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IRREDUCIBLE REPRESENTATIONS OF FREE PRODUCTS OF INFINITE GROUPS

BY

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1. Introduction. Let I be a nonempty index set and let $\{G_i\}_{i\in I}$ be a family of discrete groups. Then we can consider the *free product group* $G = \ast_{i\in I} G_i$ in which each element x can be uniquely represented as a *reduced word*

$$x = g_1 g_2 \ldots g_n\,,\qquad n\ge 0,\ g_k\in G_{i_k}\setminus\{e\},\ i_1\ne\ldots\ne i_n. \tag{1}$$

For such an element x we define its *type* as the formal word $t(x) = i_1 i_2 \ldots i_n$ and its *length* to be $|x| = n$, as introduced by J.-P. Serre in his book [Se]. A function f on G whose value $f(x)$ depends only on the type (resp. the length) of x will be called *type-dependent* (resp. *radial*).

Note in passing that if all G_i's are isomorphic to the group $\mathbb{Z}$ of integers then G can be regarded as the free group with I as the set of generators. In this case we can define another length putting $\ell(x) = |g_1| + \ldots + |g_n|$, where $|g_k|$ denotes the absolute value of the integer g_k. Then one can study radial functions and spherical functions with respect to ℓ as it was done in [FP1, 2 and PS].

Now let $\{P_i\}_{i\in I}$ be an arbitrary family of (not necessarily orthogonal) bounded projections on a Hilbert space H_0. We construct a representation π of G acting on a Hilbert space H containing H_0 in such a way that for every $x\in G$ the restriction of $\pi(x)$ to H_0 is $P_{i_1}\ldots P_{i_n}$, where $i_1\ldots i_n = t(x)$. Therefore if we pick a vector ζ_0 lying in H_0 then the corresponding coefficient $x\mapsto \langle \pi(x)\zeta_0,\zeta_0\rangle$ of π is a type-dependent function. The construction is presented in Section 2 where we also establish some relations between certain properties of the family $\{P_i\}_{i\in I}$ and those of π. In particular, if all P_i's are orthogonal then π turns out to be unitary. The construction gains in interest in view of Theorem 3.3 which, together with Proposition 3.1, says that if all G_i's are infinite then every type-dependent positive definite function on G is a coefficient of such a representation π.

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In [M3] we have described the class of all type-dependent positive definite functions on G in the following way. For $i \in I$ define $\tau(i) = 1/(|G_i| - 1)$. Then we endow the linear space of finitely supported functions on the set of types $S(I) = \{i_1 \ldots i_n : n \geq 0,\ i_k \in I \text{ and } i_1 \neq \ldots \neq i_n\}$ with a *τ-convolution* defined by

$$\text{(i)} \qquad \delta_i *_\tau \delta_i = (1 - \tau(i))\delta_i + \tau(i)\delta_e,$$

where e denotes the empty word in $S(I)$ and

$$\text{(ii)} \qquad \delta_{i_1} *_\tau \ldots *_\tau \delta_{i_n} = \delta_{i_1 \ldots i_n} \quad \text{for } n \geq 2,\ i_k \in I,\ i_1 \neq \ldots i_n,$$

and with an involution $f^*(i_1 \ldots i_n) := \overline{f(i_n \ldots i_1)}$, thus obtaining a $*$-algebra $\mathcal{A}(\tau)$. A complex function ϕ on $S(I)$ is said to be *τ-positive definite* if $\sum_{u \in S(I)} \phi(u)(f^* *_\tau f)(u) \geq 0$ for any $f \in \mathcal{A}(\tau)$. In particular, if all G_i's are infinite then $\tau \equiv 0$ and this notion coincides with the positive definiteness on $S(I)$ regarded as the free $*$-semigroup generated by I and defined by the relations $ii = i^* = i$ for $i \in I$ (cf. [BCR]). It was proved in [M3] that a type-dependent function (which obviously can be uniquely expressed as composition of a function ϕ on $S(I)$ and the type t), $t \circ \phi$, is positive definite on G if and only if ϕ is τ-positive definite on $S(I)$. This allows us to study functions on $S(I)$ instead of on G, in particular to prove positive definiteness of 1) spherical functions on the free product $\mathbb{Z}_k * \ldots * \mathbb{Z}_k$ of cyclic groups of the same order [M3, Theorem 5.8] (see [IP]) and 2) spherical functions on the free product $\mathbb{Z}_r * \mathbb{Z}_s$ of two cyclic groups [M3, Theorem 4.5] (see [CS]). The proofs use the fact that, having the index set I fixed, all the algebras $\mathcal{A}(\tau)$ are mutually isomorphic.

In this paper we prove that if all G_i's are infinite and ϕ is an extreme point in the convex cone of *type-dependent* positive definite functions on $G = \ast_{i \in I} G_i$ then, in fact, ϕ is an extreme point in the convex cone of *all* positive definite functions on G, unless $\phi = c\delta_e$, $c > 0$ (Theorem 3.3). The same question without the assumption that all G_i's are infinite presents a more delicate problem (because the representations involved are more complicated) and will be studied in a forthcoming paper.

In Section 4 we construct a family π_z, $z \in \mathbb{C}$, of representations of $G = G_1 * \ldots * G_N$, $N \geq 2$, related to a family $\{\zeta_i(z) \otimes \zeta_i(\overline{z})\}_{i=1}^N$ of one-dimensional projections on $\mathbb{C}^N$. The radial function ϕ_z defined by

$$\phi_z(x) = \begin{cases} 1 & \text{for } x = e\ , \\ z\left(\dfrac{Nz - 1}{N - 1}\right)^{|x|-1} & \text{for } x \neq e, \end{cases}$$

turns out to be a coefficient of π_w if $w^2 = z$. This function ϕ_z can be viewed as a spherical function on a free product $G = G_1 * \ldots * G_N$ of infinite groups. Namely, let $G^k = G_1^k * \ldots * G_N^k$ be the free product of finite groups of order k. Then a radial function ϕ_z^k is said to be *spherical* with eigenvalue z if $\phi_z^k(e) = 1$ and $\phi_z^k * \mu_1 = z\phi_z^k$, where μ_1 denotes the probability measure equidistributed over the set $W_1^k = \{x \in G_k : |x| = 1\}$ (see [IP]). Such a function is unique and given by $\phi_z^k(x) = P_{|x|}(z; k, N)$, where $P_n(\,\cdot\,; k, N)$ is a polynomial of degree n defined in [M2]. Now taking k to be infinite we cannot define spherical functions in the same way since the set W_1^∞ is also infinite. But putting

$$\phi_z^\infty(x) = \lim_{k\to\infty} P_{|x|}(z; k, N)$$

we get the function ϕ_z. For finite k the related representations were studied by Iozzi and Picardello [IP] and for $k = \infty$ by Wysoczański [W2] (see also Szwarc [Sz1]), whose construction was based on the ideas of Pytlik and Szwarc [PS] (cf. also [B1, FP1, FP2, Va and Sz2]). In the last section we prove that our representations π_z are topologically equivalent to those constructed by Wysoczański [W2].

