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A NOTE ON SUMS OF INDEPENDENT UNIFORMLY DISTRIBUTED RANDOM VARIABLES

BY

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Introduction. Let (t_i) be a sequence of independent random variables uniformly distributed on $[-1,1]$. We are looking for the best constants A_p and B_p such that for every sequence (a_i) of real numbers the following inequalities hold:

$$A_p\Big(E\Big|\sum_{i=1}^n a_it_i\Big|^2\Big)^{1/2} \le \Big(E\Big|\sum_{i=1}^n a_it_i\Big|^p\Big)^{1/p} \le B_p\Big(E\Big|\sum_{i=1}^n a_it_i\Big|^2\Big)^{1/2}.$$

These inequalities with the best possible constants have some importance for geometric problems and elsewhere. Some estimates for A_p and B_p were found by K. Ball [1]. The values of B_{2m} for m positive integers are known (cf. [4], Chapter 12.G).

Let g be a standard normal variable and

$$\gamma_p = (E|g|^p)^{1/p} = \sqrt{2}\bigg(\frac{\Gamma\big(\frac{p+1}{2}\big)}{\sqrt{\pi}}\bigg)^{1/p}.$$

We will prove that

$$A_p = \begin{cases} \gamma_p & \text{for } p\in[1,2], \\ \dfrac{3^{1/2}}{(p+1)^{1/p}} & \text{for } p\ge 2, \end{cases} \qquad B_p = \begin{cases} \dfrac{3^{1/2}}{(p+1)^{1/p}} & \text{for } p\in[1,2], \\ \gamma_p & \text{for } p\ge 2. \end{cases}$$

The same inequalities for a Bernoulli sequence (ε_i), i.e. the sequence of independent symmetric random variables taking on values ±1, were studied by Haagerup [3]. We will not use Haagerup's results, but it should be pointed out that they immediately yield the values of some A_p and B_p (since $\sum_{i=1}^\infty 2^{-i}\varepsilon_i$ has the same distribution as each of t_i).

I. The inequalities in the real case. We start with some well known facts about symmetric unimodal variables (cf. [2]); we present the proofs for completeness.

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Definition 1. A random real variable X is called *symmetric unimodal* (s.u.) if it has a density with respect to the Lebesgue measure and the density function is symmetric and nonincreasing on $[0,\infty)$.

Lemma 1. *A real random variable X is s.u. if and only if there exists a probability measure μ on $[0,\infty)$ such that the density function $g(x)$ of X is*

$$g(x)=\int_0^\infty \frac{1}{2t}\chi_{[-t,t]}(x)\,d\mu(t) \quad \textit{for } x\in\mathbb{R}.$$

Proof. Let $g(x)$ be the density of some s.u. random variable. Since g is nonincreasing on $[0,\infty)$ we can assume that $g(x)$ is left-continuous for $x>0$. We define the measure ν on $[0,\infty)$ by $\nu[x,\infty)=g(x)$ for $x>0$ and let $\mu(t)=2t\nu(t)$. We have, for $x>0$,

$$g(x)=\int_0^\infty \chi_{[-t,t]}(x)\,d\nu(t)=\int_0^\infty \frac{1}{2t}\chi_{[-t,t]}(x)\,d\mu(t).$$

For $x<0$ the above formula holds by symmetry.

Since

$$\int_0^\infty d\mu(t)=\int_0^\infty 2t\,d\nu(t)=\int_0^\infty\int \chi_{[-t,t]}(x)\,dx\,d\nu(t)=\int g(x)\,dx=1,$$

μ is a probability measure.

If μ and $g(x)$ satisfy the lemma's assumptions then $g(x)$ is obviously symmetric and monotone on $[0,\infty)$ and since as above $\int g(x)\,dx=1$, $g(x)$ is the density of some random s.u. variable.

Lemma 2. *If $X=\sum_{i=1}^n X_i$ and X_i are independent s.u. random variables, then X is s.u. In particular, if $X=\sum_{i=1}^n a_it_i$, where the t_i are independent random variables uniformly distributed on $[-1,1]$ and $a_i\in\mathbb{R}$, then X is symmetric unimodal.*

Proof. It suffices to prove the lemma for $n=2$ and proceed by induction.

