t \le 1$ [To$_1$, Theorem 2.4 with Corollary 3.3].

LEMMA 1.1. *If $X_1 \subset X_2 \subset \ldots$ is an expansive* (*resp. finitely expansive*) *tower of compact* (*resp. finite-dimensional compact*) *Z-sets in X and*

([4]) We mean $0 = (0, 0, \ldots) \in Q$.

has the mapping absorption property for compacta (*resp. finite-dimensional compacta*) *in* X, *then* $(X_n)_{n\in\mathbb{N}}$ *has the cap* (*resp. the fd-cap*) *in* X, *whence* $\bigcup_{n\in\mathbb{N}} X_n$ *is a cap set* (*resp. an fd-cap set*) *for* X.

Proof. For each (finite-dimensional) compact Z-set A in X, $\varepsilon > 0$ and $n \in \mathbb{N}$, we have a map $f : A \to X_m$ for some $m \ge n$ such that $f|A \cap X_n = \mathrm{id}$ and f is $\varepsilon/2$-close to id. On the other hand, we have a map $h : A \to Q$ ($h : A \to \mathbf{I}^k$ for some $k \in \mathbb{N}$) such that $h(A \cap X_n) = \{0\}$ and $h|A \setminus X_n$ is injective. Since $(X_n)_{n\in\mathbb{N}}$ is (finitely) expansive, there is an embedding $\varphi : X_m \times Q \to X_{m+1}$ (or $\varphi : X_m \times \mathbf{I}^k \to X_{m+k}$) such that $\varphi(x, 0) = x$ and $\operatorname{diam} \varphi(\{x\} \times Q) < \varepsilon/2$ (or $\operatorname{diam} \varphi(\{x\} \times \mathbf{I}^k) < \varepsilon/2$) for every $x \in X_m$. Then we have the embedding $g : A \to X_{m+1}$ (or $g : A \to X_{m+k}$) defined by $g(x) = \varphi(f(x), h(x))$, which is ε-close to id. ∎

The following is due to Anderson [An] (cf. [Ch, Lemma 4.3]):

LEMMA 1.2. *If M is a cap set* (*resp. an fd-cap set*) *for* Q, *then* $(Q, M) \approx (Q, \Sigma)$ (*resp.* $(Q, M) \approx (Q, \sigma)$). *Moreover*, *if* $M \subset s$ *in the above*, *then* $(Q, s, M) \approx (Q, s, \Sigma)$ (*resp.* $(Q, s, M) \approx (Q, s, \sigma)$).

As is well-known, the pseudo-boundary $Q \setminus s$ is a cap set for Q. Then $(Q, Q \setminus s) \approx (Q, \Sigma)$, whence $(Q, Q \setminus \Sigma) \approx (Q, s)$.

To prove (a) in the Main Theorem, we apply the following characterization due to Sakai and Wong [SW]:

THEOREM 1.3. *In order that* $(X, M, N) \approx (Q, \Sigma, \sigma)$ (*or* $(X, M, N) \approx (\ell_2, \ell_2^Q, \ell_2^f)$), *it is necessary and sufficient that* $X \approx Q$ (*or* $X \approx \ell_2$) *and* X *has a tower* $X_1 \subset X_2 \subset \ldots$ *of compacta such that*

(i) $X_n \approx Q$ *for each* $n \in \mathbb{N}$,
(ii) *each* X_n *is a* Z-*set in* X_{n+1},
(iii) $M = \bigcup_{n\in\mathbb{N}} X_n$ *is a cap set for* X *and*
(iv) *each* $X_n \cap N$ *is an fd-cap set for* X_n.

A *Z-matrix* in X is a double sequence $(A_i^n)_{n,i\in\mathbb{N}}$ of Z-sets in X such that $A_i^{n+1} \subset A_i^n \subset A_{i+1}^n$ for all $n, i \in \mathbb{N}$ ([5]), that is,

$$\begin{array}{ccccc}
A_1^1 & \subset A_2^1 & \subset & A_3^1 & \subset \ldots \\
\cup & \cup & & \cup & \\
A_1^2 & \subset A_2^2 & \subset & A_3^2 & \subset \ldots \\
\cup & \cup & & \cup & \\
A_1^3 & \subset A_2^3 & \subset & A_3^3 & \subset \ldots \\
\cup & \cup & & \cup & \\
\vdots & \vdots & & \vdots &
\end{array}$$

([5]) For a technical reason, it is assumed in [vM$_2$] that $A_1^n = \emptyset$ for each $n \in \mathbb{N}$. One can add $A_0^n = \emptyset$ to the matrix if necessary.