2. The construction. Assume that $\{G_i\}_{i\in I}$ is a family of discrete groups, $G = \ast_{i\in I} G_i$, and $\{P_i\}_{i\in I}$ is a family of bounded (not necessarily orthogonal) projections in a fixed Hilbert space H_0. If $x \in G \setminus \{e\}$ is as in (1) then we put $i(x) = i_n$. Define

$$H = \Big\{ f : G \to H_0 : \sum_{w\in G} \|f(w)\|^2 < \infty \text{ and } \\ \text{if } w \in G \setminus \{e\} \text{ then } f(w) \in \operatorname{Ker} P_{i(w)} \Big\}.$$

For any $w \in G$ and any vector $\xi \in H_0$ lying in $\operatorname{Ker} P_{i(w)}$ whenever $w \neq e$, we denote by (w, ξ) the function in H which has the value ξ at w and 0 elsewhere. H_w will stand for the space of all functions in H vanishing outside $\{w\}$, i.e. the set of all admissible pairs (w, ξ). Then we have $H = \bigoplus_{w\in G} H_w$. By abuse of notation we shall identify H_0 with $H_e \subseteq H$.

Now we are going to define a representation π of G acting on H. To do that, for every $i \in I$, $g \in G_i \setminus \{e\}$ and $f \in H$, we define

$$(\pi_i(g)f)(w) = \begin{cases} f(g^{-1}) + P_i f(e) & \text{if } w = e, \\ (\mathrm{Id} - P_i) f(e) & \text{if } w = g, \\ f(g^{-1}w) & \text{otherwise}, \end{cases} \tag{2a}$$

or, in terms of the vectors (w, ξ),

$$\pi_i(g)(w, \xi) = \begin{cases} (e, P_i\xi) + (g, (\mathrm{Id} - P_i)\xi) & \text{if } w = e, \\ (gw, \xi) & \text{otherwise}. \end{cases} \tag{2a$'$}$$

Note in particular that $\|\pi_i(g)\| \le \|P_i\| + \|\mathrm{Id} - P_i\|$. Putting $\pi_i(e) = \mathrm{Id}$ it is easy to see that π_i is a representation of the group G_i. More precisely, let P_0 denote the orthogonal projection of H onto $H_e = H_0$ and set $T_i = P_i P_0$

(T_i is a projection of H onto $\operatorname{Im} P_i$). Then the operator $\pi_i(g)$ acts as the identity on $\operatorname{Im} T_i = \operatorname{Im} P_i$ and $\pi_i(g)$ acts in $\operatorname{Ker} T_i = (\operatorname{Ker} P_i) \oplus \bigoplus_{w\neq e} H_w$ as a multiple of the regular representation. Moreover, if P_i is orthogonal then the direct decomposition $H = \operatorname{Im} T_i + \operatorname{Ker} T_i$ is also orthogonal and the representation π_i of G_i is unitary.

In this way for every $i \in I$ we have defined a representation π_i of G_i. By the definition of the free product of groups (see [Se]) the π_i's extend uniquely to a representation π of G. Namely,

$$\pi(x) = \pi_{i_1}(g_1)\ldots\pi_{i_n}(g_n) \tag{2b}$$

if x is as in (1). Note that if all the projections P_i are orthogonal then we have $\pi(x)^* = \pi_{i_n}(g_n)^*\ldots\pi_{i_1}(g_1)^* = \pi_{i_n}(g_n^{-1})\ldots\pi_{i_1}(g_1^{-1}) = \pi(x^{-1})$ so π is unitary.

LEMMA 2.1. *If x is as in* (1) *and $\xi \in H_0$, then*

$$\pi(x)(e,\xi) = (e, P_{i_1}\ldots P_{i_n}\xi) + \sum_{k=1}^{n}(g_1\ldots g_k, (\mathrm{Id} - P_{i_k})P_{i_{k+1}}\ldots P_{i_n}\xi).$$

Proof. If $n = 0$ then the formula is obvious. Assume that it holds for elements of length n and pick x as in (1). We shall consider an element g_0x of length $n+1$ with $g_0 \in G_{i_0} \setminus \{e\}$, $i_0 \neq i_1$. By our assumption and (2) we have

$$\begin{aligned}\pi(g_0x)(e,\xi) &= \pi(g_0)(e, P_{i_1}\ldots P_{i_n}\xi)\\ &\quad + \sum_{k=1}^{n}(g_0g_1\ldots g_k, (\mathrm{Id} - P_{i_k})P_{i_{k+1}}\ldots P_{i_n}\xi)\\ &= (e, P_{i_0}P_{i_1}\ldots P_{i_n}\xi)\\ &\quad + \sum_{k=0}^{n}(g_0g_1\ldots g_k, (\mathrm{Id} - P_{i_k})P_{i_{k+1}}\ldots P_{i_n}\xi),\end{aligned}$$

which completes the proof.

Let $\mathcal{A}$ be a family of bounded operators on some Hilbert space. A closed subspace M is called *invariant* for $\mathcal{A}$ if $AM \subseteq M$ for each $A \in \mathcal{A}$. Note that if M is invariant for $\mathcal{A}$ then $M^\perp$ is invariant for $\mathcal{A}^* = \{A^* : A \in \mathcal{A}\}$. The family $\mathcal{A}$ is called *topologically irreducible* (cf. [Di]) if there is no nontrivial closed invariant subspace for $\mathcal{A}$. Hence if $\mathcal{A}$ is irreducible then so is $\mathcal{A}^*$.

THEOREM 2.2. *Let $\{P_i\}_{i\in I}$ be a family of bounded projections in a fixed Hilbert space H_0 and let π be the representation of $G = \ast_{i\in I} G_i$ defined by* (2). *Then*

(i) *if all* P_i *are orthogonal then* π *is unitary*;

(ii) *if* $x \in G$, $t(x) = i_1 \ldots i_n$ *and* $\xi \in H_0$ *then* $P_0 \pi(x)\xi = P_{i_1} \ldots P_{i_n} \xi$, *where* P_0 *denotes the orthogonal projection of* H *onto* H_0;

(iii) *if the family* $\{P_i\}_{i \in I}$ *is nontrivial* (*i.e.* $P_i \neq 0$ *for some* $i \in I$) *and topologically irreducible* (*on* H_0) *then* π *is also topologically irreducible* (*on* H) *provided that all* G_i*'s are infinite*;

(iv) *assume that* $\|(\mathrm{Id} - P_{i_0})P_{i_1} \ldots P_{i_n}\| \leq a_n$ *and* $\|P_{i_1} \ldots P_{i_n}\| \leq a_n$ *for any* $n \geq 0$ *and any sequence* $i_0, i_1, \ldots, i_n \in I$ *satisfying* $i_0 \neq i_1 \neq \ldots \neq i_n$; *then*

$$\|\pi(x)\| \leq \sum_{s=0}^{|x|} a_s. \tag{3}$$

In particular, *if the series* $\sum a_n$ *is convergent then* π *is uniformly bounded.*

