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A NOTE ON TOLERANCE STABLE DYNAMICAL SYSTEMS

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We study the tolerance stability of homeomorphisms of a compact metric space M. The notion of tolerance stability was considered by F. Takens [3], who has proved several results in the direction of Zeeman's tolerance stability conjecture. We first recall this conjecture as it plays an important role here.

Let M be a compact metric space with the distance d. $H(M)$ is the set of homeomorphisms of M with the C^0-topology. $\mathcal{O} \subseteq H(M)$ is some subset of $H(M)$, endowed with a topology finer than that induced from $H(M)$ (we denote the metric on $\mathcal{O}$ by ϱ). We define the *φ-orbit* of m, $O_\varphi(m)$, to be the set $\{\varphi^i(m) : i \in \mathbb{Z}\}$. Denote the closure of $O_\varphi(m)$ by $C_\varphi(m)$. The *ε-neighbourhood* of A, $U_\varepsilon(A)$, is the set $\{x : d(x, A) < \varepsilon\}$.

DEFINITION 1. The homeomorphism $\varphi \in \mathcal{O}$ is *$\mathcal{O}$-tolerance stable* if for each $\varepsilon > 0$, there is an open neighbourhood $U \subseteq \mathcal{O}$ of φ such that for each $\psi \in U$ and for each $m \in M$:

(a) there is an $m' \in M$ such that $O_\psi(m') \subseteq U_\varepsilon(O_\varphi(m))$,
(b) there is an $m'' \in M$ such that $O_\varphi(m'') \subseteq U_\varepsilon(O_\psi(m))$.

Remark 1. Since $C_\varphi(m)$ is closed, it may be viewed as an element of the metric space $\mathfrak{C}(M)$ of all closed subsets of M with the Hausdorff metric d_{H}.

$\mathfrak{C}(M)$ is again a compact metric space. In general, we will identify points of $\mathfrak{C}(M)$ and closed subsets of M. The metric space of all closed subsets of $\mathfrak{C}(M)$ is denoted by $\mathfrak{C}(\mathfrak{C}(M))$ and the Hausdorff metric on $\mathfrak{C}(\mathfrak{C}(M))$ by d_{HH}. The closure of the set $\{C_\varphi(m) : m \in M\}$ in $\mathfrak{C}(M)$ is denoted by C_φ. Being a closed subset of $\mathfrak{C}(M)$, C_φ may be viewed as an element of $\mathfrak{C}(\mathfrak{C}(M))$. Let $C : \mathcal{O} \to \mathfrak{C}(\mathfrak{C}(M))$ map φ to C_φ .

Observe that $\varphi \in \mathcal{O}$ is $\mathcal{O}$-tolerance stable if $C : \mathcal{O} \to \mathfrak{C}(\mathfrak{C}(M))$ is continuous at φ.

EXAMPLE. Let $\mathcal{O}$ be the space of C^1 diffeomorphisms of a compact differentiable manifold M with C^1-topology. Since C^1-close diffeomorphisms

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satisfying Axiom A and the strong transversality condition are conjugate by a homeomorphism close to the identity [1], it follows that such diffeomorphisms are $\mathcal{O}$-tolerance stable. But for less regular spaces $\mathcal{O}$ the question of tolerance stability is more complicated.

ZEEMAN'S TOLERANCE STABILITY CONJECTURE. Let M be a metric space and $\mathcal{O}$ some subset of $H(M)$ as above. There is a residual subset $\mathcal{R} \subseteq \mathcal{O}$ such that each $\varphi \in \mathcal{R}$ is $\mathcal{O}$-tolerance stable.

W. White [4] found a pathological counterexample to this conjecture but there arises a problem of characterizing the spaces $\mathcal{O} \subseteq H(M)$ for which the subset of $\mathcal{O}$-tolerance stable homeomorphisms is residual in $\mathcal{O}$.

F. Takens [3] modified Zeeman's conjecture slightly (in a way suggested by possible applications, when we consider a physical system and we take into account the possibility of very small unknown exterior perturbations) and introduced the notion of extended orbits.

DEFINITION 2. A sequence $\{m_i : m_i \in M, i \in \mathbb{Z}\}$ is called an *ε-pseudoorbit* of φ if $d(\varphi(m_i), m_{i+1}) \le \varepsilon$ for every $i \in \mathbb{Z}$. We denote by A_ε each subset of M which is the closure of an ε-pseudoorbit of φ.

