C O L L O Q U I U M   M A T H E M A T I C U M

VOL. 71 | 1996 | NO. 2

QUASI-COMMUTATIVE POLYNOMIAL ALGEBRAS AND THE POWER PROPERTY OF 2×2 QUANTUM MATRICES

BY

PIOTR JĘDRZEJEWICZ (TORUŃ)

Let K be a field. Recall (e.g. [4], 3.1, [5], 4.2.1) that a *quadratic algebra* is a graded associative K-algebra

$$A = \bigoplus_{k=0}^{\infty} A_k,$$

where $A_0 = K$, $\dim_K A_1 < \infty$ and A is generated by A_1 with the ideal of relations generated by quadratic ones:

$$A = T(A_1)/(R_A),$$

where $T(A_1)$ is the tensor algebra of A_1 and $R_A \subset A_1^{\otimes 2}$. It is convenient to write

$$A \leftrightarrow \{A_1, R_A\}.$$

In this paper we consider quadratic algebras of a special type, with the relations quite similar to ordinary commutativity relations. This approach generalizes different examples of quadratic algebras. In the case of two generators we can unify the definitions of algebras

$$A_q^{2|0} = K\langle x_1, x_2\rangle/(x_1x_2 - q^{-1}x_2x_1)$$

and

$$A_J = K\langle x_1, x_2\rangle/(x_1x_2 - x_2x_1 - x_1^2)$$

(notation from [4], 1.2, [5], 4.2.8, 4.4.3). This will allow us to have a common view of some properties of quantum matrices connected with these two algebras. In particular, our Theorem is a generalization of [5], 1.3.8(v) and [3], (ii).

Definition 1. Let V be an m-dimensional linear space over K. Denote by R the subspace of $V^{\otimes 2}$ spanned by elements $x \otimes y - y \otimes x$ for $x, y \in V$. For each $P \in GL(V)$ we define the quadratic algebra

$$A^P = A^P[V] \leftrightarrow \{V, (I \otimes P)(R)\},$$

1991 *Mathematics Subject Classification*: Primary 16W30.

where I is the identity operator. A *quasi-commutative polynomial algebra* is a quadratic algebra of the form $A^P[V]$ for some V and $P \in GL(V)$.

LEMMA 1. *Quadratic algebras* $A^{P_1}[V]$ *and* $A^{P_2}[V]$ *are isomorphic if and only if there exist* $C \in \mathrm{GL}(V)$ *and* $\alpha \in K \setminus \{0\}$ *such that* $P_2 = \alpha \cdot CP_1C^{-1}$.

Proof. Any isomorphism $A^{P_1}[V] \to A^{P_2}[V]$ is an extension of a linear automorphism $C : V \to V$ such that $(C \otimes C)(I \otimes P_1)(R) = (I \otimes P_2)(R)$, and this condition is equivalent to $(I \otimes CP_1C^{-1})(R) = (I \otimes P_2)(R)$, which means that $\alpha \cdot CP_1C^{-1} = P_2$ for some $\alpha \neq 0$.

Now we obtain a linear basis of $A^P[V]$. Choose a basis $x_1, \ldots, x_m$ of V and its dual basis $x^1, \ldots, x^m$ of V^*. We have

$$A^P = \bigoplus_{k=0}^{\infty} A_k^P = K\langle x_1, \ldots, x_m\rangle/(x_iP(x_j) - x_jP(x_i), 1 \leq i < j \leq m).$$

One easily verifies that in A_k^P the following relations hold:

$$x_{i_{\sigma(1)}}P(x_{i_{\sigma(2)}})\ldots P^{k-1}(x_{i_{\sigma(k)}}) = x_{i_1}P(x_{i_2})\ldots P^{k-1}(x_{i_k})$$

for all $i_1, \ldots, i_k \in \{1, \ldots, m\}$ and $\sigma \in S_k$. This implies that the monomials

$$x_{i_1}P(x_{i_2})\ldots P^{k-1}(x_{i_k})$$

with $1 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq m$ span A_k^P. Since they are linearly independent (proof by induction on k), this is a basis of A_k^P. It makes this algebra very similar to the algebra of commutative polynomials. In particular, we have

$$\dim A_k^P = \binom{m+k-1}{k}.$$

Note that any quadratic algebra with two generators and one non-degenerate relation is quasi-commutative polynomial, but some well known quadratic algebras with more than two generators are not. This is discussed in the following two lemmas.