Proof. We have already noted statement (i). Moreover, (ii) is a consequence of Lemma 2.1. Assume that the family of projections $\{P_i\}_{i \in I}$ on H_0 is nontrivial and irreducible and that all G_i's are infinite. For each $i \in I$ let $\{g_{k,i}\}_{k=1}^{\infty}$ be a sequence of distinct elements of the group G_i. For any $i \in I$ and a natural number n define the operator $T_{n,i}$ on H by

$$T_{n,i} = \frac{1}{n} \sum_{k=1}^{n} \pi(g_{k,i}).$$

Then $\|T_{n,i}\| \leq \|P_i\| + \|\mathrm{Id} - P_i\|$. Moreover, for $\xi \in H_0$,

$$T_{n,i}(e, \xi) = (e, P_i \xi) + \frac{1}{n} \sum_{k=1}^{n} (g_{k,i},\, (\mathrm{Id} - P_i)\xi)$$

and for any $w \neq e$ and any $(w, \xi) \in H_w$,

$$T_{n,i}(w, \xi) = \frac{1}{n} \sum_{k=1}^{n} (g_{k,i} w, \xi).$$

Now, fix $f \in H$, $\varepsilon > 0$ and decompose $G = B_0 \dot{\cup} B_1 \dot{\cup} B_2$ and $f = f_0 + f_1 + f_2$, $\operatorname{supp} f_s \subseteq B_s$, in such a way that $B_0 = \{e\}$, B_1 is finite and $\|f_2\| \leq \varepsilon(2\|P_i\| + 2\|\mathrm{Id} - P_i\|)^{-1}$. We obtain

$$\begin{aligned}
&\|T_{n,i} f - (e, P_i f(e))\| \\
&\quad \leq \|T_{n,i}(e, (\mathrm{Id} - P_i) f(e))\| + \sum_{w \in B_1} \|T_{n,i}(w, f(w))\| + \|T_{n,i} f_2\| \\
&\quad \leq \frac{1}{\sqrt{n}} \|(\mathrm{Id} - P_i) f(e)\| + \frac{1}{\sqrt{n}} \sum_{w \in B_1} \|f(w)\| + \frac{\varepsilon}{2} \leq \varepsilon
\end{aligned}$$

for n sufficiently large. Therefore the sequence $T_{n,i}$ is strongly convergent to the operator $T_i = P_i P_0$.

Let M be a closed subspace invariant for the representation π. Then $T_{n,i}M \subseteq M$ for all natural numbers n and so $T_iM \subseteq M$ for all $i \in I$. If $T_iM \neq \{0\}$ for some $i \in I$ then $M \cap H_0$ is a nonzero invariant subspace for the family $\{P_i\}_{i\in I}$ (as T_i restricted to H_0 is just P_i) so $M \cap H_0 = H_0$ and $H_0 \subseteq M$. Then for any $x \in G$ and $(x, \xi) \in H_x$ we have $(x, \xi) = \pi(x)(e, \xi) \in M$ (as M is invariant). This implies $H_x \subseteq M$ for all $x \in G$ and so $M = H$.

Assume that $T_iM = \{0\}$ for all $i \in I$ and let $m : G \to H_0$ be any function in $M \subseteq H$. Then we have $0 = T_im = P_iP_0m = P_im(e)$ for all $i \in I$. Since the subspace $\bigcap_{i\in I} \operatorname{Ker} P_i$ of H_0 is invariant for $\{P_i\}_{i\in I}$ and the family is nontrivial we have $m(e) = 0$. We are going to prove that $m(w) = 0$ for all $w \in G$. Assume that this holds for all $m \in M$ and all $w \in G$ such that $|w| < n$ $(n \geq 1)$. Take x as in (1). As $m(e) = 0$ and M is invariant we have $m(x) = (\pi(g_1^{-1})m)(g_2 \ldots g_n) = 0$.

We now turn to (iv). Let x be a fixed element as in (1) and for $1 \leq r \leq n$ put $w_r = x^{-1}g_1 \ldots g_r = (g_{r+1} \ldots g_n)^{-1}$. By Lemma 2.1 we have

$$\begin{aligned}\pi(x)(w_r, \xi) &= \pi(g_1 \ldots g_r)(e, \xi)\\ &= (e, P_{i_1} \ldots P_{i_r}\xi) + \sum_{k=1}^{r}(g_1 \ldots g_k,\ (\mathrm{Id} - P_{i_k})P_{i_{k+1}} \ldots P_{i_r}\xi)\end{aligned}$$

and if w is none of w_r, $1 \leq r \leq n$, then $\pi(x)(w, \xi) = (xw, \xi)$. Hence

$$(4)\qquad (\pi(x)f)(w) = \begin{cases} f(x^{-1}) + \sum_{r=1}^{n} P_{i_1} \ldots P_{i_r} f(w_r) & \text{if } w = e,\\ \sum_{r=k}^{n}(\mathrm{Id} - P_{i_k})P_{i_{k+1}} \ldots P_{i_r} f(w_r) & \text{if } w = g_1 \ldots g_k,\ 1 \leq k \leq n,\\ f(x^{-1}w) & \text{otherwise.}\end{cases}$$

For $0 \leq s \leq n$ define the operator A_s acting on H in the following way:

$$(A_0f)(w) = \begin{cases} (\mathrm{Id} - P_{i_k})f(w_k) & \text{if } w = g_1 \ldots g_k,\ 1 \leq k \leq n,\\ f(x^{-1}w) & \text{otherwise,}\end{cases}$$

and if $1 \leq s \leq n$ then we put

$$(A_sf)(w) = \begin{cases} P_{i_1}P_{i_2} \ldots P_{i_s} f(w_s) & \text{if } w = e,\\ (\mathrm{Id} - P_{i_k})P_{i_{k+1}} \ldots P_{i_{k+s}} f(w_{k+s}) & \text{if } w = g_1 \ldots g_k,\ 1 \leq k \leq n - s,\\ 0 & \text{otherwise}\end{cases}$$

(in particular, $(A_nf)(e) = P_{i_1} \ldots P_{i_n} f(e)$, and for $w \neq e$, $(A_nf)(w) = 0$). Then $\|A_s\| \leq a_s$ and by (4), $\pi(x) = \sum_{s=0}^{n} A_s$, which gives us (3) and completes the proof.

Remark. Note that if $P_i = 0$ for every $i \in I$ and $H_0 = \mathbb{C}$ then π is just the regular representation of G, so the first assumption in (iii) is essential.

Corollary 2.3. *Let $G = \ast_{i\in I} G_i$ and let $\{P_i\}_{i\in I}$ be a family of orthogonal projections in a Hilbert space H_0. Then*

(a) *the operator-valued function U on G given by $U(e) = \mathrm{Id}$ and $U(x) = P_{i_1} \dots P_{i_n}$ for x as in* (1) *is positive definite*;

(b) *for any vector $\xi_0 \in H_0$ the complex-valued function $x \mapsto \langle \xi_0, P_{i_1} \dots P_{i_n} \xi_0 \rangle$ for x as in* (1) *is positive definite.*

Proof. The statement (a) is an obvious consequence of (i) and (ii) in Theorem 2.2 (see [NF, Theorem 7.1]) and it easily entails (b).

Remark. Let us note that the operator-valued function U is a free product function (see [Bo2]). Hence Corollary 2.4 can also be obtained as a consequence of [Bo2, Theorem 7.1].

