COLLOQUIUM MATHEMATICUM

VOL. LXVIII 1995 FASC. 1

PLANAR RATIONAL COMPACTA

BY

L. E. FEGGOS, S. D. ILIADIS AND S. S. ZAFIRIDOU (PATRAS)

1. Introduction. In this paper we consider rational subspaces of the plane. A *rational space* is a space which has a basis of open sets with countable boundaries. In the special case where the boundaries are finite, the space is called *rim-finite*.

G. Nöbeling [8] has proved that the family of all rim-finite spaces does not contain a universal element. The same is true even for the family of planar rim-finite spaces. This fact is included in a wider result (see [1] and [4]) concerning some families of planar rim-scattered spaces.

S. Iliadis [3] (see also [7]) proved that there exists a universal rational space. Therefore there exists a rational space which contains topologically all rational compacta.

In [6] J. Mayer and E. Tymchatyn constructed a planar continuum of rim-type $\alpha+1$ which is a containing space for all planar compacta of rim-type $\leq \alpha$, where α is a countable ordinal.

In this paper we give a simple, direct and visualized example of a planar rational connected and locally connected space which is a containing space for all planar rational compacta. This provides an affirmative answer to Problem 5(2) of [2].

2. Definitions and notations. Let E^2 be the plane with a system Oxy of orthogonal coordinates. By a *simple closed curve* we mean a subset of E^2 which is homeomorphic to the set $\{(x,y) : x^2+y^2 = 1\}$, and by a *disk* a subset of E^2 homeomorphic to $\{(x,y) : x^2+y^2 \leq 1\}$. An *arc* is a subset A of E^2 for which there exists a homeomorphism h of $I \equiv [0,1]$ onto A. The points $h(0)$ and $h(1)$ are the *endpoints* of the arc and the set $A \setminus h(\{0,1\})$ is its *interior*.

Let $G \subseteq D \subseteq E^2$. By $\mathrm{Cl}_D(G)$, $\mathrm{Int}_D(G)$ and $\mathrm{Bd}_D(G)$ we denote the closure, interior, and boundary of G, respectively, in D. We omit the subscript "D" if $D = E^2$. For each $\varepsilon > 0$ we denote by $N(G,\varepsilon)$ the set of all points of E^2 whose distance from G is less than ε. By ω we denote the set $\{0,1,2,\ldots\}$ of all non-negative integers.

1991 *Mathematics Subject Classification*: 54C25, 54F50.

A space Y is called a *containing space* for a family $\mathcal{F}$ of spaces if for every $X \in \mathcal{F}$, there exists a homeomorphism of X onto a subset of Y. If in addition $Y \in \mathcal{F}$, then Y is called a *universal space* for the family $\mathcal{F}$.

We denote by L_n, $n = 1, 2, \ldots$, the set of all ordered n-tuples $i_1 \ldots i_n$, where $i_t = 0$ or 1, for every $t = 1, \ldots, n$, and by L_0 the set $\{\emptyset\}$. By $I_{\bar{i}}$, where $\bar{i} = i_1 \ldots i_n \in L_n$, $n \geq 1$, we denote the set of all points of $I \equiv [0, 1]$ for which the kth digit of the dyadic expansion, $k = 1, \ldots, n$, coincides with i_k. Also we set $I_\emptyset = I$.

Let $\mathcal{W}_n = \{I_{\bar{i}} \times I_{\bar{j}} : \bar{i}, \bar{j} \in L_n\}$, $n \in \omega$. Obviously for every $n \in \omega$ the family $\mathcal{W}_n$ is a finite closed covering of I^2. If a is an endpoint of $I_{\bar{i}}$ and b is an endpoint of $I_{\bar{j}}$, then the sets $\{a\} \times I_{\bar{j}}$ and $I_{\bar{i}} \times \{b\}$ are called *edges* and the point $(a, b) \in E^2$ is called a *vertex* of $\mathcal{W}_n$. The sets of all edges and of all vertices of $\mathcal{W}_n$ are denoted by $E(\mathcal{W}_n)$ and $V(\mathcal{W}_n)$, respectively. We set $\mathrm{Bd}(\mathcal{W}_n) = \bigcup\{\mathrm{Bd}(F) : F \in \mathcal{W}_n\} = \bigcup\{e : e \in E(\mathcal{W}_n)\}$.