LEMMA 2. *Let* $A \leftrightarrow \{A_1, R_A\}$, *where* $\dim A_1 = 2$ *and* $\dim R_A = 1$. *If for* $x, y \in V$ *we have* $R_A \neq K(x \otimes y)$, *then* A *is a quasi-commutative polynomial algebra.*

Proof. Take $a_{ij} \in K$ such that the generating relation is

$$\begin{aligned} r &= a_{11}x_1 \otimes x_1 + a_{12}x_1 \otimes x_2 + a_{21}x_2 \otimes x_1 + a_{22}x_2 \otimes x_2 \\ &= x_1 \otimes (a_{11}x_1 + a_{12}x_2) - x_2 \otimes (-a_{21}x_1 - a_{22}x_2). \end{aligned}$$

Since r is not of the form $x \otimes y$, the operator P defined by $P(x_1) = (-a_{21}x_1 - a_{22}x_2)$ and $P(x_2) = a_{11}x_1 + a_{12}x_2$ is non-degenerate and $R_A = (I \otimes P)(x_1 \otimes x_2 - x_2 \otimes x_1)$. Hence $A = A^P$.

LEMMA 3. *The algebra* $A = K\langle x_1, \ldots, x_m\rangle/(x_i x_j - q_{ij}^{-1} x_j x_i, 1 \le i < j \le m)$ *is a quasi-commutative polynomial algebra if and only if there exist* $q_1, \ldots, q_m$ *such that* $q_{ij} = q_i^{-1} q_j$ *for all* $i < j$.

Proof. Suppose that $A = A^P$. Let (p_k^i) be the matrix of P with respect to the basis $x_1, \ldots, x_m$. Take any i, j such that $1 \le i < j \le m$. We have

$$r_{ij} = \sum_{l=1}^{m} p_j^l x_i \otimes x_l - \sum_{k=1}^{m} p_i^k x_j \otimes x_k = x_i \otimes P(x_j) - x_j \otimes P(x_i) \in R_A,$$
$$x^i \otimes x^j + q_{ij} x^j \otimes x^i \in R_A^{\perp},$$

so that $p_j^j - q_{ij} p_i^i = (x^i \otimes x^j + q_{ij} x^j \otimes x^i)(r_{ij}) = 0$.

On the other hand, if there exist $q_1, \ldots, q_m$ such that $q_{ij} = q_i^{-1} q_j$ for all $i < j$, then for $p_k^i = \delta_k^i q_k$ we get $A = A^P$.

As a consequence, note that for $m > 2$ and $q \neq 1$ the algebra

$$A_q^{m|0} = K\langle x_1, \ldots, x_m\rangle/(x_i x_j - q^{-1} x_j x_i, 1 \le i < j \le m)$$

is not quasi-commutative polynomial. The algebra from Lemma 3 is considered in [8] and the case of $q_{ij} = q_i^{-1} q_j$ is connected with the version of the power property given at the end of that paper.

Now recall some general constructions of “quantum endomorphism semigroups”.

Let $A \leftrightarrow \{V, R_A\}$. Put

$$E(A) \leftrightarrow \{V^* \otimes V, S_{23}((R_A)^{\perp} \otimes R_A)\},$$

where $(R_A)^{\perp} \subset (V \otimes V)^* \simeq (V^* \otimes V^*)$ is the annihilator of R_A and $S_{23} : V^* \otimes V^* \otimes V \otimes V \to V^* \otimes V \otimes V^* \otimes V$ is the isomorphism interchanging the 2nd and 3rd components (see [4], 4.5b, [5], 4.2.6). The canonical map

$$V \to (V^* \otimes V) \otimes V : x_k \mapsto \sum_{i=1}^{m} z_k^i \otimes x_i,$$

where $z_k^i = x^i \otimes x_k$ for $1 \le i, k \le m$, extends to a homomorphism of algebras

$$\delta_A : A \to E(A) \otimes A.$$

$E(A)$ is far from any kind of commutativity, it has m^2 generators and only $\binom{m}{2} \cdot (m^2 - \binom{m}{2}) = \frac{1}{2}\binom{m^2}{2}$ relations. To obtain a good analogue of a commutative algebra we have to add the “second half” of the relations.