3. The $\ast$-semigroup $S(I)$ and free product of infinite groups. Let I be a set and let $S(I)$ denote the set of all formal words of the form

$$u = i_1 \dots i_n, \quad \text{where } n \ge 0,\ i_k \in I,\ i_1 \ne \dots \ne i_n. \tag{5}$$

We shall regard $S(I)$ as a unital $\ast$-semigroup generated by I with the empty word e as unit and defined by the following relations:

$$ii = i^* = i \quad \text{for any } i \in I.$$

In particular, if $u = i_1 \dots i_n$ and $v = j_1 \dots j_m$ then $u^* = i_n \dots i_1$ and $uv = i_1 \dots i_n j_2 \dots j_m$ provided $n \ne 0 \ne m$ and $i_n = j_1$; otherwise $uv = i_1 \dots i_n j_1 \dots j_m$.

Proposition 3.1. *Let ϕ be a complex function on $S(I)$. Then ϕ is positive definite if and only if there exists a family $\{P_i\}_{i\in I}$ of orthogonal projections on some Hilbert space H_0 and a vector $\zeta_0 \in H_0$ such that for any $u = i_1 \dots i_n \in S(I)$,*

$$\phi(u) = \langle \zeta_0, P_{i_1} P_{i_2} \dots P_{i_n} \zeta_0 \rangle.$$

Proof. By [BCR, Theorem 4.1.14] it is enough to prove that if ϕ is positive definite then $|\phi(u)| \le \phi(e)$ for any $u \in S(I)$. Let ϕ be a positive definite function on $S(I)$ and let $u = i_1 \dots i_n \in S(I)$. Then we set $u_k = i_{k+1} \dots i_n$, $0 \le k \le n$. By [BCR, Remark 4.1.6] for any $u, v \in S(I)$ we have $\phi(u^*u) \ge 0$ and $\phi(v^*u)\phi(u^*v) \le \phi(v^*v)\phi(u^*u)$. Therefore $\phi(u_{k+1}^* u_k)\phi(u_k^* u_{k+1}) \le \phi(u_{k+1}^* u_{k+1})\phi(u_k^* u_k)$ for $0 \le k \le n$. But $u_k^* u_{k+1} = u_{k+1}^* u_k = u_k^* u_k$, hence $0 \le \phi(u_k^* u_k) \le \phi(u_{k+1}^* u_{k+1})$. Since $u_n = e$ and $u_0 = u$ we get $\phi(u^*u) \le \phi(e)$. So $|\phi(u)|^2 = \phi(e^*u)\phi(u^*e) \le \phi(e)\phi(u^*u) \le \phi^2(e)$.

Corollary 3.2. *Let $\{G_i\}_{i\in I}$ be any family of groups, $G = \ast_{i\in I} G_i$ and let ϕ be a positive* (*resp. negative*) *definite function on the $\ast$-semigroup*

$S(I)$. Then the composite function $\phi \circ t$ (i.e. $\phi \circ t(x) = \phi(t(x))$) is positive (resp. negative) definite on G.

P r o o f. If ϕ is a positive definite function then by Corollary 2.3(b) so is $\phi \circ t$. Suppose that ϕ is negative definite on $S(I)$. Then, by Schoenberg's theorem (see [BCR, Theorem 3.2.2]) for any positive λ the function $\phi_\lambda = \exp(-\lambda\phi)$ is positive definite on $S(I)$. Hence $\phi_\lambda \circ t$ is positive definite on G. Applying Schoenberg's theorem to $\phi_\lambda \circ t$ we see that $\phi \circ t$ is negative definite on G.

We conclude with the following theorem stating the correspondence between the class of positive definite functions on a free product of infinite groups and the class of positive definite functions on the $*$-semigroup $S(I)$. The first statement is in fact a special case of [M3, Theorem 3.2.]. Note that each type-dependent function on $G = \mathop{\ast}_{i\in I} G_i$ can be uniquely expressed as a composition of the form $\phi \circ t$.

THEOREM 3.3. *Let $\{G_i\}_{i\in I}$ be any family of infinite groups, $G = \mathop{\ast}_{i\in I} G_i$, and let ϕ be any complex function on $S(I)$. Then*

(i) *$\phi \circ t$ is positive (resp. negative) definite on G if and only if ϕ is positive (resp. negative) definite on $S(I)$;*

(ii) *if ϕ is an extreme point in the convex cone of positive definite functions on $S(I)$ and ϕ is not of the form $c\delta_e$, $c > 0$, then $\phi \circ t$ is an extreme point in the convex cone of all positive definite functions on G.*

P r o o f. (i) By the last corollary we need to show only one implication. Suppose that $\phi\circ t$ is positive definite. For any $i \in I$ and any natural number p we choose a subset $A(i,p)$ of $G_i \setminus \{e\}$ of cardinality p (recall that G_i's are infinite). If $u = i_1 \ldots i_n \in S(I)$ then we put

$$A(u,p) = \{g_1 \ldots g_n \in G : g_k \in A(i_k,p)\}.$$

Note that $\operatorname{Card} A(u,p) = p^{|u|}$, where $|u|$ denotes the length of u. We are going to prove that for any $u, v \in S(I)$,

$$S_p(u,v) := \sum_{\substack{x\in A(u,p)\\ y\in A(v,p)}} \phi(t(y^{-1}x))p^{-|u|}\,p^{-|v|} \to \phi(v^*u) \tag{6}$$

as $p \to \infty$. First of all, note that if x and y have the first letters distinct (though they may be of the same type) then $t(y^{-1}x) = t(y)^*t(x)$. Therefore if u and v have the first letters distinct or one of them is e then $S_p(u,v) = \phi(v^*u)$. Suppose that $u = i_1 \ldots i_n \neq e$, $v = j_1 \ldots j_m \neq e$ and $i_1 = j_1$ and let C denote the set of all pairs $(x,y) \in A(u,p) \times A(v,p)$ such that the first letters of x and y are the same. It is clear that $\operatorname{Card} C = p^{|u|+|v|-1}$. Then

$\phi(t(y^{-1}x)) = \phi(v^*u)$ for $(x,y) \in A(u,p) \times A(v,p) \setminus C$. Hence

$$\begin{aligned}
&\Big|\phi(v^*u) - \sum_{\substack{x\in A(u,p)\\ y\in A(v,p)}} \phi(t(y^{-1}x))p^{-|u|}p^{-|v|}\Big| \\
&\qquad = \Big|p^{-1}\phi(v^*u) - \sum_{(x,y)\in C} \phi(t(y^{-1}x))p^{-|u|}p^{-|v|}\Big| \\
&\qquad \le p^{-1}|\phi(v^*u)| + \sum_{(x,y)\in C} |\phi(t(y^{-1}x))|p^{-|u|}p^{-|v|} \le 2p^{-1}\phi(e)
\end{aligned}$$

(the last inequality holds because $|\phi(u)| \le \phi(e)$ for any $u \in S(I)$, as $\phi \circ t$ is positive definite on G). This proves (6).