To prove (b) in the Main Theorem, we apply the following theorem due to van Mill, which is a combination of Theorem 3.6 and Corollaries 2.3 and 4.2 of [vM$_2$]:

THEOREM 1.4. *Suppose that* $X \approx Q$. *Let* $(A_i^n)_{n,i\in\mathbb{N}}$ *be a* Z*-matrix in* X *which has the following properties* ([6]):

(i) *each* $(A_i^n)_{i\in\mathbb{N}}$ *has the cap for* X ([7]),
(ii) $\bigcap_{j=1}^m A_{i_j}^{n_j} \approx Q$ *for each* $n_1 < \ldots < n_m$ *and* $i_1, \ldots, i_m \in \mathbb{N}$,
(iii) *for each* $n_1 < \ldots < n_m$ *and* $i_1, \ldots, i_m, p \in \mathbb{N}$, $(A_i^{n_m+p} \cap \bigcap_{j=1}^m A_{i_j}^{n_j})_{i\in\mathbb{N}}$ *has the cap in* $\bigcap_{j=1}^m A_{i_j}^{n_j}$ *and*
(iv) *for each* $n_1 < \ldots < n_m$ *and* $i_1, \ldots, i_m, n, i \in \mathbb{N}$, $\bigcap_{j=1}^m A_{i_j}^{n_j} \not\subset A_i^n$ *implies that* $A_i^n \cap \bigcap_{j=1}^m A_{i_j}^{n_j}$ *is a* Z*-set in* $\bigcap_{j=1}^m A_{i_j}^{n_j}$.

Then $(X, \bigcap_{n\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} A_i^n) \approx (Q^\omega, \Sigma^\omega)$, *hence* $\bigcap_{n\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} A_i^n \approx (\ell_2^f)^\omega$.

For any collection $\mathcal{U}$ of open sets in X, two maps $f, g : A \to X$ are $\mathcal{U}$*-close* if for each $x \in A$, $f(x) = g(x)$ or $\{f(x), g(x)\}$ is contained in some $U \in \mathcal{U}$. Let M be a Z_σ-set in X, that is, a countable union of Z-sets. We call M a Z*-absorber* for X [DM] (cf. [We]) if for any Z-set A in X and any collection $\mathcal{U}$ of open sets in X, there exists a homeomorphism $h : X \to X$ such that h is $\mathcal{U}$-close to id and $h(A \cap \bigcup\mathcal{U}) \subset M$. The following is due to West [We] (cf. [Dij, 1.2.11]):

THEOREM 1.5. *Suppose that* X *is completely metrizable. If* M *and* N *are* Z*-absorbers for* X, *then for any collection* $\mathcal{U}$ *of open sets in* X, *there exists a homeomorphism* $h : X \to X$ $\mathcal{U}$*-close to* id *with* $h(M \cap \bigcup\mathcal{U}) = N \cap \bigcup\mathcal{U}$. *In particular*, $(X, M) \approx (X, N)$.

It is known that $\ell_2 \times \Sigma$ and $\ell_2 \times \sigma$ are Z-absorbers for $\ell_2 \times Q$. Since $\ell_2 \times Q \approx \ell_2$, we have the following:

COROLLARY 1.6. *In order that* $(X, M) \approx (\ell_2 \times Q, \ell_2 \times \Sigma)$, *it is necessary and sufficient that* $X \approx \ell_2$ *and* M *is a* Z*-absorber for* X.

We apply this to prove (c) in the Main Theorem, but it is a little hard to check the condition in the definition of Z-absorbers, where the existence of homeomorphisms of X onto itself is required. So we give here a characterization of Z-absorbers for ℓ_2-manifolds which can be easily applied. An embedding $f : A \to X$ is called a Z*-embedding* if $f(A)$ is a Z-set in X.

THEOREM 1.7. *Let* X *be an* ℓ_2*-manifold and* $M \subset X$. *Then the following are equivalent*:

([6]) A Z-matrix with these properties is called a Q*-matrix* in [vM$_2$].

([7]) In case $X \approx Q$ (or X is a Q-manifold), a tower of compact Z-sets in X with the cap is called a *skeleton* in [vM$_2$].

(a) *M is a Z-absorber for X;*
(b) *M is a Z_σ-set in X and for each open set W in X and each Z-set A in W and each map $\alpha : W \to (0,1)$, there exists a Z-embedding $f : A \to M \cap W$ such that $d(f(x), x) < \alpha(x)$ for each $x \in W$, where d is an admissible metric for X;*
(c) *there exist a deformation $h : X \times \mathbf{I} \to X$ and a tower $X_1 \subset X_2 \subset \ldots$ of Z-sets in X such that $h_0 = \mathrm{id}$, $h_t(X) \subset X_n$ for $t \geq 2^{-n}$, each X_n is an ℓ_2-manifold and $M = \bigcup_{n\in\mathbb{N}} X_n$.*