Let X_1 and X_2 be independent s.u. variables with density functions g_1,g_2 and measures μ_1,μ_2 as in Lemma 1. Then X_1+X_2 has the density

$$g(x)=g_1*g_2(x)=\int_0^\infty\int_0^\infty \frac{1}{4ts}\chi_{[-t,t]}*\chi_{[-s,s]}(x)\,d\mu(t)\,d\mu(s)$$

and obviously g is symmetric and nonincreasing on $[0,\infty)$.

Corollary 1. *Let $p>q>0$ and $X_1,\ldots,X_n$ be a sequence of independent symmetric unimodal random variables. Then*

$$(p+1)^{1/p}\Big(E\Big|\sum_{i=1}^n X_i\Big|^p\Big)^{1/p}\ge (q+1)^{1/q}\Big(E\Big|\sum_{i=1}^n X_i\Big|^q\Big)^{1/q}.$$

Proof. By Lemma 2, the random variable $X = \sum_{i=1}^n X_i$ is s.u. Let $g(x)$ be the density of X and μ the measure given for X by Lemma 1. Then

$$\Big(E\Big|\sum_{i=1}^n X_i\Big|^p\Big)^{q/p} = \Big(\int_{\mathbb{R}} |x|^p \int_0^\infty \frac{1}{2t}\chi_{[-t,t]}(x)\,d\mu(t)\,dx\Big)^{q/p}$$
$$= \Big(\int_0^\infty \Big(\frac{1}{2t}\int_{\mathbb{R}} |x|^p \chi_{[-t,t]}(x)\,dx\Big)\,d\mu(t)\Big)^{q/p}.$$

So by the Jensen inequality,

$$\Big(E\Big|\sum_{i=1}^n X_i\Big|^p\Big)^{q/p} \ge \int_0^\infty \Big(\frac{1}{2t}\int_{\mathbb{R}} |x|^p \chi_{[-t,t]}(x)\,dx\Big)^{q/p} d\mu(t)$$
$$= \int_0^\infty \frac{q+1}{(p+1)^{q/p}}\Big(\frac{1}{2t}\int_{\mathbb{R}} |x|^q \chi_{[-t,t]}(x)\,dx\Big)\,d\mu(t)$$
$$= \frac{q+1}{(p+1)^{q/p}}\Big(\int_{\mathbb{R}} \int_0^\infty |x|^q \frac{1}{2t}\chi_{[-t,t]}(x)\,d\mu(t)\,dx\Big)$$
$$= \frac{q+1}{(p+1)^{q/p}}\Big(E\Big|\sum_{i=1}^n X_i\Big|^q\Big).$$

LEMMA 3. *Let $p \ge 1$ and define*

$$G(t) = \begin{cases} (p+2)\dfrac{(t+1)^{p+1}-(t-1)^{p+1}}{t^2} - \dfrac{(t+1)^{p+2}-(t-1)^{p+2}}{t^3} & \text{for } t \ge 1, \\[2ex] (p+2)\dfrac{(1+t)^{p+1}+(1-t)^{p+1}}{t^2} - \dfrac{(1+t)^{p+2}-(1-t)^{p+2}}{t^3} & \text{for } 0 < t < 1. \end{cases}$$

Then G is nondecreasing on $(0,\infty)$ if $p \ge 2$ and nonincreasing if $1 \le p \le 2$.

The proof is based on the following lemma:

LEMMA 4. *Let $p \ge 1$ and let*

$$f_1(t) = (p-1)((1+t)^p - (1-t)^p) - p((1+t)^{p-1} - (1-t)^{p-1}) \quad \text{for } t \in [0,1],$$
$$f_2(t) = (1+t)^p((p^2-1)t^2 - 3pt + 3) - (1-t)^p((p^2-1)t^2 + 3pt + 3) \quad \text{for } t \in [0,1],$$
$$f_3(t) = (t+1)^p((p^2-1)t^2 - 3pt + 3) - (t-1)^p((p^2-1)t^2 + 3pt + 3) \quad \text{for } t > 1.$$

Then f_1, f_2 and f_3 are nonnegative for $p \ge 2$ and nonpositive for $1 \le p \le 2$.