Now let $u_1, \dots, u_m$ be any distinct elements of $S(I)$ and let $\alpha_1, \dots, \alpha_m$ be any complex numbers. We have to prove that

$$\sum_{r,s=1}^{m} \phi(u_s^* u_r)\alpha_r\overline{\alpha}_s \ge 0.$$

For any natural number p we define the function f_p on G by

$$f_p(x) = \begin{cases} \alpha_r p^{-|x|} & \text{if } x \in A(u_r,p) \text{ for some } 1 \le r \le m,\\ 0 & \text{otherwise.}\end{cases}$$

The function $\phi \circ t$ is positive definite on G and so using (6) we get

$$\begin{aligned}
0 &\le \sum_{x,y\in G} \phi(t(y^{-1}x)) f_p(x)\overline{f_p(y)} \\
&= \sum_{r,s=1}^{m} S_p(u_r,u_s)\,\alpha_r\overline{\alpha}_s \to \sum_{r,s=1}^{m} \phi(u_s^*u_r)\,\alpha_r\overline{\alpha}_s
\end{aligned}$$

as $p \to \infty$ and so ϕ is positive definite on $S(I)$. In the case of a negative definite function we can apply Schoenberg's theorem as in the proof of Corollary 4.2.

Now suppose that ϕ is an extreme point in the convex cone of all positive definite functions on $S(I)$. Then ϕ is a matrix coefficient of an irreducible $*$-representation (H_0, π) of $S(I)$. Hence for $u = i_1 \dots i_n$,

$$\phi(u) = \langle \zeta_0, P_{i_1} \dots P_{i_n} \zeta_0 \rangle,$$

where $P_i = \pi(i)$ and $\{P_i\}_{i\in I}$ is an irreducible family of orthogonal projections on H_0, $\zeta_0 \in H_0$. Since ϕ is not of the form $c\delta_e$ the family is nontrivial. By Theorem 2.2(i), (ii), $\phi \circ t$ is a coefficient of an irreducible unitary representation of G, which concludes the proof.

R e m a r k. Note that the function δ_e is extreme on the $*$-semigroup $S(I)$ being its character but obviously δ_e is not extreme on G.

4. One-dimensional projections. In this section we will be concerned only with the case of one-dimensional projections. Let us start with the following

PROPOSITION 4.1. *Let H_0 be a Hilbert space and for every $i \in I$ let P_i be a one-dimensional projection on H_0, i.e. $P_i(\xi) = (\zeta_i \otimes \eta_i)\xi = \langle \xi, \eta_i \rangle \zeta_i$, for some vectors ζ_i, η_i satisfying $\langle \zeta_i, \eta_i \rangle = 1$. Then*

(i) *the family $\{P_i\}_{i\in I}$ is irreducible if and only if both the subsets $\{\zeta_i\}_{i\in I}$ and $\{\eta_i\}_{i\in I}$ are linearly dense and there is no nontrivial partition $I = I_1 \cup I_2$ such that $\{\zeta_i : i \in I_1\} \perp \{\eta_i : i \in I_2\}$;*

(ii) *for any $\zeta_0, \eta_0 \in H_0$ and $i_1, i_2, \ldots, i_n \in I$,*

$$\langle \eta_0, P_{i_1} P_{i_2} \ldots P_{i_n} \zeta_0 \rangle = \langle \eta_0, \zeta_{i_1} \rangle \langle \eta_{i_1}, \zeta_{i_2} \rangle \langle \eta_{i_2}, \zeta_{i_3} \rangle \ldots \langle \eta_{i_n}, \zeta_0 \rangle;$$

(iii) *for any $i_1, i_2, \ldots, i_n \in I$,*

$$\|P_{i_1} P_{i_2} \ldots P_{i_n}\| = \|\zeta_{i_1}\| \cdot |\langle \eta_{i_1}, \zeta_{i_2} \rangle \langle \eta_{i_2}, \zeta_{i_3} \rangle \ldots \langle \eta_{i_{n-1}}, \zeta_{i_n} \rangle| \cdot \|\eta_{i_n}\|.$$

Proof. To see (i) we note that if one of the conditions is not satisfied then one of the invariant subspaces

$$M_1 = \langle \zeta_i : i \in I \rangle, \quad M_2 = \langle \eta_i : i \in I \rangle^{\perp}, \quad M_3 = \langle \zeta_i : i \in I_1 \rangle$$

is nontrivial (for $A \subseteq H_0$, $\langle A \rangle$ denotes the closed subspace generated by A). Suppose that the conditions are satisfied and that M is a closed invariant subspace. Put $I_1 = \{i \in I : P_i M \neq \{0\}\}$, $I_2 = I \backslash I_1$. Then $\{\zeta_i : i \in I_1\} \subseteq M$ and $\{\eta_i : i \in I_2\} \perp M$ so one of I_1, I_2 is empty and the subspace M must be trivial. By induction on n one can prove (ii), and (iii) is a consequence of (ii).

Combining Theorem 2.2 and Proposition 4.1 we obtain the following generalization of [Sz1, Corollary 1] (see [M3, Example 2.3.2])

COROLLARY 4.2. *Let $\{v_i\}_{i\in I\cup\{0\}}$ be a family of unit vectors in a Hilbert space H_0 and let $a_{ij} = \langle v_i, v_j \rangle$, $i, j \in I \cup \{0\}$, $G = \ast_{i\in I} G_i$. Then the function ϕ on G given by*

$$\phi(x) = a_{0i_1} a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_n 0} \qquad \text{for } x \text{ as in } (1),$$

$\phi(e) = 1$, is positive definite. Moreover, if the family $\{v_i\}_{i\in I}$ is linearly dense in H_0, $\langle v_i, v_j \rangle \neq 0$ for $i, j \in I$ and all G_i's are infinite then ϕ is extreme.

From now on we restrict our attention to the following case. Let $I = \{1, \ldots, N\}$, $N \geq 2$, and let $\xi_1, \ldots, \xi_N$ be an orthonormal basis in $H_0 = \mathbb{C}^N$. Then we put

$$\zeta_0 = \frac{1}{\sqrt{N}}(\xi_1 + \ldots + \xi_N)$$

and for $1 \le i \le N$,

$$\zeta_i = \sqrt{\frac{N-1}{N}}\xi_i - \frac{1}{\sqrt{N(N-1)}}\sum_{\substack{j=1\\ j\neq i}}^{N} \xi_j.$$

It is easy to check that

$$\langle \zeta_i, \zeta_j\rangle = \begin{cases} 1 & \text{if } i=j,\\ 0 & \text{if } i=0 \text{ or } j=0 \text{ and } i\neq j,\\ -1/(N-1) & \text{if } i\neq j,\ 1\le i,\ j\le N \end{cases} \tag{7}$$

(in particular $\zeta_1 + \ldots + \zeta_N = 0$). For $1 \le i \le N$ and for any fixed complex number z define

$$\zeta_i(z) = z\zeta_0 + \sqrt{1-z^2}\zeta_i$$

(to avoid dealing with square roots of complex numbers one can substitute $z = \cos\alpha$ and $\sqrt{1-z^2} = \sin\alpha$, $\alpha \in \mathbb{C}$). Then, by (7), $\langle \zeta_i(z), \zeta_i(\overline{z})\rangle = 1$ and for $i \neq j$,

$$\langle \zeta_i(z), \zeta_j(\overline{z})\rangle = z^2 - \frac{1-z^2}{N-1} = \frac{Nz^2-1}{N-1}. \tag{8}$$