Proof. (b)⇒(a): Let A be a Z-set in X and $\mathcal{U}$ a collection of open sets in X. Then $W = \bigcup \mathcal{U}$ is an ℓ_2-manifold and $A \cap W$ is a Z-set in W. By [We, Lemma 2], W has an open cover $\mathcal{U}_0$ such that $\mathcal{U}_0$ refines $\mathcal{U}$ and if a homeomorphism $h : W \to W$ is $\mathcal{U}_0$-close to id then h extends to the homeomorphism $\widetilde{h} : X \to X$ with $\widetilde{h}|X \setminus W = \mathrm{id}$. Let $\mathcal{U}_1$ be an open star-refinement of $\mathcal{U}_0$. Since W is an ANR, $\mathcal{U}_1$ has an open refinement $\mathcal{U}_2$ such that any two $\mathcal{U}_2$-close maps of an arbitrary space to W are $\mathcal{U}_1$-homotopic (cf. [vM$_3$, 5.1.1]). Choose a map $\alpha : W \to (0,1)$ so that the $\alpha(x)$-neighborhood of x in X is contained in some member of $\mathcal{U}_2$. By (b), there exists a Z-embedding $f : A\cap W \to M\cap W$ such that $d(f(x), x) < \alpha(x)$ for each $x \in W$. Then f is $\mathcal{U}_1$-homotopic to id. By the Z-set Unknotting Theorem for ℓ_2-manifolds (cf. [Sa, §3]), f extends to a homeomorphism $h : W \to W$ which is $\mathcal{U}_0$-isotopic to id. Then h extends to the homeomorphism $\widetilde{h} : X \to X$ by $\widetilde{h}|X \setminus W = \mathrm{id}$, whence $\widetilde{h}$ is $\mathcal{U}$-close to id and $\widetilde{h}(A \cap \bigcup \mathcal{U}) \subset M$. Hence M is a Z-absorber for X.

(c)⇒(b): Let d be an admissible metric for X, let A be a Z-set in an open set W in X and $\alpha : W \to (0,1)$ a map. Then we have a map $\beta : W \to (0,1)$ such that $\beta(x) < 2^{-1}\alpha(x)$ and

$$(\sharp) \quad d(h(x, \beta(x)), x) < \min\{2^{-1}\alpha(x), d(x, X \setminus W)\} \quad \text{for each } x \in W.$$

By ($\sharp$), we can define a map $f_0 : A \to W \cap M$ by $f_0(x) = h(x, \beta(x))$. Then $d(f_0(x), x) < 2^{-1}\alpha(x)$ for each $x \in W$. For each $n \in \mathbb{N}$, let

$$W_n = W \cap X_n \quad \text{and} \quad A_n = \{x \in A \mid \beta(x) \geq 2^{-n}\}.$$

Then each W_n is a Z-submanifold of an ℓ_2-manifold W and each A_n is a closed set in A such that $f_0(A_n) \subset W_n$. Moreover, it follows that

$$W \cap M = \bigcup_{n\in\mathbb{N}} W_n \quad \text{and} \quad A = \bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} \operatorname{int} A_n.$$

Since each W_n is a Z-set in W, $W \cap M$ is a Z_σ-set in W.

Since W_1 is an ℓ_2-manifold, $f_0|A_1$ is 2^{-3}-homotopic to a Z-embedding $g_1 : A_1 \to W_1$ (cf. [Sa, §3]). By the Homotopy Extension Theorem (cf. [vM$_3$, 5.1.3]), f_0 is 2^{-3}-homotopic to a map $f_1 : A \to W \cap M$ such that $f_1|A_1 = g_1$, $f_1(A_2) \subset W_2$ and $f_1|A \setminus A_2 = f_0|A \setminus A_2$. Since W_2 is an

ℓ_2-manifold, $f_1|A_2$ is 2^{-4}-homotopic to a Z-embedding $g_2 : A_2 \to W_2$ such that $g_2|A_1 = g_1 = f_1|A_1$. Again by the Homotopy Extension Theorem, f_1 is 2^{-4}-homotopic to a map $f_2 : A \to W \cap M$ such that $f_2|A_2 = g_2$, $f_2(A_3) \subset W_3$ and $f_2|A \setminus A_3 = f_0|A \setminus A_3$. Thus we inductively construct maps $f_n : A \to W \cap M$ such that f_n is 2^{-n-2}-homotopic to f_{n-1}, $f_n|A_n$ is a Z-embedding into W_n, and $f_n|A \setminus A_{n+1} = f_0|A \setminus A_{n+1}$.

