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NONBASIC HARMONIC MAPS ONTO CONVEX WEDGES

BY

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We construct a nonbasic harmonic mapping of the unit disk onto a convex wedge. This mapping satisfies the partial differential equation $\overline{f_{\bar{z}}} = af_z$ where $a(z)$ is a nontrivial extreme point of the unit ball of H^∞.

1. Introduction. There are several papers in the literature dealing with harmonic mappings of the unit disk onto plane domains. For univalent harmonic mappings there are results on coefficient estimates, boundary behavior, normality of certain families and integral representations of some subfamilies of these mappings [3], [4], [7].

If D is the unit disk in $\mathbb{C}$, we let $H(D,\Omega)$ be the set of one-to-one harmonic maps $f = u + iv$, mapping D onto a simply connected domain Ω. These mappings satisfy a partial differential equation

$$\overline{f_{\bar{z}}} = af_z \tag{1.1}$$

where $a(z)$ is a function in the closed unit ball of H^∞. The following questions arise in this situation. First, assume the simply connected domain Ω is given and a function $a(z)$ in the closed unit ball of H^∞ is given. Fix $w_0 \in \Omega$ and ask whether there exists f in $H(D,\Omega)$, normalized by $f(0) = w_0$ and $f_z(0) > 0$, which satisfies (1.1). A second question in this regard is the following. Assume again that Ω is a simply connected domain and $w_0 \in \Omega$ is given. For which $a(z)$ in the closed unit ball of H^∞ can we find f in $H(D,\Omega)$ with $f(0) = w_0$, $f_z(0) > 0$, and satisfying (1.1)? The Riemann mapping theorem says that for $a(z) \equiv 0$, the first question has a unique answer and we call such analytic harmonic maps basic.

For these two questions, the best answers to date are those given in the papers by Hengartner and Schober [6] and Abu-Muhanna and Schober [1]. As an example to prove that question one does not always have a positive answer we refer to [6] where $\Omega = D$ and $a(z) = z$ and there is no function

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from $H(D, D)$ satisfying (1.1). The authors in these papers refer to this phenomenon as a collapsing effect.

Little is known about the set of functions $a(z)$ for which question two has an affirmative answer. Since nontrivial inner functions are the extreme points of the unit ball H^∞, it would be useful to know for which inner functions question two has an affirmative answer. There are several explicit examples of harmonic functions in the literature [1]. One can calculate the $a(z)$ for the simple examples and these are not inner functions. In this paper we show by construction that there are nontrivial inner functions $a(z)$ for which question one has an affirmative answer with Ω being a convex wedge of opening less than π.

Again, with Ω a convex wedge, we have examples of some functions $a(z)$ in the unit ball of H^∞ for which question two has an affirmative answer [1]. None of these are inner functions. Our construction provides another positive answer to this question, with $a(z)$ being an extreme point of the unit ball in H^∞.

We thank Peter Duren for pointing out the Kneser proof [9] of the Choquet Theorem [2] and indicating that it should hold in more general contexts. We also include another example of harmonic maps of D onto a convex wedge, shown to us by Don Marshall.

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2. Definitions and notation. Let $f = u + iv$ be harmonic in D and let $F = u + i\widetilde{u}$ and $G = v + i\widetilde{v}$ be analytic completions of u and v. Then $f = h + \overline{g}$ where $h = (F + iG)/2$ and $g = (F - iG)/2$ are analytic in D. It is known [4] that f is locally one-to-one and sense preserving in D if and only if the function

$$S(z) = \frac{g'(z)}{h'(z)} = \frac{F'(z) - iG'(z)}{F'(z) + iG'(z)}$$

is analytic in D and satisfies $|S(z)| < 1$ for z in D. Functions f that are harmonic, one-to-one and sense preserving in D will be called *harmonic maps* for brevity. A harmonic map $f = u + iv$ will be called *basic* if $v = a\widetilde{u} + b$ where a and b are real numbers.

We are concerned with constructing a nonbasic map of D onto a cone of aperture opening less than π. Since harmonic maps remain harmonic under affine change of variables $w \to aw + b\overline{w} + c$, with $|b| < |a|$, we may assume without loss of generality that K is the cone

$$K = \{(x, y) : x > 0,\ -x < y < 0\}.$$

The spaces H^∞ and H^1 are the usual Hardy spaces on the disk and the space of Cauchy transforms of finite Borel measures on $\partial D = \Pi$ is written as $\mathcal{K}$. A function of S in H^∞ is called an *inner function* if $|S(e^{i\theta})| = 1$ a.e. on Π. In the case that we will be considering, since $u > 0$ and $v < 0$ in D, we know that the completions F and G are in $\mathcal{K}$. Also, there is an at most countable set $\{\zeta_j\} \subset \Pi$ such that F is continuous on $\overline{D} \setminus \{\zeta_j\}$. This latter statement follows by applying a technique used by the authors in an earlier paper [3].