P r o o f. Assume first that $p \ge 2$. We have

- $f_1(0) = 0$ and
$$f_1'(t) = p(p-1)t((1+t)^{p-2} - (1-t)^{p-2}) \ge 0 \quad \text{for } t \in [0,1],$$
- $f_2(0) = 0$ and
$$f_2'(t) = (p+2)(p+1)tf_1(t) \ge 0 \quad \text{for } t \in [0,1].$$
- $$\begin{aligned} f_3(t) &= 3(t^2-1)^2((t+1)^{p-2} - (t-1)^{p-2}) \\ &\quad + (p-2)t[((p+2)t^2-3)((t+1)^{p-1} - (t-1)^{p-1}) \\ &\quad + (p-1)t((t+1)^{p-1} + (t-1)^{p-1})] \ge 0 \quad \text{for } t > 1. \end{aligned}$$

For $p \in [1,2]$ the proof is analogous.

P r o o f o f L e m m a 3. Since $G(t)$ is continuous it suffices to show that $G(t)$ is nondecreasing (nonincreasing for $p \in [1,2]$) on $(0,1)$ and $(1,\infty)$. But
$$G'(t) = \begin{cases} t^{-4}f_3(t) & \text{if } t > 1, \\ t^{-4}f_2(t) & \text{if } 0 < t < 1. \end{cases}$$
Hence $G'(t) \ge 0$ for $p \ge 2$ and $G'(t) \le 0$ for $1 \le p \le 2$, by Lemma 4.

LEMMA 5. *If t_1, t_2, t_3 are independent random variables uniformly distributed on $[-1,1]$ and $a,b,c,d > 0$, $a^2+b^2 = c^2+d^2$ with $c \ge a \ge b \ge d$, then*
$$E|t_1 + at_2 + bt_3|^p \le E|t_1 + ct_2 + dt_3|^p \quad \textit{for } p \in [1,2]$$
and
$$E|t_1 + at_2 + bt_3|^p \ge E|t_1 + ct_2 + dt_3|^p \quad \textit{for } p \ge 2.$$

P r o o f. Since
$$|x|^p = \frac{d^3}{dx^3}\left(\frac{x^3|x|^p}{(p+1)(p+2)(p+3)}\right)$$
we easily check by integrating by parts that for
$$c_p = \frac{1}{4(p+1)(p+2)(p+3)}$$
we have
$$\begin{aligned} E|t_1 + at_2 + bt_3|^p &= \frac{1}{8}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} |x_1 + ax_2 + bx_3|^p \, dx_1 \, dx_2 \, dx_3 \\ &= c_p\bigg(\frac{(a+b+1)^3|a+b+1|^p + (a-b-1)^3|a-b-1|^p}{ab} \\ &\quad - \frac{(a-b+1)^3|a-b+1|^p + (a+b-1)^3|a+b-1|^p}{ab}\bigg) \end{aligned}$$

Let $k = a^2 + b^2$, $s = 2ab$. Then $a - b = \sqrt{k-s}$, $a + b = \sqrt{k+s}$ and

$$\begin{aligned}
f(s) &= E|t_1 + at_2 + bt_3|^p \\
&= 2c_p\bigg(\frac{(\sqrt{k+s}+1)^3|\sqrt{k+s}+1|^p + (\sqrt{k-s}-1)^3|\sqrt{k-s}-1|^p}{s} \\
&\quad - \frac{(\sqrt{k-s}+1)^3|\sqrt{k-s}+1|^p + (\sqrt{k+s}-1)^3|\sqrt{k+s}-1|^p}{s}\bigg) \\
&= 2c_p\frac{g(s)}{s}.
\end{aligned}$$

We are to show that for fixed k, $f(s)$ is nondecreasing if $p \ge 2$ (nonincreasing if $p \in [1,2]$) on $(0,k)$.