Let D be a disk of the plane. A finite closed covering $\mathcal{V}$ of D is said to be an *n-subdivision* (or *subdivision*) of D, where $n \in \omega$, if there exists a homeomorphism h of D onto I^2 such that $\mathcal{V} = \{h^{-1}(F) : F \in \mathcal{W}_n\}$. Every such homeomorphism is called a *$\mathcal{V}$-homeomorphism*. The sets $h^{-1}(e)$, where $e \in E(\mathcal{W}_n)$, are called *edges* of $\mathcal{V}$ and the points $h^{-1}(v)$, where $v \in V(\mathcal{W}_n)$, are called *vertices* of $\mathcal{V}$. We denote by $E(\mathcal{V})$ and $V(\mathcal{V})$ the sets of all edges and of all vertices of $\mathcal{V}$, respectively. We set $\mathrm{Bd}(\mathcal{V}) = \bigcup\{\mathrm{Bd}(F) : F \in \mathcal{V}\} = \bigcup\{e : e \in E(\mathcal{V})\}$ and $\mathrm{mesh}(\mathcal{V}) = \max\{\mathrm{diam}(F) : F \in \mathcal{V}\}$. Obviously $\mathrm{Bd}(\mathcal{V}) = h^{-1}(\mathrm{Bd}(\mathcal{W}))$. Also, for $G \subseteq D$ we set

$$\mathrm{st}(G, \mathcal{V}) = \bigcup\{F \in \mathcal{V} : F \cap G \neq \emptyset\}.$$

We say that a subdivision $\mathcal{V}$ of D is *rational with respect to a set* $X \subseteq D$ if for every edge e of $\mathcal{V}$ the set $e \cap X$ is a countable subset of the interior of e. Note that in this case no point of X is a vertex of $\mathcal{V}$.

Let $n_1, n_2 \in \omega$, $n_1 \leq n_2$. We say that an n_2-subdivision $\mathcal{V}_2$ of D is *inscribed* in an n_1-subdivision $\mathcal{V}_1$ of D if: (α) each element of $\mathcal{V}_2$ is contained in some element of $\mathcal{V}_1$ and (β) for every $F \in \mathcal{V}_1$ the set of all elements of $\mathcal{V}_2$ which are contained in F is an $(n_2 - n_1)$-subdivision of the disk F. We observe that in this case $\mathrm{Bd}(\mathcal{V}_1) \subseteq \mathrm{Bd}(\mathcal{V}_2)$.

3. Containing space. Let

$$Q_\Delta = \{p/2^n \in I \setminus \{0, 1\} : p, n \in \omega\}, \qquad Q_T = \{p/3^n \in I : p, n \in \omega\}$$

and

$$Y = I^2 \setminus (((I \setminus Q_T) \times Q_\Delta) \cup (Q_\Delta \times (I \setminus Q_T))).$$

We shall prove that Y is a containing space for the family of all planar rational compacta. It is easy to verify that $I^2 \setminus \bigcup\{\mathrm{Bd}(\mathcal{W}_n) : n \in \omega\} \subseteq Y$.

We observe that this remains true if Q_Δ and Q_T are replaced by any pair of disjoint countable dense subsets of I.

4. LEMMA. *The space Y is rational, connected and locally connected.*

Proof. We observe that the set $K \equiv (Q_T \times I) \cup (I \times Q_T)$ is connected and $K \subseteq Y \subseteq I^2 = \mathrm{Cl}(K)$. So Y is connected.

For $y \in Y$ and $i \in \omega$ we set

$$U_i(y) \equiv Y \cap \mathrm{Int}_{I^2}(\mathrm{st}(y, \mathcal{W}_i)).$$

It is easy to verify (as for the space Y) that $U_i(y)$ is connected. Moreover, $\mathrm{Bd}_Y(U_i(y))$ is countable. Also it is easy to see that $\{U_i(y) : i \in \omega\}$ is a basis of open neighbourhoods of y in Y. Thus Y is a planar rational connected and locally connected space.

5. LEMMA. *Let D be a disk of the plane, $a, b \in \mathrm{Bd}(D)$, $a \neq b$, and $X \subseteq D \setminus \{a, b\}$ be a rational compact space. Then there exists an arc $A \subseteq D$ with endpoints a, b such that $A \cap X$ is countable.*

Proof. Let A_1, A_2 be the arcs of D with endpoints a, b such that $A_1 \cup A_2 = \mathrm{Bd}(D)$. It is clear that $X \cap A_1$ and $X \cap A_2$ are closed disjoint subsets of $X \cap D$. Thus there exists a closed countable subset F of $X \cap D$ which separates (in $X \cap D$) the sets $X \cap A_1$ and $X \cap A_2$ (see [5], §51, IV, Th. 9). Let G_1, G_2 be disjoint open subsets of $X \cap D$ such that $(X \cap D) \setminus F = G_1 \cup G_2$, $X \cap A_1 \subseteq G_1$ and $X \cap A_2 \subseteq G_2$.