We define $f : A \to W \cap M$ by $f|A_n = f_n|A_n$ for each $n \in \mathbb{N}$. Since $(f_n)_{n\in\mathbb{N}}$ is uniformly Cauchy, f is the uniform limit of $(f_n)_{n\in\mathbb{N}}$, whence f is continuous. Since each pair of points of A are contained in some A_n and $f_n|A_n$ is injective, it follows that f is injective. For each $x \in A_n \setminus A_{n-1}$, $f(x) = f_n(x)$ and $f_{n-2}(x) = f_0(x)$, whence

$$\begin{aligned} d(f(x), x) &\leq d(f_n(x), f_{n-1}(x)) + d(f_{n-1}(x), f_{n-2}(x)) + d(f_0(x), x) \\ &< 2^{-n-2} + 2^{-n-1} + 2^{-1}\alpha(x) < 2^{-n} + 2^{-1}\alpha(x) \\ &\leq \beta(x) + 2^{-1}\alpha(x) < \alpha(x). \end{aligned}$$

To see that f is closed, let $(x_i)_{i\in\mathbb{N}}$ be a sequence in A such that $f(x_i)$ converges to y in W. Assume that $\liminf \alpha(x_i) = 0$. Then $(x_i)_{i\in\mathbb{N}}$ has a subsequence $(x_{n_i})_{i\in\mathbb{N}}$ such that $\lim \alpha(x_{n_i}) = 0$, whence x_{n_i} converges to y because

$$d(x_{n_i}, y) \leq d(x_{n_i}, f(x_{n_i})) + d(f(x_{n_i}), y) < \alpha(x_{n_i}) + d(f(x_{n_i}), y).$$

This contradicts $\alpha(y) \neq 0$. Therefore $\liminf \alpha(x_i) > 2^{-n}$ for some $n \in \mathbb{N}$, which means that $\alpha(x_i) \geq 2^{-n}$ for sufficiently large $i \in \mathbb{N}$, whence $f(x_i) = f_n(x_i)$ because $x_i \in A_n$. Since $f_n|A_n$ is a closed embedding, x_i converges to some x in A_n. This means that f is closed. Since $f(A)$ is a closed set in W and $f(A) \subset W \cap M$, $f(A)$ is a Z-set in W ([vM$_3$, 6.2.2(3)]), hence f is a Z-embedding.

(a)$\Rightarrow$(c): For each $n \in \mathbb{N}$, let $Q_n = [-1 + 2^{-n}, 1 - 2^{-n}]^\omega \subset Q$. Note that $X \approx X \times \ell_2 \approx X \times \ell_2 \times Q \approx X \times Q$ by the Stability Theorem for ℓ_2-manifolds (cf. [Sa, §2]), $X \times \Sigma = \bigcup_{n\in\mathbb{N}} X \times Q_n$ and each $X \times Q_n$ is a Z-set in $X \times Q$, which in turn is an ℓ_2-manifold. We have the deformation $h : X \times Q \times \mathbf{I} \to X \times Q$ defined by $h_t(x, y) = (x, (1-t)y)$. Then $h_0 = \mathrm{id}$ and $h_t(X \times Q) \subset X \times Q_n$ for $t \geq 2^{-n}$. Thus $X \times \Sigma$ satisfies the condition (c) for $X \times Q$. The implication (c)$\Rightarrow$(a) has already been proved. Hence $X \times \Sigma$ is a Z-absorber for $X \times Q$. Since $(X, M) \approx (X \times Q, X \times \Sigma)$ by Theorem 1.5, M also satisfies the condition (c). ∎

2. Proofs of Theorems. Let $h : A \times \mathbf{I} \to X$ be a deformation of $A \subset X$. We define a deformation $h^* : P_{\mathfrak{F}}(A) \times \mathbf{I} \to P_{\mathfrak{F}}(X)$ as follows:

$$h_t^*(\mu) = \sum_{i=1}^k s_i \delta_{h_t(x_i)} \quad \text{for each } \mu = \sum_{i=1}^k s_i \delta_{x_i} \in P_{\mathfrak{F}}(A).$$

In other words,

$$\int_X f\,dh_t^*(\mu) = \int_X fh_t\,d\mu = \sum f(h_t(x))\mu(x) \quad \text{for each } f \in C_{\mathrm{b}}(X).$$

Then the continuity of h^* is obvious. Note that $h_t^*(P_k(A)) \subset P_k(h_t(A)) \subset P_k(X)$ for every $t \in \mathbf{I}$ and $k \in \mathbb{N}$. If $h_0 = \mathrm{id}$ then $h_0^* = \mathrm{id}$.

Here we observe that P_k satisfies the conditions (1)–(10) in Remark 3. Indeed, as mentioned in the Introduction, P_k satisfies (1) and (10). Using the open base in [Va$_2$, Part II, Remark 3 to Theorem 2] (or [NT, Proposition 2.1]), it can be shown that $P_k(A)$ is closed in $P_k(X)$ if A is closed in X, that is, P_k satisfies (2). And as seen in the above, P_k satisfies (3). Obviously P_k satisfies (4)–(7). We have the continuous surjection $\pi : X^k \times \Delta^{k-1} \to P_k(X)$ defined by

$$\pi(x_1, \ldots, x_k; s_1, \ldots, s_k) = \sum_{i=1}^{k} s_i \delta_{x_i},$$

where Δ^{k-1} is the standard $(k-1)$-simplex. Observe that $\pi^{-1}(\mu)$ is finite for each $\mu \in P_k(X)$. If X is a finite-dimensional compactum, then so is $P_k(X)$, that is, P_k satisfies (8). It has been shown in [Va$_1$] that if X is separable completely metrizable then so is $P(X)$, hence $P_k(X)$, which means that P_k satisfies (9).