Since $g(0) = 0$ it suffices to prove that $g'(s)$ is nondecreasing (nonincreasing). We have

$$g''(s) = \frac{p+3}{4}(G(\sqrt{k+s}) - G(\sqrt{k-s})),$$

where $G(t)$ was defined in Lemma 3. Hence $g''(s) \ge 0$ for $p \ge 2$ and $g''(s) \le 0$ for $p \in [1,2]$ (by Lemma 3) and the proof is complete.

COROLLARY 2. *If* X, t_1, t_2 *are independent random variables*, t_1, t_2 *are uniformly distributed on* $[-1,1]$, X *is symmetric unimodal and* $a,b,c,d > 0$, $a^2 + b^2 = c^2 + d^2$ *with* $c \ge a \ge b \ge d$, *then*

$$E|X + at_1 + bt_2|^p \le E|X + ct_1 + dt_2|^p \quad \textit{for } p \in [1,2]$$

and

$$E|X + at_1 + bt_2|^p \ge E|X + ct_1 + dt_2|^p \quad \textit{for } p \ge 2.$$

Proof. Let $g(x)$ be the density function of X and μ be the measure given by Lemma 1. Let t_3 be a random variable independent of t_1, t_2 uniformly distributed on $[-1,1]$. We have, for $p \in [1,2]$,

$$\begin{aligned}
E|X + at_1 + bt_2|^p &= \int_{-\infty}^{\infty} E|x + at_1 + bt_2|^p g(x)\,dx \\
&= \int_0^{\infty} \frac{1}{2s} \int_{-s}^{s} E|t + at_1 + bt_2|^p \,dt\,d\mu(s) \\
&= \int_0^{\infty} E|st_3 + at_1 + bt_2|^p \,d\mu(s) \\
&\le \int_0^{\infty} E|st_3 + ct_1 + dt_2|^p \,d\mu(s) = E|X + ct_1 + dt_2|^p.
\end{aligned}$$

The second equality follows from Fubini's theorem, and the inequality is a consequence of Lemma 5.

For $p \geq 2$ we proceed in the same way.

DEFINITION 2. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two sequences of real numbers. We say that x is *majorized* by y and write $x \prec y$ if $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$ and $\sum_{i=1}^k x_i^* \leq \sum_{i=1}^k y_i^*$ for $k = 1, \ldots, n$, where (x_i^*) and (y_i^*) are the nonincreasing rearrangements of x and y.

PROPOSITION 1. *Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be two sequences of real numbers such that $(a_i^2) \prec (b_i^2)$ and $t_1, \ldots, t_n$ be a sequence of independent random variables uniformly distributed on $[-1, 1]$. Then*

$$\Big(E\Big|\sum_{i=1}^n a_i t_i\Big|^p\Big)^{1/p} \leq \Big(E\Big|\sum_{i=1}^n b_i t_i\Big|^p\Big)^{1/p} \qquad \textit{for } p \in [1,2]$$

and

$$\Big(E\Big|\sum_{i=1}^n a_i t_i\Big|^p\Big)^{1/p} \geq \Big(E\Big|\sum_{i=1}^n b_i t_i\Big|^p\Big)^{1/p} \qquad \textit{for } p \geq 2.$$

Proof. By the lemma of Muirhead (cf. [4], Chapter 1.B) it suffices to prove the inequalities if $a_i^2 = b_i^2$ for $i \neq j, k$, $a_j^2 = tb_j^2 + (1-t)b_k^2$ and $a_k^2 = tb_k^2 + (1-t)b_j^2$ for some $j, k \in \{1, \ldots, n\}$ and $t \in (0,1)$. By symmetry we can also assume that a_i and b_i are nonnegative. So finally Proposition 1 follows from Corollary 2 if we set $X = \sum_{i \neq j,k} a_i t_i$.