Let $F_1 = \mathrm{Cl}(G_1) \cup A_1$, $F_2 = \mathrm{Cl}(G_2) \cup A_2$, $x \in A_1 \setminus \{a, b\}$ and $y \in A_2 \setminus \{a, b\}$. Since F_1 and F_2 are compact and $F_1 \cap F_2 \subseteq F \cup \{a, b\}$ is totally disconnected, there exists (see [9], p. 108, Th. (3.1)) a simple closed curve J which separates the points x and y in the plane such that $J \cap (F_1 \cup F_2) \subseteq F \cup \{a, b\}$.

From the above it follows that $J \cap (A_1 \cup A_2) = \{a, b\}$. Since J separates x and y, the simple closed curve J intersects the disk D in an arc A with endpoints a, b. We have $A \cap X \subseteq (J \cap D) \cap X \subseteq J \cap (F_1 \cup F_2 \cup F) \subseteq F \cup \{a, b\}$. Hence $A \cap X$ is countable. Thus A is the required arc.

6. THEOREM. *The space Y is a containing space for all planar rational compacta.*

Proof. Let X be a planar rational compact space and D be a disk of the plane such that $X \subseteq \mathrm{Int}(D)$. We construct a homeomorphism $h : D \to I^2$ such that $h(X) \subseteq Y$. For every $i \in \omega$ we shall define by induction a natural number n_i, an n_i-subdivision $\mathcal{V}_i$ of D, rational with respect to X, and a $\mathcal{V}_i$-homeomorphism h_i such that:

(1) $\mathcal{V}_{i+1}$ is inscribed in $\mathcal{V}_i$,
(2) $\mathrm{mesh}(\mathcal{V}_{i+1}) < 1/2^{i+1}$,
(3) $h_{i+1}|_{\mathrm{Bd}(\mathcal{V}_i)} = h_i|_{\mathrm{Bd}(\mathcal{V}_i)}$,

(4) $h_i(\mathrm{Bd}(\mathcal{V}_i) \cap X) \subseteq Y$.

Let $i = 0$. We set $n_0 = 0$ and $\mathcal{V}_0 = \{D\}$. Let h_0 be a homeomorphism of D onto I^2. Obviously h_0 is a $\mathcal{V}_0$-homeomorphism and (4) is satisfied for $i = 0$ because $\mathrm{Bd}(\mathcal{V}_0) \cap X = \emptyset$. The other properties concern the case $i > 0$.

Suppose that for every $i \le k$ we have defined a natural number n_i, an n_i-subdivision $\mathcal{V}_i$ of D, rational with respect to X, and a $\mathcal{V}_i$-homeomorphism h_i such that (1)–(3) are satisfied if $i + 1 \le k$, and (4) is satisfied if $i \le k$. We define a natural number n_{k+1}, an n_{k+1}-subdivision $\mathcal{V}_{k+1}$ of D, rational with respect to X, and a $\mathcal{V}_{k+1}$-homeomorphism h_{k+1} such that (1)–(3) are satisfied if $i + 1 \le k + 1$, and (4) is satisfied if $i \le k + 1$.

There exists an integer $j \in \omega$ such that $\mathrm{diam}(h_k^{-1}(F)) < 1/2^{k+1}$ for every $F \in \mathcal{W}_{n_k+j}$. Since $I^2 \setminus h_k(X)$ is a dense subset of I^2, $\mathcal{V}_k$ is rational with respect to X and since $V(\mathcal{W}_{n_k+j}) \cap Y = \emptyset$ there exists a $\mathcal{V}_k$-homeomorphism h'_k such that $h'_k|_{\mathrm{Bd}(\mathcal{V}_k)} = h_k|_{\mathrm{Bd}(\mathcal{V}_k)}$, $h'_k(X) \cap V(\mathcal{W}_{n_k+j}) = \emptyset$ and $\mathrm{diam}((h'_k)^{-1}(F)) < 1/2^{k+1}$ for every $F \in \mathcal{W}_{n_k+j}$.