DEFINITION 3. We say that a closed subset $A \subseteq M$ is an *extended φ-orbit* if for each $\varepsilon > 0$ and $\delta > 0$, there is a set A_ε for φ such that $d_{\mathrm{H}}(A, A_\varepsilon) < \delta$.

For each $\varphi \in \mathcal{O}$ we denote by E_φ the closure of the set of all extended φ-orbits; then $E_\varphi \in \mathfrak{C}(\mathfrak{C}(M))$. Let $E : \mathcal{O} \to \mathfrak{C}(\mathfrak{C}(M))$ map φ to E_φ.

DEFINITION 4. The homeomorphism φ is said to be *extended $\mathcal{O}$-stable* if the map E is continuous at the point φ.

F. Takens [3] proved the following theorem:

THEOREM 1. *Let M and $\mathcal{O} \subseteq H(M)$ be as above, and suppose there is a residual subset $R \subseteq \mathcal{O}$ such that $C_\varphi = E_\varphi$ for each $\varphi \in R$. Then there exists a residual subset $R' \subseteq \mathcal{O}$ of $\mathcal{O}$-tolerance stable homeomorphisms on M.*

We shall prove some relations between the tolerance stability and the property $E_\varphi = C_\varphi$. We expect that tolerance stability is a necessary condition for the equality $E_\varphi = C_\varphi$.

For $\varphi \in H(M)$ let $\Omega(\varphi)$ denote the set of nonwandering points of φ [1].

THEOREM 2. *Let M be a compact manifold and φ a homeomorphism of M. If φ is $H(M)$-tolerance stable, then for each $A \in E_\varphi$ there is an $O \in C_\varphi$ such that $A \subseteq O$.*

DEFINITION 5. We say that φ *has no C^0 Ω-explosion* if for each $\varepsilon > 0$ there is a neighbourhood $U(\varphi)$ of φ in $\mathcal{O}$ such that $\Omega(\psi) \subseteq U_\varepsilon(\Omega(\varphi))$ for any $\psi \in U(\varphi)$.

THEOREM 3. *If $\varphi \in \mathcal{O}$ is $\mathcal{O}$-tolerance stable, then φ has no C^0 Ω-explosion.*

DEFINITION 6. We define a relation $\vdash$ on $M \times M$, induced by φ, as follows: $x \vdash y$ if and only if for each $\varepsilon > 0$ there exists an ε-pseudoorbit $\{x_j\}_{j=0}^n$ with $x_0 = x$, $x_n = y$ and $n \geq 1$. The set $N(\varphi) = \{x \in M : x \vdash x\}$ is called the *chain recurrent set.*

K. Sawada [2] showed that if φ has no C^0 Ω-explosion then $N(\varphi) = \Omega(\varphi)$. Hence

COROLLARY. *If φ is $\mathcal{O}$-tolerance stable, then $N(\varphi) = \Omega(\varphi)$.*

Finally, we shall prove

THEOREM 4. *Let (M, d) be a compact metric space. If $\varphi \in H(M)$ is minimal (i.e. $C_\varphi(x) = M$ for each $x \in M$) then it is $H(M)$-tolerance stable.*

Proof of Theorem 2. Take $A \in E_\varphi$. There is a sequence $\{A'_{\delta_n}\}$ of δ_n-pseudoorbits of φ with $\delta_n \to 0$ such that the sequence of their closures converges to A in $\mathfrak{C}(M)$. We write $A'_{\delta_n} = \{x_n^k : k \in \mathbb{Z}\}$. Fix $x_0 \in A$. We may assume that $x_n^0 \to x_0$ as $n \to \infty$. For each A'_{δ_n} we find a homeomorphism ψ_n mapping x_n^i to x_n^{i-1} for each $i \in \mathbb{Z}$ satisfying $|i| < [1/\delta_n]$. Since φ is $H(M)$-tolerance stable and $\varrho(\varphi, \psi_n) < \delta_n$, for the sequence of orbits $O_{\psi_n}(x_n^0)$ there is a sequence of points $\{x_{0,n}\}$ for which $d_{\mathrm{H}}(C_{\psi_n}(x_n^0), C_\varphi(x_{0,n})) < \varepsilon(\delta_n)$ for a sequence $\{\varepsilon(\delta_n)\}_{n=1}^\infty$ converging to 0.

Let O be the limit of the closures of $O_\varphi(x_{0,n})$. It is easy to see that $A \subseteq O$.