In the following, we use only these properties (1)–(10).

LEMMA 2.1. *If A is a Z-set in an ANR X, then each $P_k(A)$ is a Z-set in $P_k(X)$.*

Proof. First note that $P(A)$ is a closed set in $P(X)$. Since A is a Z-set in an ANR X, there is a deformation $h : X \times \mathbf{I} \to X$ such that $h_0 = \mathrm{id}$ and $h_t(X) \subset X \setminus A$ if $0 < t \le 1$ [To$_1$, Theorem 2.4 with Corollary 3.3]. Then h induces the deformation $h^* : P_k(X) \times \mathbf{I} \to P_k(X)$ such that $h_0^* = \mathrm{id}$ and $h_t^*(P(X)) \subset P_k(X \setminus A) \subset P_k(X) \setminus P_k(A)$ for $0 < t \le 1$. Therefore $P_k(A)$ is a Z-set in $P_k(X)$. ∎

For each $n \in \mathbb{N}$, $P_k(Q_n) \approx Q$ [Fe$_1$], where $Q_n = [-1+2^{-n}, 1-2^{-n}]^\omega \subset Q$. Since $Q_n \subset (-1+2^{-n-1}, 1-2^{-n-1})^\omega$, Q_n is a Z-set in Q_{n+1} [vM$_3$, 6.2.4]. Then each $P_k(Q_n)$ is a Z-set in $P_k(Q_{n+1})$ by Lemma 2.1. Thus we have a tower $P_k(Q_1) \subset P_k(Q_2) \subset \ldots$ which satisfies the conditions (i) and (ii) in Theorem 1.3. To prove (a), it remains to show that $P_k(\Sigma) = \bigcup_{n\in\mathbb{N}} P_k(Q_n)$ is a cap set for $P_k(Q)$ and each $P_k(Q_n) \cap P_k(\sigma)$ is an fd-cap set for $P_k(Q_n)$.

LEMMA 2.2. *For each $k \in \mathbb{N}$, $(P_k(Q_n))_{n\in\mathbb{N}}$ has the cap in both $P_k(Q)$ and $P_k(s)$, whence $P_k(\Sigma) = \bigcup_{n\in\mathbb{N}} P_k(Q_n)$ is a cap set for both $P_k(Q)$ and $P_k(s)$.*

Proof. First note that $P_k(\Sigma) = \bigcup_{n\in\mathbb{N}} P_k(Q_n)$ and $P_k(Q_n) \approx Q$ for each $n \in \mathbb{N}$. By Lemma 2.1, each $P_k(Q_n)$ is a Z-set for $P_k(Q_{n+1})$. By the Z-set Unknotting Theorem [vM$_3$, 6.4.6], we have

$$(P_k(Q_{n+1}), P_k(Q_n)) \approx (Q \times Q, Q \times \{0\}) \approx (P_k(Q_n) \times Q, P_k(Q_n) \times \{0\}),$$

hence the tower $(P_k(Q_n))_{n\in\mathbb{N}}$ is expansive. Let $\theta : [-1,1] \times \mathbf{I} \to [-1,1]$ be the deformation defined by

$$\theta_t(s) = \begin{cases} s & \text{if } s \le 1-t, \\ 1-t & \text{if } s \ge 1-t. \end{cases}$$

We define a deformation $h : Q \times \mathbf{I} \to Q$ by $h_t(x_1, x_2, \ldots) = (\theta_t(x_1), \theta_t(x_2), \ldots)$. Then h induces the deformation $h^* : P_k(Q) \times \mathbf{I} \to P_k(Q)$ such that $h^*(P_k(s) \times \mathbf{I}) \subset P_k(s)$, $h^*_0 = \mathrm{id}$ and each $h^*_{2^{-n}}$ is a retraction onto $P_k(Q_n)$. Hence $(P_k(Q_n))_{n\in\mathbb{N}}$ has the mapping absorption property in both $P_k(Q)$ and $P_k(s)$. By Lemma 1.1, we have the result. ∎

LEMMA 2.3. *For each $k, n \in \mathbb{N}$, $P_k(Q_n) \cap P_k(\sigma) = P_k(Q_n \cap \sigma)$ is an fd-cap set for $P_k(Q_n)$.*

Proof. For each $i \in \mathbb{N}$, let $X^i_n = [-1+2^{-n}, 1-2^{-n}]^i \times \{0\} \subset Q_n$. Then $Q_n \cap \sigma = \bigcup_{i\in\mathbb{N}} X^i_n$. Each $P_k(X^i_n)$ is a finite-dimensional compactum, which is a Z-set in $P_k(Q_n)$ by Lemma 2.1. We define a deformation $\varphi : X^i_n \times \mathbf{I} \to X^{i+1}_n$ by

$$\varphi_t(x_1, \ldots, x_i, 0, 0, \ldots) = (x_1, \ldots, x_i, t/2, 0, \ldots).$$