Let $n_{k+1} = n_k + j$ and

$$\mathcal{V}'_{k+1} = \{(h'_k)^{-1}(F) : F \in \mathcal{W}_{n_{k+1}}\}.$$

Then $\mathcal{V}'_{k+1}$ is an n_{k+1}-subdivision of D with $\mathrm{mesh}(\mathcal{V}'_{k+1}) < 1/2^{k+1}$, which is inscribed in $\mathcal{V}_k$. However, this subdivision is not, in general, rational with respect to X. The n_{k+1}-subdivision $\mathcal{V}_{k+1}$ of D will be obtained by some modification of $\mathcal{V}'_{k+1}$.

For every edge $e \in E(\mathcal{V}'_{k+1}) \setminus \mathrm{Bd}(\mathcal{V}_k)$ we denote by D_e a disk such that: (α) $e \subseteq D_e$, (β) $D_e \cap \mathrm{Bd}(\mathcal{V}_k) \subseteq e \cap \mathrm{Bd}(\mathcal{V}_k)$, ($\gamma$) $D_{e_1} \cap D_{e_2} \subseteq e_1 \cap e_2$ if $e_1 \ne e_2$, and (δ) for every $F \in \mathcal{V}'_{k+1}$, $\mathrm{diam}(F \cup \bigcup\{D_e : e \subseteq F\}) < 1/2^{k+1}$.

For every $e \in E(\mathcal{V}'_{k+1})$ we define an arc $\widetilde{e}$ as follows: (α) if $e \subseteq \mathrm{Bd}(\mathcal{V}_k)$, then $\widetilde{e} = e$, and (β) if $e \not\subseteq \mathrm{Bd}(\mathcal{V}_k)$, then $\widetilde{e}$ is the arc A of Lemma 5, where D is the disk D_e and a, b are the endpoints of e. Thus every $F \in \mathcal{V}'_{k+1}$ defines a simple closed curve J_F, which is the union of arcs $\widetilde{e}$, where $e \subseteq \mathrm{Bd}(F)$. Let $\widetilde{F}$ be the disk having as boundary the simple closed curve J_F. We set

$$\mathcal{V}_{k+1} = \{\widetilde{F} : F \in \mathcal{V}'_{k+1}\}.$$

For every $e \in E(\mathcal{V}'_{k+1})$ we define a homeomorphism $h^{\tilde{e}}_{k+1}$ of $\widetilde{e}$ into $\mathrm{Bd}(\mathcal{W}_{n_{k+1}})$ as follows: (α) if $\widetilde{e} = e \subseteq \mathrm{Bd}(\mathcal{V}_k)$, then $h^{\tilde{e}}_{k+1} = h_k|_e$, and ($\beta$) if $e \not\subseteq \mathrm{Bd}(\mathcal{V}_k)$, then $h^{\tilde{e}}_{k+1}$ is a homeomorphism of $\widetilde{e}$ onto $h'_k(e)$ such that

$$h^{\tilde{e}}_{k+1}|_{\{a,b\}} = h'_k|_{\{a,b\}},$$

where a, b are the endpoints of $\widetilde{e}$, and

$$h^{\tilde{e}}_{k+1}(\widetilde{e} \cap X) \subseteq Y \cap h'_k(e).$$

The existence of such a homeomorphism is based on the fact that $\widetilde{e} \cap X$ is countable, and $h'_k(e) \cap Y$ is countable and dense in $h'_k(e)$. For every

$\widetilde{F} \in \mathcal{V}_{k+1}$ we denote by $h_{k+1}^{\tilde{F}}$ a homeomorphism of $\widetilde{F}$ onto $h'_k(F)$ such that $h_{k+1}^{\tilde{F}}|_{\tilde{e}} = h_{k+1}^{\tilde{e}}$ for every $\widetilde{e} \subseteq \widetilde{F}$. Let h_{k+1} be a homeomorphism of D onto I^2 for which $h_{k+1}|_{\tilde{F}} = h_{k+1}^{\tilde{F}}$ for every $\widetilde{F} \in \mathcal{V}_{k+1}$.

It is easy to verify that $\mathcal{V}_{k+1}$ is an n_{k+1}-subdivision of D, rational with respect to X, and h_{k+1} is a $\mathcal{V}_{k+1}$-homeomorphism with properties (1)–(4).