Proof of Theorem 3. Since φ is $\mathcal{O}$-tolerance stable, for each $\varepsilon > 0$ there is a $\delta > 0$ such that if $\varrho(\psi, \varphi) < \delta$, then $d_{\mathrm{H}}(C_\psi, C_\varphi) < \varepsilon/3$. Let ψ be a homeomorphism such that $\varrho(\psi, \varphi) < \delta$ and let $x' \in \Omega(\psi)$.

Set $B_n(x', 1/n) = \{y : d(y, x') < 1/n\}$. Since $x' \in \Omega(\psi)$, there are x_1^n, $x_2^n \in B_n(x', 1/n)$ such that $\psi^{k_n}(x_1^n) = x_2^n$ for some $k_n \in \mathbb{N}$. Hence there are $x_1'^n$, $x_2'^n \in B_n(x', 1/n + \varepsilon/3)$ such that $\varphi^{l_n}(x_1'^n) = x_2'^n$ for some $l_n \in \mathbb{Z}$ and $d(x_1^n, x_1'^n) < \varepsilon/3$, $d(x_2'^n, x_2^n) < \varepsilon/3$. It follows that there are x_1, $x_2 \in B(x', \varepsilon)$ and $l \in \mathbb{N}$ such that $\varphi^l(x_1) = x_2$.

If φ has the C^0 Ω-explosion property, then there is an $\varepsilon > 0$ and a sequence $\{\psi_n\}$ converging to φ in $\mathcal{O}$ such that $\Omega(\psi_n) \subsetneq U_\varepsilon(\Omega(\varphi))$ for all n. Hence there is a sequence $\{a_n : n \in \mathbb{N}\}$ with $a_n \in \Omega(\psi_n)$ and $d_{\mathrm{H}}(a_n, \Omega(\varphi)) > \varepsilon$. Since $\varrho(\psi_n, \varphi) < 1/n$ we see that $d_{\mathrm{HH}}(C_{\psi_n}, C_\varphi) \to 0$ as $n \to \infty$. So the sequence $\{\varepsilon_n = 2d_{\mathrm{HH}}(C_\varphi, C_{\psi_n}) : n \in \mathbb{N}\}$ converges to 0 as $n \to \infty$.

There are a_1^n, a_2^n in $B(a_n, \varepsilon_n)$ and $s_n \in \mathbb{N}$ such that $\varphi^{s_n}(a_1^n) = a_2^n$. Since M is compact, we also have a point a which is the limit of some subsequences of $\{a_1^n\}$, $\{a_2^n\}$, $\{a_n\}$. Hence $a \in \Omega(\varphi)$ and $d_{\mathrm{H}}(a, \Omega(\varphi)) \geq \varepsilon$, so we have a contradiction. This completes the proof of Theorem 3.

Proof of Theorem 4. Fix $\varepsilon > 0$. We need to show that there is a $\delta > 0$ such that if $\varrho(\varphi, \eta) < \delta$, then $d_{\mathrm{HH}}(C_\varphi, C_\eta) < \varepsilon$. Suppose that this is not true. First suppose that there is a sequence $\{\psi_n : n \in \mathbb{N}\}$ such that $\varrho(\varphi, \psi_n) < 1/n$ and $C_{\psi_n} \subsetneq U_\varepsilon(C_\varphi)$. This means that there is a sequence of points $\{x_n\}$ such that $C_{\psi_n}(x_n) \subsetneq U_\varepsilon(O_\varphi(x))$ for any $x \in M$. But this is impossible, because the closure of the orbits $\{\varphi^k(x_n) : k \in \mathbb{Z}\}$ equals M. So we have a contradiction.

Now suppose that there is a sequence $\{\psi_n : n \in \mathbb{N}\}$ such that $\varrho(\varphi, \psi_n) < 1/n$ for each $n \in \mathbb{N}$, and $C_\varphi \subsetneq U_\varepsilon(C_{\psi_n})$. This means that there is a sequence of points $\{y_n\}$ such that $C_\varphi(y_n) \subsetneq U_\varepsilon(O_{\psi_n}(y))$ for any $y \in M$. Fix $y_0 \in M$. Let A be a limit point of the sequence $\{C_{\psi_n}(y_0) : n \in \mathbb{N}\}$; then $A \in E_\varphi$. It is clear that there is no orbit of φ in the ε-neighbourhood of A. Because φ is minimal and each extended orbit of φ is invariant it follows that $A = M$. We arrive at a contradiction.

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