Let $A \leftrightarrow \{V, R_A\}$, $B \leftrightarrow \{V^*, R_B\}$. Put

$$E(A, B) \leftrightarrow \{V^* \otimes V, S_{23}((R_A)^{\perp} \otimes R_A + R_B \otimes (R_B)^{\perp})\}$$

(compare [4], 6.2, [5], 4.2.7, [6], 1.4). The canonical maps $V \to (V^* \otimes V) \otimes V$

(as above) and

$$V^* \to (V^* \otimes V) \otimes V^* : x^i \mapsto \sum_{k=1}^{m} z_k^i \otimes x^k$$

extend to homomorphisms of algebras

$$\delta^1_{A,B} : A \to E(A,B) \otimes A, \qquad \delta^2_{A,B} : B \to E(A,B) \otimes B.$$

$E(A,B)$ can be thought of as the "greatest common factor" of $E(A)$ and $E(B)$, both of them being generated by $V^* \otimes V$, the latter via the canonical isomorphism $V \otimes V^* \simeq V^* \otimes V$.

Now, we apply these constructions to quasi-commutative polynomial algebras and write down the relations in terms of the basis $z_k^i = x^i \otimes x_k$ of $V^* \otimes V$, which can be considered as a matrix $Z = (z_k^i)$.

Definition 2. Let $P \in \mathrm{GL}(V)$. Put

$$E^P = E^P[V^* \otimes V] = E(A^P[V]).$$

The relations of $E^P[V^* \otimes V]$ in the above basis are the following:

$$z_k^i t_l^i = z_l^i t_k^i, \qquad 1 \le i \le m, 1 \le k < l \le m,$$
$$z_k^i t_l^j - z_l^i t_k^j = z_l^j t_k^i - z_k^j t_l^i, \qquad 1 \le i < j \le m, 1 \le k < l \le m,$$

where t_k^i are the entries of the matrix $T = P^{-1}ZP$.

It is useful to write these relations in matrix form:

$$\begin{pmatrix} z_k^i & z_l^i \\ z_k^j & z_l^j \end{pmatrix} \cdot \begin{pmatrix} t_l^j & -t_l^i \\ -t_k^j & t_k^i \end{pmatrix} = \begin{pmatrix} D_{kl}^{ij} & 0 \\ 0 & D_{kl}^{ij} \end{pmatrix}$$

for all $i < j, k < l$ and suitable D_{kl}^{ij} (i.e. defined by these relations).

Definition 3. Let $P, Q \in \mathrm{GL}(V)$. Put

$$E^{P,Q} = E^{P,Q}[V^* \otimes V] = E(A^P[V], A^{Q^*}[V^*]).$$

The relations of $E^{P,Q}[V^* \otimes V]$ consist of the ones of $E^P[V^* \otimes V]$ (above) and the ones of $E^{Q^*}[V \otimes V^*]$:

$$z_k^i s_k^j = z_k^j s_k^i, \qquad 1 \le i < j \le m, 1 \le k \le m,$$
$$z_k^i s_l^j - z_k^j s_l^i = z_l^j s_k^i - z_l^i s_k^j, \qquad 1 \le i < j \le m, 1 \le k < l \le m,$$

where $(s_k^i) = QZQ^{-1}$, or in matrix form:

$$\begin{pmatrix} z_l^j & -z_l^i \\ -z_k^j & z_k^i \end{pmatrix} \cdot \begin{pmatrix} s_k^i & s_l^i \\ s_k^j & s_l^j \end{pmatrix} = \begin{pmatrix} D_{kl}'^{ij} & 0 \\ 0 & D_{kl}'^{ij} \end{pmatrix}$$

for all $i < j, k < l$.