In particular, $P_i = \zeta_i(z) \otimes \zeta_i(\overline{z})$ is a projection. Applying Theorem 2.2 and Proposition 4.1 we easily obtain

THEOREM 4.3. *Let $G = G_1 * \ldots * G_N$ be a free product of arbitrary groups, $z \in \mathbb{C}$, and let π_z be the representation of G in $\mathbb{C}^N$ given by the family $\{P_i = \zeta_i(z) \otimes \zeta_i(\overline{z})\}_{i=1}^N$ and defined by (2). Then*

(i) *if $z \in [-1,1]$ then π_z is unitary*;

(ii) $\langle \pi_z(x)\zeta_0, \zeta_0\rangle = \begin{cases} 1 & \textit{if } x = e,\\ z^2\left(\dfrac{Nz^2-1}{N-1}\right)^{|x|-1} & \textit{if } x \neq e; \end{cases}$

(iii) *if all G_i are infinite, $z \in \mathbb{C}$ and $z^2 \neq 0, 1, 1/N$ then π_z is topologically irreducible*;

(iv) *if $z \in \mathbb{C}$ and $|Nz^2 - 1| < N-1$ then π_z is uniformly bounded and for any $x \in G$,*

$$\|\pi_z(x)\| \le (|z^2| + |1-z^2|)\left(1 + \frac{|z^2| + |1-z^2|}{1 - \left|\frac{Nz^2-1}{N-1}\right|}\right).$$

In particular, for $z \in [0,1]$ the function ϕ_z given by

$$\phi_z(x) = \begin{cases} 1 & \textit{for } x = e,\\ z\left(\dfrac{Nz-1}{N-1}\right)^{|x|-1} & \textit{for } x \neq e, \end{cases}$$

is a positive definite function on $G = G_1 * \ldots * G_N$; it is an extreme positive definite function provided $z \neq 0, 1/N$ and all G_i's are infinite.

Proof. If $z \in [-1,1]$ then P_i's are orthogonal, which gives us (i). Both (ii) and (iii) are consequences of (8) because $\{\zeta_i(z)\}_{i=1}^N$ is a linear basis of H_0 unless $z = 0, 1$ or -1. Finally, by (8),

$$\|P_{i_1} \ldots P_{i_n}\| = (|z^2| + |1 - z^2|)\left|\frac{Nz^2 - 1}{N-1}\right|^{n-1} \quad \text{for } n \ge 1 \text{ and } i_1 \ne \ldots \ne i_n.$$

Moreover, one can easily check that if P is a one-dimensional projection on a Hilbert space then $\|\mathrm{Id} - P\| = \|P\|$. Therefore, in the notation of Theorem 2.2(iv), $a_0 = |z^2| + |1 - z^2|$ and

$$a_n \le (|z^2| + |1 - z^2|)^2 \left|\frac{Nz^2 - 1}{N-1}\right|^{n-1} \quad \text{for } n \ge 1,$$

which leads to (iv) and completes the proof.

Let us change our parameter putting

$$u = \frac{Nz^2 - 1}{N-1}, \quad \text{i.e.} \quad z^2 = \frac{(N-1)u + 1}{N}$$

(this parametrization was used in [M1, Sz1, W1 and W2]). Writing $\Pi_u = \pi_z$ we can rephrase the last theorem as follows:

THEOREM 4.3′. (i′) *If* $u \in [-1/(N-1), 1]$ *then* Π_u *is unitary*;

(ii′) $$\langle \Pi_u(x)\zeta_0, \zeta_0\rangle = \begin{cases} 1 & \text{if } x = e, \\ \dfrac{(N-1)u + 1}{N} u^{|x|-1} & \text{if } x \ne e; \end{cases}$$

(iii′) *if all* G_i *are infinite*, $u \in \mathbb{C}$ *and* $u \ne 0, 1, -1/(N-1)$, *then* Π_u *is irreducible*;

(iv′) *if* $|u| < 1$ *then* Π_u *is uniformly bounded and for any* $x \in G$

$$\|\Pi_u(x)\| \le \frac{|(N-1)u + 1| + (N-1)|1-u|}{N} \times \left(1 + \frac{|(N-1)u + 1| + (N-1)|1-u|}{N(1 - |u|)}\right).$$

In particular, *for* $u \in [-1/(N-1), 1]$ *the function* ψ_u *given by*

$$\psi_u(x) = \begin{cases} 1 & \text{for } x = e, \\ \dfrac{(N-1)u + 1}{N} u^{n-1} & \text{for } x \ne e,\ |x| = n, \end{cases}$$

is a positive definite function on $G = G_1 * \ldots * G_N$; *it is an extreme positive definite function provided* $z \ne -1/(N-1), 0$ *and all* G_i*'s are infinite.*

Remarks. (a) The positive definiteness of ψ_u, $u \in [-1/(N-1), 1]$, was first proved in [M1] and the fact that for $u \ne -1/(N-1), 0$ the function ψ_u is extreme is due to Szwarc [Sz1]. An analytic series of representations giving ψ_u's as coefficients was constructed by Wysoczański [W1, W2]. In the next section we will show that our series π_z is topologically equivalent to his.

(b) Let us mention that Wysoczański [W1] has proved that if $G = G_1 * \ldots * G_N$ and $|u| < 1$ then

$$\|\psi_u\|_{B_2} \le \frac{N-1}{N}|1-u| + \frac{|[(N-1)u+1](1-u)|}{N(1-|u^2|)}\left\{\left|u+\frac{1}{N-1}\right| + \frac{N-2}{N-1}\right\} \tag{9}$$

($\|\cdot\|_{B_2}$ denotes the norm in the algebra of Herz–Schur multipliers—see [BF] for instance) and that the equality holds provided all G_i are infinite.

5. Relation to Wysoczański's construction. In this section we prove that the representations π_z of $G = G_1 * \ldots * G_N$ are equivalent to those studied by Wysoczański [W2]. Firstly we present a brief exposition of his construction. We will, however, change the parameter by substituting $(Nz^2-1)/(N-1)$ instead of z in all formulas of [W2] indicating this by a tilde, so that $\widetilde{\pi}_z$ will stand for π_u of [W2], $u = (Nz^2-1)/(N-1)$, while (π_z, H) will denote the representations defined in the previous section.