Note that φ is an embedding. Let $\varphi^* : P_k(X^i_n) \times \mathbf{I} \to P_k(X^{i+1}_n)$ be the deformation induced by φ. Then $\varphi^*_0 = \mathrm{id}$ and φ^* is obviously injective by the definition, that is, φ^* is an embedding. Hence the tower $(P_k(X^i_n))_{i\in\mathbb{N}}$ is finitely expansive. We define a deformation $h : Q_n \times \mathbf{I} \to Q_n$ as follows: $h_0 = \mathrm{id}$ and

$$h_t(x_1, x_2, \ldots) = (x_1, \ldots, x_i, (2 - 2^i t)x_{i+1}, 0, 0, \ldots) \quad \text{if } 2^{-i} < t \le 2^{-i+1}.$$

Then h induces the deformation $h^* : P_k(Q_n) \times \mathbf{I} \to P_k(Q_n)$ such that $h^*_0 = \mathrm{id}$ and each $h^*_{2^{-i}}$ is a retraction onto $P_k(X^i_n)$. Hence $(P_k(X^i_n))_{i\in\mathbb{N}}$ has the mapping absorption property. By Lemma 1.1, $P_k(Q_n) \cap P(\sigma) = P_k(Q_n \cap \sigma) = \bigcup_{i\in\mathbb{N}} P_k(X^i_n)$ is an fd-cap set for $P_k(Q_n)$. ∎

It is known that $P_k(\ell_2) \approx \ell_2$ [NT]. But we will give a short proof.

LEMMA 2.4. *For each $k \in \mathbb{N}$, $(P_k(Q), P_k(s)) \approx (Q, s)$, hence $P_k(\ell_2) \approx \ell_2$.*

Proof. We show that $P_k(Q) \setminus P_k(s)$ is a cap set for $P_k(Q)$. Then the result will follow from Lemma 1.2 because $P_k(Q) \approx Q$. It has been shown in [Va$_1$] that $P(X)$ is separable completely metrizable if so is X. Then $P_k(s)$ is completely metrizable, so $P_k(Q) \setminus P_k(s)$ is F_σ in $P_k(Q)$. Let $h : Q \times \mathbf{I} \to Q$ be the deformation defined by $h_t(x) = (1-t)x$. Then h induces the deformation

$h^* : P_k(Q) \times \mathbf{I} \to P_k(Q)$ such that $h_0^* = \mathrm{id}$ and $h_t^*(P_k(Q)) \subset P_k(s)$ for $0 < t \le 1$. Therefore $P_k(Q) \setminus P_k(s)$ is a Z_σ-set in $P_k(Q)$. Observe that

$$P_k(Q) \setminus P_k(s) = \{\mu \in P_k(Q) \mid \operatorname{supp} \mu \not\subset s\} \supset P_k(Q \setminus s).$$

Since $(Q, Q \setminus s) \approx (Q, \Sigma)$, we have $(P_k(Q), P_k(Q \setminus s)) \approx (P_k(Q), P_k(\Sigma))$, whence $P_k(Q \setminus s)$ is a cap set for $P_k(Q)$ by Lemma 2.2. Since any Z_σ-set containing a cap set is itself a cap set [Ch, Lemma 4.2 or Theorem 6.6], $P_k(Q) \setminus P_k(s)$ is a cap set for $P_k(Q)$. ■

Remark 4. As for the above lemmas, 2.1 follows from (1)–(3); 2.2 from (1)–(4) and (10); 2.3 from (1)–(5) and (8); 2.4 from (1)–(4), (6), (9) and (10).

Proof of the Main Theorem. First we show (a). Since $P_k(Q) \approx Q$, we can apply Theorem 1.3 with Lemmas 2.1–2.3 to obtain $(P_k(Q), P_k(\Sigma), P_k(\sigma)) \approx (Q, \Sigma, \sigma)$. In particular, $(P_k(\Sigma), P_k(\sigma)) \approx (\Sigma, \sigma)$. On the other hand, $(P_k(Q), P_k(s)) \approx (Q, s)$ by Lemma 2.4. By Lemmas 1.2 and 2.2, $(P_k(Q), P_k(s), P_k(\Sigma)) \approx (Q, s, \Sigma)$. Applying Theorem 2.4 of [CDM], we have

$$(P_k(Q), P_k(s), P_k(\Sigma), P_k(\sigma)) \approx (Q, s, \Sigma, \sigma).$$

Next we prove (b) by applying Theorem 1.4. For each $n, i \in \mathbb{N}$, let

$$A_i^n = \underbrace{Q_i \times \ldots \times Q_i}_{n \text{ times}} \times Q \times Q \times \ldots \subset Q^\omega.$$