From now on we assume that $m = 2$ and we consider only 2×2 matrices.

Define

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^s = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Note that for any 2×2 matrix M we have $(M^s)^s = M$, $\operatorname{tr} M^s = \operatorname{tr} M$, $M + M^s = (\operatorname{tr} M)\cdot I$ and $M\cdot M^s = (\det M)\cdot I$. Also, $(M^s)^k = (M^k)^s$ for any positive integer k. If M is invertible, then $(M^s)^{-1} = (M^{-1})^s$. If the entries of matrices M and N commute, then $(MN)^s = N^s M^s$.

Let $P, Q \in \mathrm{GL}_2(K)$. The relations of $E^P[V^*\otimes V]$ reduce to one matrix equation

$$Z(P^{-1}ZP)^s = \mathrm{DET}\cdot I,$$

where $\mathrm{DET} = D^{12}_{12}$. The relations of $E^{P,Q}[V^*\otimes V]$ are the following:

$$Z(P^{-1}ZP)^s = \mathrm{DET}_1\cdot I, \qquad Z^s QZQ^{-1} = \mathrm{DET}_2\cdot I,$$

where $\mathrm{DET}_1 = D^{12}_{12}$ and $\mathrm{DET}_2 = D'^{12}_{12}$.

Lemma 4. *If* $\operatorname{tr}(QP)\neq 0$, *then* $\dim R_{E^{P,Q}} = 6$ *and* $\mathrm{DET}_1 = \mathrm{DET}_2$. *If* $\operatorname{tr}(QP) = 0$, *then* $\dim R_{E^{P,Q}} = 5$.

Proof. Let $R = K(x_1\otimes x_2 - x_2\otimes x_1)$ and $R' = K(x^1\otimes x^2 - x^2\otimes x^1)$. Since

$$(x^1\otimes Q^*(x^2) - x^2\otimes Q^*(x^1))(x_1\otimes P(x_2) - x_2\otimes P(x_1)) = \operatorname{tr}(QP),$$

we have $(I\otimes Q^*)(R')\subset ((I\otimes P)(R))^\perp$ if and only if $\operatorname{tr}(QP) = 0$. Therefore

$$\dim((I\otimes P)(R))^\perp\otimes(I\otimes P)(R)\cap(I\otimes Q^*)(R')\otimes((I\otimes Q^*)(R'))^\perp = \begin{cases}1 & \text{if } \operatorname{tr}(QP)=0,\\ 0 & \text{if } \operatorname{tr}(QP)\neq 0.\end{cases}$$

Finally, $\dim R_{E^{P,Q}} = 5$ if $\operatorname{tr}(QP) = 0$ and $\dim R_{E^{P,Q}} = 6$ if $\operatorname{tr}(QP)\neq 0$. Now suppose that $\operatorname{tr}(QP)\neq 0$. For any 2×2 matrices A, B we have $\operatorname{tr}(AB^s) = \operatorname{tr}(A\cdot\operatorname{tr} B - AB) = \operatorname{tr}((\operatorname{tr} A)\cdot B - AB) = \operatorname{tr}(A^s B)$. Since $Z(QZP)^s = (QP)^s\cdot\mathrm{DET}_1$ and $Z^sQZP = QP\cdot\mathrm{DET}_2$, we get

$$\begin{aligned}\operatorname{tr}(QP)\cdot\mathrm{DET}_1 &= \operatorname{tr}((QP)^s)\cdot\mathrm{DET}_1 = \operatorname{tr}(Z(QZP)^s)\\ &= \operatorname{tr}(Z^sQZP) = \operatorname{tr}(QP)\cdot\mathrm{DET}_2,\end{aligned}$$

and hence $\mathrm{DET}_1 = \mathrm{DET}_2$.