Let

$$X_1 = \{(x,j) : x \in G,\ j \in I \text{ and if } x \ne e \text{ then } j \ne i(x)\}$$

(recall that for $x \ne e$ as in (1) we have defined $i(x) = i_n$; here and subsequently $I = \{1,\ldots,N\}$, $N \ge 2$). Then, for every $z \in \mathbb{C}$, $i \in I$, we define a representation $\widetilde{A}_z(g)$ of G_i acting on $\ell^2(X_1)$ putting $\widetilde{A}_z(e) = \mathrm{Id}$ and for $g \in G_i \setminus \{e\}$,

$$\widetilde{A}_z(g)(e,i) = (e,i), \tag{10a}$$

$$\widetilde{A}_z(g)(e,j) = \frac{Nz^2-1}{N-1}(e,i) + (g,j) \quad \text{if } j \ne i, \tag{10b}$$

$$\widetilde{A}_z(g)(g^{-1},j) = (e,j) - \frac{Nz^2-1}{N-1}(e,i), \tag{10c}$$

$$\widetilde{A}_z(g)(x,j) = (gx,j) \quad \text{if } x \ne e,\ g^{-1} \tag{10d}$$

(we will identify X_1 with the natural orthonormal basis of $\ell^2(X_1)$). By the definition of the free product $\widetilde{A}_z$ extends uniquely to the whole of G. From now on we assume that $z \ne 0, 1, -1$. We define an operator $\widetilde{V}_z$ acting on $\ell^2(X_1)$ by putting for $j \in I$,

$$\widetilde{V}_z(e,j) = (e,j) + \left(\frac{-1}{N} + \frac{1}{Nz}\sqrt{\frac{1-z^2}{N-1}}\right)\sum_{k=1}^{N}(e,k), \tag{11a}$$

and for $x \neq e$ such that $t(x) = i_1 \dots i_n$, and $j \neq i_n$,

$$\widetilde{V}_z(x,j) = (x,j) + \left(\frac{-1}{N-1} + \frac{1}{(N-1)z\sqrt{N}}\right) \sum_{k \neq i_n} (x,k). \tag{11b}$$

This operator is bounded, invertible [W2, Lemma 10] and

$$\widetilde{V}_z^{-1}(e,j) = (e,j) + \left(\frac{-1}{N} + \frac{z}{N}\sqrt{\frac{N-1}{1-z^2}}\right) \sum_{k=1}^{N} (e,k), \tag{12a}$$

$$\widetilde{V}_z^{-1}(x,j) = (x,j) + \left(\frac{-1}{N-1} + \frac{z\sqrt{N}}{N-1}\right) \sum_{k \neq i_n} (x,k). \tag{12b}$$

Now Wysoczański's family of representations of G is given by

$$\widetilde{\pi}_z(x) = \widetilde{V}_z^{-1} \widetilde{A}_z(x) \widetilde{V}_z$$

(see [W2, Theorem 11]). We are in a position to formulate the main result of this section stating that this construction is topologically equivalent to that presented in the previous section.

THEOREM 5.1. *Let $z \in \mathbb{C} \setminus \{0, 1, -1\}$. Then there exists a bounded, invertible operator $T_z : \ell^2(X_1) \to H$ intertwining $\widetilde{\pi}_z$ and π_z. This operator satisfies $\|T_z\| = \sqrt{|z^2| + |1 - z^2|}$, $\|T_z^{-1}\| = 1$ and is an isometry for $z \in (-1, 0) \cup (0, 1)$.*

Proof. Fix $z \in \mathbb{C} \setminus \{0, 1, -1\}$. For any $i \in I$, $j \in I \setminus \{i\}$ we define a vector in $H_0 = \mathbb{C}^N$ by

$$\begin{aligned} \eta_j^{(i)}(z) = {} & \frac{1 - z\sqrt{N}}{\sqrt{(N-1)(1-z^2)}} (\zeta_0 - z\zeta_i(z)) \\ & + \sqrt{\frac{N-1}{N(1-z^2)}} \left(\zeta_j(z) - \frac{Nz^2 - 1}{N-1} \zeta_i(z)\right). \end{aligned} \tag{13a}$$

By the definition of $\zeta_i(z)$, $\zeta_j(z)$ we have

$$\eta_j^{(i)}(z) = \sqrt{\frac{1-z^2}{N-1}} \zeta_0 + \frac{1 - z\sqrt{N}}{\sqrt{N(N-1)}} \zeta_i + \sqrt{\frac{N-1}{N}} \zeta_j, \tag{13b}$$

or, more explicitly,

$$\begin{aligned} \eta_j^{(i)}(z) = {} & \left(\sqrt{\frac{1-z^2}{N(N-1)}} - \frac{z}{\sqrt{N}}\right) \xi_i \\ & + \left(\sqrt{\frac{1-z^2}{N(N-1)}} + \frac{z}{(N-1)\sqrt{N}} + \frac{N-2}{N-1}\right) \xi_j \end{aligned}$$

$$+\left(\sqrt{\frac{1-z^2}{N(N-1)}}+\frac{z}{(N-1)\sqrt{N}}-\frac{1}{N-1}\right)\sum_{k\neq i,j}\xi_k\,.$$

By (7), (8) and (13a) we have $\langle \eta_j^{(i)}(z), \zeta_i(\overline{z})\rangle = 0$. Moreover,

$$\langle \eta_j^{(i)}(z), \eta_j^{(i)}(z)\rangle = \frac{|z^2|+|1-z^2|-1}{N-1}+1, \tag{14a}$$

and, if $N \geq 3$, $j, k \in I \setminus \{i\}$, $j \neq k$, then

$$\langle \eta_j^{(i)}(z), \eta_k^{(i)}(z)\rangle = \frac{|z^2|+|1-z^2|-1}{N-1} \tag{14b}$$

(to see this one can use (7) and (13b)). Therefore for any linear combination $u = \sum_{j\neq i}\alpha_j\eta_j^{(i)}(z)$ we have

$$\langle u, u\rangle = \sum_{j\neq i}|\alpha_j|^2 + \frac{|z^2|+|1-z^2|-1}{N-1}\Big|\sum_{j\neq i}\alpha_j\Big|^2. \tag{15a}$$

In particular, $\{\eta_j^{(i)}(z)\}_{j\neq i}$ is a linear basis of $\operatorname{Ker} P_i$ and for $z^2 \in (0,1)$ this is an orthonormal basis. Using the Schwarz inequality we get

$$\langle u, u\rangle \leq (|z^2|+|1-z^2|)\sum_{j\neq i}|\alpha_i|^2. \tag{15b}$$

Fix $i \in I = \{1,\ldots,N\}$ and define $T_i : \ell^2(I \setminus \{i\}) \to \operatorname{Ker} P_i$ by putting $T_i(j) = \eta_j^{(i)}(z)$. By (15b) we have $\|T_i\| \leq \sqrt{|z^2|+|1-z^2|}$ and by (15a), T_i is invertible and $\|T_i^{-1}\| \leq 1$. It is easy to verify that both estimates are sharp. Now we define $T_z : \ell^2(X_1) \to H$ by

$$T_z(e,i) = (e,\xi_i) \tag{16a}$$

(recall that $\{\xi_1,\ldots,\xi_N\}$ is the orthonormal basis of $H_0 = \mathbb{C}^N$) and for $x \neq e$, $t(x) = i_1\ldots i_n$ and $j \neq i_n$,

$$T_z(x,j) = (x, \eta_j^{(i_n)}(z)). \tag{16b}$$

Fix $x \neq e$ and assume that $t(x) = i_1\ldots i_n$. Then T_z maps $\ell^2(\{(x,j) : j \in I \setminus \{i_n\}\}$ onto $H_x \cong \operatorname{Ker} P_{i_n}$ so that the restriction of T_z to $\ell^2(\{(x,j) : j \in I \setminus \{i_n\}\}$ can be identified with T_{i_n}. Therefore $\|T_z\| = \sqrt{|z^2|+|1-z^2|}$, T_z is invertible, $\|T_z^{-1}\| = 1$ and for $z \in (0,1)$, T_z is an isometry.