3. The mapping construction. Let $\pi/2 > \beta_1 > \beta_2 > \ldots > 0$ decrease to 0 in such a way that

$$\sum_{j=1}^{\infty} j \operatorname{Log} j (\beta_j - \beta_{j+1})$$

converges and if $\varepsilon_j = \beta_{j-1} - \beta_j$ there exists a constant $c > 0$ such that $\varepsilon_j / \beta_j \geq c$ for all j. (For example $\beta_j = 1/2^j$.) Define F for $|z| < 1$ by

$$F(z) = \frac{i}{\pi} \sum_{j=1}^{\infty} \operatorname{Log} \left(\overline{\zeta}_j \frac{z - \zeta_j}{z - \overline{\zeta}_j} \right) \tag{3.1}$$

where $\overline{\zeta}_j = e^{i\beta_j}$ and the principal branch is taken for Log.

We first prove that the series converges uniformly on compact subsets of D. Let

$$F_n(z) = \frac{i}{\pi} \sum_{j=1}^{n+1} \operatorname{Log} \left(\overline{\zeta}_j \frac{z - \zeta_j}{z - \overline{\zeta}_j} \right).$$

Then

$$F_n'(z) = -\frac{2i}{\pi} \sum_{j=1}^{n+1} \frac{\sin \beta_j}{(z - \zeta_j)(z - \overline{\zeta}_j)}.$$

If $|z| \leq r < 1$ and $m > n$,

$$\begin{aligned} |F_m'(z) - F_n'(z)| &\leq \frac{2}{\pi} \sum_{j=n+2}^{m+1} \frac{\sin \beta_j}{|(z - \zeta_j)(z - \overline{\zeta}_j)|} \\ &\leq \frac{2}{\pi(1-r)^2} \sum_{j=n+1}^{m+1} \beta_j . \end{aligned}$$

Since $\sum_{j=1}^{\infty} \beta_j$ converges, it follows that the sequence $F_n'(z)$ converges uniformly on compact subsets of D. Thus the series in (3.1) also converges uniformly on compact subsets of D.

It is easily seen that $\operatorname{Re}(1 - z^2) F'(z) > 0$ for z in D. Using a result of Hengartner and Schober [6, Theorem 1] it follows that $F(z)$ is univalent on D and maps D onto a domain which is convex in the direction of the imaginary axis. An examination of the boundary values gives that F maps the arcs

$$I_j = (e^{i\beta_j}, e^{i\beta_{j+1}}) \subset \Pi, \qquad j = 1, 2, \ldots,$$

onto line segments of the form $\{z = j + iy : y \geq n(j) > 0\}$. The reflected arc $\bar{I}_j$ is mapped by F onto $\{z = j + iy : y \leq -n(j)\}$.

Let $u(z) = \operatorname{Re} F(z)$. Since $u(z) \geq 0$ and

$$\int_0^{2\pi} u(e^{it}) \log^+ u(e^{it})\, dt = 2 \sum_{j=1}^{\infty} j(\operatorname{Log} j)(\beta_j - \beta_{j+1}) < \infty,$$

we may apply a theorem of Zygmund [10, p. 135] once we have proven the following lemma.

Lemma. *The function $u(z)$ is the Poisson integral of $u(e^{it})$.*

Proof. By the Herglotz theorem, since $u(z) = \operatorname{Re} F(z) > 0$, we know there is a positive measure μ such that

$$u(z) = Pz * d\mu$$

where Pz is the Poisson kernel. Because of the form of u (u is locally bounded near $e^{i\theta} \neq 1$) we see that $d\mu(\theta) = u(e^{i\theta})\frac{d\theta}{2\pi} + c d\delta$, where $c \geq 0$ and $d\delta$ is the point mass at $\zeta = 1$. We prove $c = 0$. It is sufficient to produce a sequence x_J in (0, 1) tending to 1 such that

$$\lim_{J\to\infty} (1 - x_J)u(x_J) = 0.$$

A computation shows

$$\arg\left(\bar{\zeta}_j \frac{x - \zeta_j}{x - \bar{\zeta}_j}\right) = \arg \omega(\beta_j, x)$$

where

$$\begin{aligned}\omega(\beta_j, x) &= ((1 + x^2)\cos\beta_j - 2x) + i(1 - x^2)\sin\beta_j \\ &\equiv (b\cos\beta_j - 2x) + i(a\sin\beta_j).\end{aligned}$$