So, when $\operatorname{tr}(QP)\neq 0$, the relations of $E^{P,Q}[V^*\otimes V]$ take the form

$$Z(P^{-1}ZP)^s = Z^sQZQ^{-1} = \mathrm{DET}\cdot I.$$

Take any $p, q\in K\setminus\{0\}$, $pq\neq -1$. For

$$P = \begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}, \qquad Q = \begin{pmatrix}1 & 0\\ 0 & q\end{pmatrix},$$

we get the quantum matrix $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ of the algebra $M_{p,q}(2)$ with the relations

$$\begin{gathered}ba = pab,\quad dc = pcd,\quad ca = qac,\quad db = qbd,\\ cb = p^{-1}qbc,\qquad da = ad + (q - p^{-1})bc,\end{gathered}$$

which is discussed in [6]–[8] and, for $p = q$, in [1], [2], [4], [5], [9], [10].

The *power property*, first noticed for $M_{p,q}(2)$, states that if the entries of the matrix Z satisfy the conditions with parameters p, q, then the entries of Z^n satisfy analogous conditions with p^n, q^n.

Let $\operatorname{char} K \neq 2$ and $p, q \in K$. For

$$P = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix},$$

we obtain the algebra $M^J_{p,q}(2)$ (considered in [3], [5], [6]) with the following relations:

$$ac = ca + qc^2, \qquad dc = cd + pc^2,$$
$$da = ad + pca - qcd, \qquad bc = cb + pqc^2 + pca + qcd,$$
$$ba = ab + pqcd + pcb + pa^2 - pad, \qquad bd = db + q^2cd + qcb - qad + qd^2.$$

We observe that the algebra with these relations (for $p, q \neq 0$) is isomorphic to one with $p' = 1$, $q' = p^{-1}q$ ($a' = pa$, $b' = b$, $c' = p^2c$, $d' = pb$).

In the case of the quantum matrix Z with the relations of $M^J_{p,q}(2)$, the entries of its nth power Z^n satisfy the relations given by parameters np, nq (see [3]).

This phenomenon is clear from the following theorem.

THEOREM. *Let* $\dim V = 2$, $P, Q \in \mathrm{GL}(V)$, $PQ = QP$, $\operatorname{tr}(QP) \neq 0$. *For any positive integer n the following equalities hold in* $E^{P,Q}[V^* \otimes V]$:

$$Z^n(P^{-n}Z^nP^n)^s = (Z^n)^sQ^nZ^nQ^{-n} = \mathrm{DET}^n \cdot I.$$

The proof will follow from Lemmas 5 and 6. The theorem remains true also in the case of $\operatorname{tr}(QP) = 0$, provided we add the relation $\mathrm{DET}_1 = \mathrm{DET}_2$.

The power property seems to be possible because these quantum matrices have enough commutation relations, namely 6 relations for 4 generators of $E^{P,Q}$. But it turns out that we need only 3 relations of E^P with one additional cubic relation to prove that the nth power satisfies the corresponding relations.

The equality $Z(P^{-1}ZP)^s = \mathrm{DET} \cdot I$ is equivalent to $Z(ZP)^s = P^s \cdot \mathrm{DET}$. We have $Z^2P^2 = Z(ZP + (ZP)^s)P - Z(ZP)^sP = ZP \cdot \operatorname{tr}(ZP) - \mathrm{DET} \cdot \det P$, i.e. putting $\mathrm{TR}_P = \operatorname{tr}(ZP)$, $\mathrm{DET}_P = \mathrm{DET} \cdot \det P$, we get an analogue of the Hamilton–Cayley Formula:

$$Z^2P^2 - ZP \cdot \mathrm{TR}_P + \mathrm{DET}_P = 0.$$

For $M_{p,q}(2)$ this formula was stated in [2], [9], and for $M^J(2)$ in [3]. Note that this implies the formula

$$Z^nP^n = Z^{n-1}P^{n-1} \cdot \operatorname{tr}(ZP) - Z^{n-2}P^{n-2} \cdot \mathrm{DET} \cdot \det P,$$

for $n \geq 2$, which will be useful below.