Now we are going to prove that T_z intertwines $\widetilde{\pi}_z$ with π_z, i.e. $T_z\widetilde{\pi}_z(x) = \pi_z(x)T_z$ for any $x \in G$. All we have to do is to check that for any $i \in I$, $g \in G_i \setminus \{e\}$ and $(x,j) \in X_1$,

$$T_z\widetilde{V}_z^{-1}\widetilde{A}_z(g)(x,j) = \pi_z(g)T_z\widetilde{V}_z^{-1}(x,j). \tag{17}$$

We will need the following two formulas (cf. (12)):

$$\xi_i+\left(\frac{-1}{N}+\frac{z}{N}\sqrt{\frac{N-1}{1-z^2}}\right)\sum_{k=1}^{N}\xi_k=\sqrt{\frac{N-1}{N(1-z^2)}}\zeta_i(z),\tag{18}$$

$i\in I$, and, for $j\neq i$,

$$\eta_j^{(i)}(z)+\left(\frac{-1}{N-1}+\frac{z\sqrt{N}}{N-1}\right)\sum_{k\neq i}\eta_k^{(i)}(z)=\sqrt{\frac{N-1}{N(1-z^2)}}(\mathrm{Id}-P_i)\zeta_j(z).\tag{19}$$

The first formula is easy to check. To prove the second one recall that $\sum_{j=1}^{N}\zeta_j=0$. Hence, by (13b),

$$\sum_{k\neq i}\eta_k^{(i)}(z)=\sqrt{(N-1)(1-z^2)}\zeta_0-z\sqrt{N-1}\zeta_i=\sqrt{\frac{N-1}{1-z^2}}(\zeta_0-z\zeta_i(z)),$$

which, upon using (13a), easily leads to (19). Therefore for $j\in I$ we have

$$T_z\widetilde{V}_z^{-1}(e,j)=\sqrt{\frac{N-1}{N(1-z^2)}}(e,\zeta_j(z));\tag{20}$$

and for $x\neq e$ as in (1) and $j\neq i_n$,

$$T_z\widetilde{V}_z^{-1}(x,j)=\sqrt{\frac{N-1}{N(1-z^2)}}(x,(\mathrm{Id}-P_{i_n})\zeta_j(z)).\tag{21}$$

Now we can prove (17). If $x=e$, $j=i$ then

$$\begin{aligned}T_z\widetilde{V}_z^{-1}\widetilde{A}_z(g)(e,i)=T_z\widetilde{V}_z^{-1}(e,i)&=\sqrt{\frac{N-1}{N(1-z^2)}}(e,\zeta_i(z))\\&=\sqrt{\frac{N-1}{N(1-z^2)}}\pi_z(g)(e,\zeta_i(z))=\pi_z(g)T_z\widetilde{V}_z^{-1}(e,i).\end{aligned}$$

For $j\neq i$ we get

$$\begin{aligned}&T_z\widetilde{V}_z^{-1}\widetilde{A}_z(g)(e,j)\\&\quad=T_z\widetilde{V}_z^{-1}\left(\frac{Nz^2-1}{N-1}(e,i)+(g,j)\right)\\&\quad=\frac{Nz^2-1}{N-1}\sqrt{\frac{N-1}{N(1-z^2)}}(e,\zeta_i(z))+\sqrt{\frac{N-1}{N(1-z^2)}}(g,(\mathrm{Id}-P_i)\zeta_j(z))\\&\quad=\sqrt{\frac{N-1}{N(1-z^2)}}[(e,P_i\zeta_j(z))+(g,(\mathrm{Id}-P_i)\zeta_j(z))]\end{aligned}$$

$$= \sqrt{\frac{N-1}{N(1-z^2)}}\pi_z(g)(e,\zeta_j(z)) = \pi_z(g)T_z\widetilde{V}_z^{-1}(e,j).$$

Now take $x = g^{-1}$ and $j \neq i$. Then

$$\begin{aligned} T_z\widetilde{V}_z^{-1}\widetilde{A}_z(g)(g^{-1},j) &= T_z\widetilde{V}_z^{-1}\left((e,j) - \frac{Nz^2-1}{N-1}(e,i)\right) \\ &= \sqrt{\frac{N-1}{N(1-z^2)}}\left(e,\zeta_j(z) - \frac{Nz^2-1}{N-1}\zeta_i(z)\right) \\ &= \sqrt{\frac{N-1}{N(1-z^2)}}(e,(\mathrm{Id}-P_i)\zeta_j(z)) \\ &= \pi_z(g)T_z\widetilde{V}_z^{-1}(g^{-1},j). \end{aligned}$$

Finally, if $x \neq e$, g^{-1} is as in (1) then

$$\begin{aligned} T_z\widetilde{V}_z^{-1}\widetilde{A}_z(g)(x,j) &= T_z\widetilde{V}_z^{-1}(gx,j) \\ &= \sqrt{\frac{N-1}{N(1-z^2)}}(gx,(\mathrm{Id}-P_{i_n})\zeta_j(z)) \\ &= \sqrt{\frac{N-1}{N(1-z^2)}}\pi_z(g)(x,(\mathrm{Id}-P_{i_n})\zeta_j(z)) \\ &= \pi_z(g)T_z\widetilde{V}_z^{-1}(x,j), \end{aligned}$$

which finishes the proof.

Remarks. 1) We have obtained the family π_z, $z \in \mathbb{C}$, of representations of the group $G = G_1 * \ldots * G_N$ as a special case of the construction presented in Section 2. We could do this a little bit more generally taking for example $\{\zeta_i(z_i) \otimes \zeta_i(\overline{z}_i)\}_{i=1}^N$, $z_i \in \mathbb{C}$, as the initial family of projections, with z_i's not necessarily all equal.

2) In view of Theorem 5.1 and Theorem 4.3(iv) we have, for any complex z satisfying $|Nz^2 - 1| < N - 1$, the following estimate of Wysoczański's representation:

$$\|\widetilde{\pi}_z(x)\| \le (|z^2| + |1-z^2|)^{3/2}\left(1 + \frac{|z^2| + |1-z^2|}{1 - \left|\frac{Nz^2-1}{N-1}\right|}\right).$$

Therefore, coming back to his parametrization, for $u \in \mathbb{C}$, $|u| < 1$, the right hand side of [W2, Theorem 11] can be replaced by

$$\left(\frac{|(N-1)u+1|+(N-1)|1-u|}{N}\right)^{3/2} \times\left(1+\frac{|(N-1)u+1|+(N-1)|1-u|}{N(1-|u|)}\right)$$

or, as $|(N-1)u+1| \le N|u|+|1-u|$, by

$$(|u|+|1-u|)^{3/2}\frac{1+|1-u|}{1-|u|},$$

which no longer depends on N.

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