Let g be a standard normal variable and

$$\gamma_p = (E|g|^p)^{1/p} = \sqrt{2}\bigg(\frac{\Gamma\big(\frac{p+1}{2}\big)}{\sqrt{\pi}}\bigg)^{1/p}.$$

We have the following

THEOREM 1. *If $t_1, \ldots, t_n$ is a sequence of independent random variables uniformly distributed on $[-1, 1]$, and $a_1, \ldots, a_n$ are real numbers, then*

$$\Big(E\Big|\sum_{i=1}^n a_i t_i\Big|^2\Big)^{1/2} \leq \gamma_p^{-1}\Big(E\Big|\sum_{i=1}^n a_i t_i\Big|^p\Big)^{1/p} \qquad \textit{for } p \in [1,2]$$

and

$$\Big(E\Big|\sum_{i=1}^n a_i t_i\Big|^p\Big)^{1/p} \leq \gamma_p\Big(E\Big|\sum_{i=1}^n a_i t_i\Big|^2\Big)^{1/2} \qquad \textit{for } p \in [2,\infty).$$

The above constants are the best possible.

Proof. Let $p \in [1, 2]$. By Proposition 1,

$$E\Big|\sum_{i=1}^n a_i t_i\Big|^p \geq \Big(\sum_{i=1}^n |a_i|^2\Big)^{p/2} E\Big|\sum_{i=1}^n \frac{1}{\sqrt{n}} t_i\Big|^p.$$

But by the central limit theorem $\lim_{n\to\infty} E|\sum_{i=1}^n (1/\sqrt{n})t_i|^p = (\sqrt{1/3}\,\gamma_p)^p$ so

$$\Big(E\Big|\sum_{i=1}^n a_i t_i\Big|^p\Big)^{1/p} \ge \Big(\sum_{i=1}^n |a_i|^2\Big)^{1/2}\sqrt{1/3}\,\gamma_p = \gamma_p\Big(E\Big|\sum_{i=1}^n a_i t_i\Big|^2\Big)^{1/2}.$$

This proves the first inequality of the theorem. The second one can be established in an analogous way.

The central limit theorem shows that these constants cannot be improved. As a corollary from Proposition 1 we get the following answer to a question posed by A. Pełczyński:

PROPOSITION 2. *If $t_1, \ldots, t_n$ is a sequence of independent random variables uniformly distributed on $[-1,1]$, $\varepsilon_1, \ldots, \varepsilon_n$ is a Bernoulli sequence of random variables and $a_1, \ldots, a_n$ are real numbers, then*

$$\frac{1}{2}E\Big|\sum_{i=1}^n a_i\varepsilon_i\Big| \le E\Big|\sum_{i=1}^n a_i t_i\Big| \le \frac{2}{3}E\Big|\sum_{i=1}^n a_i\varepsilon_i\Big|.$$

The above constants are optimal.

Proof. Since for fixed a the function $b \mapsto E|at_1 + bt_2|$ is symmetric and convex it takes its maximal value on $[-|a|,|a|]$ at $b = |a|$. Hence

$$E|at_1 + bt_2| \le \max(|a|,|b|)E|t_1 + t_2| = \frac{2}{3}\max(|a|,|b|).$$

Let us first prove the second inequality of the proposition. By symmetry we can assume that $a_1 \ge \ldots \ge a_n \ge 0$. There are two possibilities:

Case 1: $a_1^2 \ge \sum_{i=2}^n a_i^2$. Proposition 1 then yields

$$E\Big|\sum_{i=1}^n a_i t_i\Big| \le E\Big|a_1 t_1 + \Big(\sum_{i=2}^n a_i^2\Big)^{1/2} t_2\Big|.$$

Hence since $E|\sum_{i=1}^n a_i\varepsilon_i| \ge a_1$, by (1) the inequality holds.

Case 2: $a_1^2 < \sum_{i=2}^n a_i^2$. From Proposition 1 we deduce that

$$E\Big|\sum_{i=1}^n a_i t_i\Big| \le E\Big|\sqrt{\frac{\sum_{i=1}^n a_i^2}{2}}\,t_1 + \sqrt{\frac{\sum_{i=1}^n a_i^2}{2}}\,t_2\Big| = \frac{\sqrt{2}}{3}\sqrt{\sum_{i=1}^n a_i^2}.$$

This combined with the Khinchin inequality

$$\sqrt{\sum_{i=1}^n a_i^2} \le \sqrt{2}E\Big|\sum_{i=1}^n a_i\varepsilon_i\Big| \qquad \text{(cf. [5])}$$

completes the proof in this case.