Let us add to E^P the cubic relation we need.

Definition 4. Denote by $E_+^P = E_+^P[V^* \otimes V]$ the algebra generated by $V^* \otimes V$ with the relations

$$Z(ZP)^s = P^s \cdot \mathrm{DET}, \qquad \mathrm{DET} \cdot \mathrm{tr}(ZP) = \mathrm{tr}(ZP) \cdot \mathrm{DET}.$$

Lemma 5. *Let* $\dim V = 2$ *and* $P \in \mathrm{GL}(V)$. *For any positive integer* n *the following equalities hold in* $E_+^P[V^* \otimes V]$:

$$Z^n(Z^nP^n)^s = (P^n)^s \cdot \mathrm{DET}^n \quad \textit{and} \quad \mathrm{DET} \cdot \mathrm{tr}(Z^nP^n) = \mathrm{tr}(Z^nP^n) \cdot \mathrm{DET}.$$

Proof. Induction on n. For $n = 0$ and $n = 1$ the formulas are obvious. Take any $n \geq 2$. Assume that the formulas hold for $n-1$ and $n-2$. We have

$$\begin{aligned} &Z^n(Z^nP^n)^s \\ &\quad = Z^n(Z^{n-1}P^{n-1})^s \cdot \mathrm{tr}(ZP) - Z^n(Z^{n-2}P^{n-2})^s \cdot \mathrm{DET} \cdot \mathrm{det}P \\ &\quad = Z(P^{n-1})^s \cdot \mathrm{DET}^{n-1} \cdot \mathrm{tr}(ZP) - Z^2(P^{n-2})^s \cdot \mathrm{DET}^{n-2} \cdot \mathrm{DET} \cdot \mathrm{det}P \\ &\quad = (Z \cdot \mathrm{tr}(ZP) - Z^2P)(P^{n-1})^s \cdot \mathrm{DET}^{n-1} = Z(ZP)^s(P^{n-1})^s \cdot \mathrm{DET}^{n-1} \\ &\quad = (P^n)^s \cdot \mathrm{DET}^n. \end{aligned}$$

Since $\mathrm{tr}(Z^nP^n) = \mathrm{tr}(Z^{n-1}P^{n-1}) \cdot \mathrm{tr}(ZP) - \mathrm{tr}(Z^{n-2}P^{n-2}) \cdot \mathrm{DET} \cdot \mathrm{det}P$, we get $\mathrm{DET} \cdot \mathrm{tr}(Z^nP^n) = \mathrm{tr}(Z^nP^n) \cdot \mathrm{DET}$.

Note that applying Lemma 5 to $E_+^{Q^*}[V \otimes V^*]$, we get

$$(Z^t)^n((Z^t)^n(Q^t)^n)^s = ((Q^t)^n)^s \cdot \mathrm{DET}^n,$$

and $(Z^t)^n$ is of course very different from $(Z^n)^t$, so this is not what we need. But we can get what we need by a dual argument.

Definition 5. Denote by $E_-^{Q^*} = E_-^{Q^*}[V \otimes V^*]$ the algebra generated by $V \otimes V^*$ with the relations

$$(Q^{-1}Z)^sZ = (Q^{-1})^s \cdot \mathrm{DET}, \qquad \mathrm{DET} \cdot \mathrm{tr}(Q^{-1}Z) = \mathrm{tr}(Q^{-1}Z) \cdot \mathrm{DET}.$$

Lemma 6. *Let* $\dim V = 2$ *and* $Q \in \mathrm{GL}(V)$. *For any positive integer* n *the following equalities hold in* $E_-^{Q^*}[V \otimes V^*]$:

$$(Q^{-n}Z^n)^sZ^n = (Q^{-n})^s \cdot \mathrm{DET}^n \quad \textit{and} \quad \mathrm{DET} \cdot \mathrm{tr}(Q^{-n}Z^n) = \mathrm{tr}(Q^{-n}Z^n) \cdot \mathrm{DET}.$$