Then observe that for each $n_1 < \ldots < n_m$ and $i_1, \ldots, i_m \in \mathbb{N}$,

$$(*) \qquad \bigcap_{j=1}^m A_{i_j}^{n_j} = \underbrace{Q_{p_1} \times \ldots \times Q_{p_1}}_{n_1 \text{ times}} \times \underbrace{Q_{p_2} \times \ldots \times Q_{p_2}}_{n_2 - n_1 \text{ times}} \times \ldots \times \underbrace{Q_{p_m} \times \ldots \times Q_{p_m}}_{n_m - n_{m-1} \text{ times}} \times Q \times Q \times \ldots,$$

where $p_k = \min\{i_k, \ldots, i_m\}$. It is proved in [vM$_2$, Thm. 4.1] that $(A_i^n)_{n,i\in\mathbb{N}}$ is a Z-matrix in Q^ω which has all the properties of Theorem 1.4. Therefore $\bigcap_{n\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} A_i^n \approx (\ell_2^f)^\omega$. Then it follows that

$$P_k((\ell_2^f)^\omega) \approx P_k\Big(\bigcap_{n\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} A_i^n\Big) = \bigcap_{n\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} P_k(A_i^n).$$

Since $P_k(Q^\omega) \approx Q$ and $(P_k(A_i^n))_{n,i\in\mathbb{N}}$ is a Z-matrix in $P_k(Q^\omega)$ by Lemma 2.1, it suffices to show that $(P_k(A_i^n))_{n,i\in\mathbb{N}}$ has all the properties of Theorem 1.4.

Let $n_1 < \ldots < n_m$ and $i_1, \ldots, i_m \in \mathbb{N}$. Since $\bigcap_{j=1}^m A_{i_j}^{n_j} \approx Q$, we have $\bigcap_{j=1}^m P_k(A_{i_j}^{n_j}) = P_k(\bigcap_{j=1}^m A_{i_j}^{n_j}) \approx Q$, that is, 1.4(ii). For each $p, i \in \mathbb{N}$, we also have $P_k(A_i^{n_m+p}) \cap \bigcap_{j=1}^m P_k(A_{i_j}^{n_j}) \approx Q$. Since Q_i is a Z-set in Q_{i+1}, $A_i^{n_m+p} \cap \bigcap_{j=1}^m A_{i_j}^{n_j}$ is a Z-set in $A_{i+1}^{n_m+p} \cap \bigcap_{j=1}^m A_{i_j}^{n_j}$ (see $(*)$). Then by

Lemma 2.1,

$$P_k(A_i^{n_m+p}) \cap \bigcap_{j=1}^m P_k(A_{i_j}^{n_j}) = P_k\Big(A_i^{n_m+p} \cap \bigcap_{j=1}^m A_{i_j}^{n_j}\Big)$$

is a Z-set in $P_k(A_{i+1}^{n_m+p} \cap \bigcap_{j=1}^m A_{i_j}^{n_j}) = P_k(A_{i+1}^{n_m+p}) \cap \bigcap_{j=1}^m P_k(A_{i_j}^{n_j})$. By the same proof as for Lemma 2.2, it follows that $(P_k(A_i^{n_m+p}) \cap \bigcap_{j=1}^m P_k(A_{i_j}^{n_j}))_{i\in\mathbb{N}}$ has the cap for $P_k(\bigcap_{j=1}^m A_{i_j}^{n_j})$, that is, 1.4(iii) holds. Similarly, 1.4(i) holds, that is, $(P_k(A_i^n))_{i\in\mathbb{N}}$ has the cap for $P_k(Q^\omega)$. To see 1.4(iv), suppose

$$\bigcap_{j=1}^m P_k(A_{i_j}^{n_j}) = P_k\Big(\bigcap_{j=1}^m A_{i_j}^{n_j}\Big) \not\subset P_k(A_i^n).$$

Then $\bigcap_{j=1}^m A_{i_j}^{n_j} \not\subset A_i^n$, which implies that $A_i^n \cap \bigcap_{j=1}^m A_{i_j}^{n_j}$ is a Z-set in $\bigcap_{j=1}^m A_{i_j}^{n_j}$. By Lemma 2.1, it follows that $P_k(A_i^n) \cap \bigcap_{j=1}^m P_k(A_{i_j}^{n_j}) = P_k(A_i^n \cap \bigcap_{j=1}^m A_{i_j}^{n_j})$ is a Z-set in $P_k(\bigcap_{j=1}^m A_{i_j}^{n_j}) = \bigcap_{j=1}^m P_k(A_{i_j}^{n_j})$, that is, we have 1.4(iv).