Let $\sigma = \sigma(\mathrm{sign}(t_1), \ldots, \mathrm{sign}(t_n))$. Then $E((t_1, \ldots, t_n) \mid \sigma)$ has the same distribution as $\frac{1}{2}(\varepsilon_1, \ldots, \varepsilon_n)$ and the first inequality of the proposition is a simple consequence of the Jensen inequality.

To see that the constants are optimal it suffices to take $n = 1$, $a_1 = 1$ for the first inequality and $n = 2$, $a_1 = a_2 = 1$ for the second.

II. The vector case. In the sequel we will consider the linear space $\mathbb{R}^n$ with a norm $\|\cdot\|$. The Lebesgue measure on $\mathbb{R}^n$ will be denoted by $|\cdot|$. We will consider some analogues in $\mathbb{R}^n$ of unimodal real variables. Our definitions are different from what can be found in the literature (cf. [2]).

DEFINITION 3. Let X be a bounded random vector with values in $\mathbb{R}^n$.

We call X *convex-uniform* (c.u.) if X is uniformly distributed on some open bounded convex symmetric set A_X, i.e. for each measurable set $B \subset \mathbb{R}^n$,

$$\Pr(X \in B) = \frac{|B \cap A_X|}{|A_X|}.$$

We say that X is *semi-convex-uniform* (s.c.u.) if X has a density g and there exist a natural number k, functions $g_1, \ldots, g_k$, and nonnegative numbers $\alpha_1, \ldots, \alpha_k$ with $\sum_{i=1}^k \alpha_i = 1$ such that $g = \sum_{i=1}^k \alpha_i g_i$ and g_i is the density of some c.u. random vector X_i for $i = 1, \ldots, k$.

X is *approximately-convex-uniform* (a.c.u.) if there exist $M > 0$ and a sequence $X_1, X_2, \ldots$ of s.c.u. random vectors bounded in norm by M converging in distribution to X.

LEMMA 6. *Let X and Y be independent convex-uniform random vectors with values in $\mathbb{R}^n$. Then $X + Y$ is a.c.u.*

Proof. Let A_X and A_Y be the convex sets from Definition 3. For $v \in \mathbb{R}^n$ define

$$P_v = \{(x, y) \in A_X \times A_Y : x + y = v\},$$
$$F_v = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x + y = v\}.$$

There exists a constant K such that $X + Y$ has a density g given by

$$g(v) = K\lambda_{2n-1}(P_v),$$

where λ_{2n-1} is the Lebesgue measure on the $(2n-1)$-dimensional subspace F_v. First we show that for each $a > 0$ the set

$$S_a = \{v \in \mathbb{R}^n : g(v) \geq a\}$$

is convex. Indeed, let $v, w \in S_a$ and $\alpha \in (0, 1)$. Since

$$P_{\alpha v + (1-\alpha)w} \supset \alpha P_v + (1 - \alpha) P_w$$

we get by the Brunn–Minkowski inequality (cf. [2])

$$g(\alpha v+(1-\alpha)w)=K\lambda_{2n-1}(P_{\alpha v+(1-\alpha)w})\geq K\lambda_{2n-1}(P_v)^\alpha\lambda_{2n-1}(P_w)^{1-\alpha}$$
$$\geq g(v)^\alpha g(w)^{1-\alpha}\geq a^\alpha a^{1-\alpha}=a.$$

Since the sets A_X and A_Y are bounded, so is the function g and there exists a number M such that $S_a=\emptyset$ for $a>M$. For a natural number j define

$$f_j=\sum_{k=0}^{jM}\Pr(g(X+Y)\in(k/j,(k+1)/j])\varrho_{k/j},$$

where ϱ_a is the density of a random vector uniformly distributed on S_a. Then f_j is the density of some semi-convex-uniform random vector Z_j. It is easy to observe that the sequence Z_j is uniformly bounded and converges in distribution to $X+Y$. And this means that $X+Y$ is a.c.u.