The proof is analogous to the proof of Lemma 5, but now we use the Hamilton–Cayley Formula for $E_-^{Q^*}[V \otimes V^*]$:

$$Q^{-2}Z^2 - \mathrm{tr}(Q^{-1}Z) \cdot Q^{-1}Z + \mathrm{DET} \cdot \mathrm{det}Q^{-1} = 0.$$

Proof of the Theorem. It is enough to prove that the equalities

$$\mathrm{DET} \cdot \mathrm{tr}(ZP) = \mathrm{tr}(ZP) \cdot \mathrm{DET}, \qquad \mathrm{DET} \cdot \mathrm{tr}(Q^{-1}Z) = \mathrm{tr}(Q^{-1}Z) \cdot \mathrm{DET}$$

hold in $E^{P,Q}[V^* \otimes V]$.

We have $ZP^{-1}Z^s = P^{-1} \cdot \mathrm{DET}$ and $Z^sQZ = Q \cdot \mathrm{DET}$, so $P^{-1}Q \cdot \mathrm{DET} \cdot Z = ZP^{-1}Z^sQZ = ZP^{-1}Q \cdot \mathrm{DET}$, therefore

$$\mathrm{DET} \cdot ZP = Q^{-1}PZP^{-1}Q \cdot \mathrm{DET} \cdot P = Q^{-1}PZPP^{-1}Q \cdot \mathrm{DET}.$$

This implies

$$\mathrm{DET} \cdot \mathrm{tr}(ZP) = \mathrm{tr}(Q^{-1}PZPP^{-1}Q) \cdot \mathrm{DET} = \mathrm{tr}(ZP) \cdot \mathrm{DET}.$$

Analogously $\mathrm{DET} \cdot \mathrm{tr}(Q^{-1}Z) = \mathrm{tr}(Q^{-1}Z) \cdot \mathrm{DET}$.

REFERENCES

[1] E. Corrigan, D. B. Fairlie, P. Fletcher and R. Sasaki, *Some aspects of quantum groups and supergroups*, J. Math. Phys. 31 (1990), 776–780.

[2] H. Ewen, O. Ogievetsky and J. Wess, *Quantum matrices in two dimensions*, Lett. Math. Phys. 22 (1991), 297–305.

[3] B. A. Kupershmidt, *The quantum group* $\mathrm{GL}_h(2)$, J. Phys. A 25 (1992), L1239–L1244.

[4] Yu. I. Manin, *Quantum Groups and Noncommutative Geometry*, Université de Montréal, 1988.

[5] —, *Topics in Noncommutative Geometry*, Princeton University Press, 1991.

[6] E. E. Mukhin, *Quantum de Rham complexes*, Comm. Algebra 22 (1994), 451–498.

[7] O. Ogievetsky and J. Wess, *Relations between* $\mathrm{GL}_{p,q}(2)$*'s*, Z. Phys. C 50 (1991), 123–131.

[8] A. Sudbery, *Consistent multiparameter quantisation of* $\mathrm{GL}(n)$, J. Phys. A 23 (1990), L697–L704.

[9] T. Umeda and M. Wakayama, *Powers of* 2×2 *quantum matrices*, Comm. Algebra 21 (1993), 4461–4465.

[10] S. Vokos, J. Wess and B. Zumino, *Analysis of the basic matrix representation of* $\mathrm{GL}_q(2,\mathcal{C})$, Z. Phys. C 48 (1990), 65–74.

Faculty of Mathematics and Informatics
Nicholas Copernicus University
Chopina 12/18
87-100 Toruń, Poland
E-mail: pjedrzej@mat.uni.torun.pl

Received 20 February 1996;
revised 15 March 1996

Added in proof. After submitting this paper for publication I found out that the quasi-commutative algebras were considered in the papers: M. Artin, J. Tate and M. Van den Bergh, *Some algebras associated to automorphisms of elliptic curves*, The Grothendieck Festschrift, Vol. I, Birkhäuser, Boston, 1990, 33–85, and M. Artin and M. Van den Bergh, *Twisted homogeneous coordinate rings*, J. Algebra 133 (1990), 249–271.