To see (c), notice that each $P_k(\ell_2 \times Q_n)$ is a Z-set in $P_k(\ell_2 \times Q)$ by Lemma 2.1, $P_k(\ell_2 \times Q_n) \approx \ell_2$ by Lemma 2.4 and $P_k(\ell_2 \times \Sigma) = \bigcup_{n\in\mathbb{N}} P_k(\ell_2 \times Q_n)$. We have the deformation $h : \ell_2 \times Q \times \mathbf{I} \to \ell_2 \times Q$ defined by $h_t(x,y) = (x,(1-t)y)$. Let $h^* : P_k(\ell_2 \times Q) \times \mathbf{I} \to P_k(\ell_2 \times Q)$ be the deformation induced by h. Then $h_0^* = \mathrm{id}$ and $h_t^*(P_k(\ell_2 \times Q)) \subset P_k(\ell_2 \times Q_n)$ for $t \ge 2^{-n}$. By Theorem 1.7, $P_k(\ell_2 \times \Sigma)$ is a Z-absorber for $P_k(\ell_2 \times Q)$. Since $P_k(\ell_2 \times Q) \approx \ell_2$, (c) follows from Corollary 1.6. ■

Remark 5. In the above, (a) follows from (1)–(6) and (8)–(10); (b) from (1)–(5), (7) and (10); (c) from (1)–(6), (9) and (10) (cf. Remark 4). Thus our Main Theorem holds if P_k is replaced by a functor $F : \mathcal{SM} \to \mathcal{SM}$ with the conditions (1)–(10).

Proof of Theorems 2 and 3. As seen in Remark 5, it suffices to see that $\mathfrak{F}_k$ and SP_G^k satisfy the conditions (1)–(10). The conditions (1), (2) and (10) have been seen in Remark 3 and the conditions (4)–(7) are obvious.

For a deformation $h : A \times \mathbf{I} \to X$ of $A \subset X$, the induced deformation $h^* : \mathfrak{F}_k(A) \times \mathbf{I} \to \mathfrak{F}_k(X)$ is defined by $h^*(F,t) = h(F \times \{t\})$, whence the continuity of h^* is easy to see. Thus $\mathfrak{F}_k$ satisfies (3). We have the natural continuous surjection $p : X^k \to \mathfrak{F}_k(X)$ defined by $p(x_1,\ldots,x_k) = \{x_1,\ldots,x_k\}$. Since p has finite fibers, if X is a finite-dimensional compactum then so is $\mathfrak{F}_k(X)$, that is, $\mathfrak{F}_k$ satisfies (8). Obviously, $\mathfrak{F}_k(U)$ is open in $\mathfrak{F}_k(X)$ for any open set U in X. If X is separable completely metrizable, then X is a G_δ-set in a metrizable compactification $\widetilde{X}$, which implies that $\mathfrak{F}_k(X)$ is a G_δ-set in the compact metrizable space $\mathfrak{F}_k(\widetilde{X}) = \widetilde{p}(\widetilde{X}^k)$, where $\widetilde{p} : \widetilde{X}^k \to \mathfrak{F}_k(\widetilde{X})$ is the natural surjection. Hence $\mathfrak{F}_k(X)$ is separable completely metrizable, that is, $\mathfrak{F}_k$ satisfies (9).

Since the quotient map $q : X^k \to \mathrm{SP}^k_G(X)$ is open, $\mathrm{SP}^k_G(U)$ is open in $\mathrm{SP}^k_G(X)$ for any open set U in X. If X is separable completely metrizable, then X is a G_δ-set in a metrizable compactification $\widetilde{X}$, which implies that $\mathrm{SP}^k_G(X)$ is a G_δ-set in the compact metrizable space $\mathrm{SP}^k_G(\widetilde{X}) = \widetilde{q}(\widetilde{X}^k)$, where $\widetilde{q} : \widetilde{X}^k \to \mathrm{SP}^k_G(\widetilde{X})$ is the quotient map. Hence $\mathrm{SP}^k_G(X)$ is separable completely metrizable, that is, SP^k_G satisfies (9). Since q has finite fibers, if X is a finite-dimensional compactum then so is $\mathrm{SP}^k_G(X)$, that is, SP^k_G satisfies (8). For a deformation $h : A \times \mathbf{I} \to X$ of $A \subset X$, the induced deformation $h^* : \mathrm{SP}^k_G(A) \times \mathbf{I} \to \mathrm{SP}^k_G(X)$ is defined by $h^*_t(q(x_1, \ldots, x_k)) = q(h_t(x_1), \ldots, h_t(x_k))$, whence the continuity of h^* is clear. Thus SP^k_G satisfies (3). ∎

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Institute of Mathematics at Hanoi
P.O. Box 631
Bo Ho, Hanoi
Vietnam

Institute of Mathematics
University of Tsukuba
Tsukuba 305, Japan
E-mail: sakaiktr@sakura.cc.tsukuba.ac.jp

Department of Mathematics
Indiana University at Bloomington
Bloomington, Indiana 47405-5701
U.S.A.
E-mail: nhnguyen@juliet.ucs.indiana.edu

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