Furthermore, for every $x \in D$ and $i, j \in \omega$, $j \geq i$, by the definition of $\mathcal{V}_i$ we have

$$h_i(\mathrm{st}(x, \mathcal{V}_i)) = \mathrm{st}(h_i(x), \mathcal{W}_{n_i}), \tag{5}$$

by (1) it follows that

$$\mathrm{st}(x, \mathcal{V}_j) \subseteq \mathrm{st}(x, \mathcal{V}_i), \tag{6}$$

and by (3) we have

$$h_j(\mathrm{st}(x, \mathcal{V}_i)) = h_i(\mathrm{st}(x, \mathcal{V}_i)). \tag{7}$$

Now we define a map $h : D \to I^2$ setting for every $x \in D$,

$$h(x) = \bigcap \{h_i(\mathrm{st}(x, \mathcal{V}_i)) : i \in \omega\}$$

and prove that h is a homeomorphism such that $h(X) \subseteq Y$.

First note that by (6) and (7), $h_{i+1}(\mathrm{st}(x, \mathcal{V}_{i+1})) \subseteq h_i(\mathrm{st}(x, \mathcal{V}_i))$ for every $x \in D$ and $i \in \omega$. On the other hand, by (5), $\lim_{i\to\infty} \mathrm{diam}(h_i(\mathrm{st}(x, \mathcal{V}_i))) = 0$. Hence $\bigcap h_i(\mathrm{st}(x, \mathcal{V}_i))$ is a singleton. Thus h is well defined.

Let $x_1, x_2 \in D$ and $x_1 \neq x_2$. By (2) there exists $i \in \omega$ such that $\mathrm{st}(x_1, \mathcal{V}_i) \cap \mathrm{st}(x_2, \mathcal{V}_i) = \emptyset$. Hence $h_i(\mathrm{st}(x_1, \mathcal{V}_i)) \cap h_i(\mathrm{st}(x_2, \mathcal{V}_i)) = \emptyset$ and therefore $h(x_1) \neq h(x_2)$, that is, h is one-to-one.

We prove that h is continuous. Let $h(x) = y$ and U be an open neighbourhood of y in I^2. There exists $i \in \omega$ such that $\mathrm{st}(y, \mathcal{W}_{n_i}) \subseteq U$. By (5), $h_i(\mathrm{st}(x, \mathcal{V}_i)) \subseteq U$. For the continuity of h it is sufficient to prove that $h(\mathrm{Int}(\mathrm{st}(x, \mathcal{V}_i))) \subseteq U$. Let $z \in \mathrm{Int}(\mathrm{st}(x, \mathcal{V}_i))$. It is easy to see that $\mathrm{st}(z, \mathcal{V}_i) \subseteq \mathrm{st}(x, \mathcal{V}_i)$. Hence $h(z) \in h_i(\mathrm{st}(z, \mathcal{V}_i)) \subseteq h_i(\mathrm{st}(x, \mathcal{V}_i)) \subseteq U$. Thus h is continuous and therefore h is a homeomorphism.

To prove that $h(X) \subseteq Y$, we observe that if $x \in \mathrm{Bd}(\mathcal{V}_i)$, then $h_i(x) = h_j(x) \in h_j(\mathrm{st}(x, \mathcal{V}_j))$ for every $j \geq i$. Thus $h(x) = h_i(x)$. Since $h_i(\mathrm{Bd}(\mathcal{V}_i)) = \mathrm{Bd}(\mathcal{W}_{n_i})$ we see that if $x \notin \bigcup\{\mathrm{Bd}(\mathcal{V}_i) : i \in \omega\}$, then $h(x) \notin \bigcup\{\mathrm{Bd}(\mathcal{W}_{n_i}) : i \in \omega\} = \bigcup\{\mathrm{Bd}(\mathcal{W}_j) : j \in \omega\}$.

Let $x \in X$. If $x \notin \bigcup_i \mathrm{Bd}(\mathcal{V}_i)$, then $h(x) \notin \bigcup_j \mathrm{Bd}(\mathcal{W}_j)$. Since $I^2 \setminus \bigcup_j \mathrm{Bd}(\mathcal{W}_j) \subseteq Y$, we have $h(x) \in Y$. If $x \in \bigcup_i \mathrm{Bd}(\mathcal{V}_i)$, then $h(x) = h_i(x)$ for some $i \in \omega$. By (4) it follows that $h(x) \in Y$. Thus $h(X) \subseteq Y$.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF PATRAS
261 10 PATRAS, GREECE

Reçu par la Rédaction le 6.4.1994