COROLLARY 3. *If $X_1,\ldots,X_k$ is a sequence of independent a.c.u. random variables with values in $\mathbb{R}^n$, then $\sum_{i=1}^k X_i$ is an a.c.u. random vector.*

P r o o f. For $k=2$ the corollary is a simple consequence of Lemma 6, for $k>2$ we proceed by induction.

LEMMA 7. *If $p>q>0$ and X is a convex-uniform random vector with values in $\mathbb{R}^n$, then*

$$\left(\frac{p+n}{n}\right)^{1/p}(E\|X\|^p)^{1/p}\geq\left(\frac{q+n}{n}\right)^{1/q}(E\|X\|^q)^{1/q}.$$

P r o o f. With the notation of Definition 3,

$$E\|X\|^p=\frac{1}{|A_X|}\int\limits_{A_X}\|x\|^p\,dx.$$

Let $\varepsilon>0$. Then

$$(1+\varepsilon)^{n+p}\int\limits_{A_X}\|x\|^p\,dx=\int\limits_{(1+\varepsilon)A_X}\|x\|^p\,dx=\Big(\int\limits_{A_X}+\int\limits_{(1+\varepsilon)A_X-A_X}\Big)\|x\|^p\,dx$$
$$\geq\int\limits_{A_X}\|x\|^p\,dx+|(1+\varepsilon)A_X-A_X|^{1-p/q}\Big(\int\limits_{(1+\varepsilon)A_X-A_X}\|x\|^q\,dx\Big)^{p/q}$$
$$=\int\limits_{A_X}\|x\|^p\,dx+(((1+\varepsilon)^n-1)|A_X|)^{1-p/q}\Big((1+\varepsilon)^{n+q}-1)\int\limits_{A_X}\|x\|^q dx\Big)^{p/q}.$$

Therefore

$$\left(\frac{(1+\varepsilon)^{n+p}-1}{(1+\varepsilon)^n-1}\right)^{1/p}\left(\frac{1}{|A_X|}\int\limits_{A_X}\|x\|^p\,dx\right)^{1/p}$$
$$\geq\left(\frac{(1+\varepsilon)^{n+q}-1}{(1+\varepsilon)^n-1}\right)^{1/q}\left(\frac{1}{|A_X|}\int\limits_{A_X}\|x\|^q\,dx\right)^{1/q}.$$

The inequality of the lemma is obtained by letting $\varepsilon \to 0$.

PROPOSITION 3. *If $p > q > 0$ and $X_1, \ldots, X_k$ are independent a.c.u. random vectors with values in $\mathbb{R}^n$, then for $S = \sum_{i=1}^k X_i$,*

$$\left(\frac{p+n}{n}\right)^{1/p} (E\|S\|^p)^{1/p} \ge \left(\frac{q+n}{n}\right)^{1/q} (E\|S\|^q)^{1/q}.$$

Proof. According to Corollary 3 we can assume that $k = 1$. By an approximation argument it suffices to prove the inequality for S a s.c.u. random vector. But in this case it is a simple consequence of Lemma 7 and the Jensen inequality.

Finally, since $x_i t_i$ is an a.c.u. random vector we obtain the following corollary:

COROLLARY 4. *If $p > q > 0$ and $t_1, \ldots, t_k$ are independent random variables uniformly distributed on $[-1, 1]$ and $x_1, \ldots, x_k$ are vectors in $\mathbb{R}^n$, then for $S = \sum_{i=1}^k t_i x_i$,*

$$\left(\frac{p+n}{n}\right)^{1/p} (E\|S\|^p)^{1/p} \ge \left(\frac{q+n}{n}\right)^{1/q} (E\|S\|^q)^{1/q}.$$

Remark. The above results are also valid for $p > q > -n$ and the proofs are very similar.

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