Having fixed an x value it is notationally more convenient to write $\omega(\beta_j, x)$ as $\omega(\beta_j)$ and we shall do so. Fix a positive integer J and choose x_J to be the solution to the equation

$$\cos\beta_J^* = \frac{2x_J}{1 + x_J^2}$$

where $\beta_j^* = \frac{1}{2}(\beta_J + \beta_{J-1})$; then $x_J \to 1$ as $J \to \infty$. Then

$$\left|\sum_{j=1}^{J-1} \arg\omega(\beta_j)\right| \leq \pi(J - 1)$$

and

$$\lim_{J\to\infty}(1-x_J)(J-1) = \lim_{J\to\infty}(J-1)\beta_{J-1}\left(\frac{1-x_J}{\beta_{J-1}}\right)$$
$$\leq \lim_{J\to\infty}\frac{1-x_J}{\beta_J^*}\cdot\lim_{J\to\infty}(J-1)\beta_{J-1}\,.$$

Since

$$\lim_{x\to 1^-}\frac{1-x}{\arccos\left(\frac{2x}{x^2+1}\right)} = 1\,,$$

it follows by the definition of x_J that

$$\lim_{J\to 1}\frac{1-x_J}{\beta_j^*} = 1\,.$$

The convergence of $\sum_{j=1}^{\infty} j(\beta_j-\beta_{j+1})$ implies that the term $(J-1)\beta_{J-1}$ goes to zero. Hence,

$$\lim_{J\to\infty}(1-x_J)\sum_{j=1}^{J-1}\arg\omega(\beta_j) = 0\,.$$

For $j \geq J$ (i.e., $\beta_j \leq \beta_J$) we have

$$\arg\omega(\beta_j) = \arctan\left(\frac{a\sin\beta_J}{b\cos\beta_j - 2x}\right).$$

Solving the equation

$$\cos\beta_J^* = \frac{2x_J}{1+x_J^2}$$

yields

$$x_J = \frac{1-\sin\beta_J^*}{\cos\beta_J^*}$$

and it follows that

$$1-x_J^2 = \frac{2(1-\sin\beta_J^*)\sin\beta_J^*}{\cos^2\beta_J^*}\,,$$
$$1+x_J^2 = \frac{2(1-\sin\beta_J^*)}{\cos^2\beta_J^*}\,.$$

Also, a calculation yields

$$(1+x_J^2)\cos\beta_J - 2x_J = \frac{2(1-\sin\beta_J^*)}{\cos^2\beta_J}[\cos\beta_J - \cos\beta_J^*]\,.$$

Further, it is easy to compute

$$\frac{\partial\omega}{\partial\beta} = \frac{a(b-2x\cos\beta)}{(b\cos\beta-2x)^2}\,,$$

which is positive if $j \geq J$. We claim there is an M independent of J for which

$$\omega(\beta_j) \leq \omega(\beta_J) \leq M\,.$$

This follows since

$$\omega(\beta_J) = \frac{(\sin\beta_J)(\sin\beta_J^*)}{\cos\beta_J - \cos\beta_J^*} \leq M\frac{\sin\beta_J}{\beta_J^* + \beta_J}\cdot\frac{\sin\beta_J^*}{\beta_J^* - \beta_J}$$

where M is independent of J. Our assumptions guarantee that

$$\frac{\sin\beta_J^*}{\beta_J^* - \beta_J} \simeq \frac{\sin\beta_J^*}{\beta_J^*}$$

is uniformly bounded independent of J and similarly

$$\frac{\sin\beta_J}{\beta_J^* + \beta_J}$$

is bounded independent of J.

Hence, there is a constant C independent of J (and independent of x_J) with

$$\arctan\omega(\beta_j) \leq C\omega(\beta_j)$$

for all $j \geq J$. Consider

$$\sum_{j=J}^{\infty}\arctan\omega(\beta_j) \leq C\sum_{j=J}^{\infty}\omega(\beta_j) \leq \frac{a}{b\cos\beta_J - 2x_J}\sum_{j=J}^{\infty}\beta_j\,.$$

Denote the series term in this last equation by $\delta(J)$ and note that $\delta(J) \to 0$ as $J \to \infty$. Using our hypothesis (and the estimates above) we conclude

$$\begin{aligned}(1 - x_J)\sum_{j=J}^{\infty}\arctan\omega(\beta_j) &\leq \frac{C(1 - x_J^2)^2}{b\cos\beta_J - 2x_J}\delta(J)\\ &\leq \frac{C(\sin\beta_J^*)^2}{(\beta_J^* - \beta_J)(\beta_J^* + \beta_J)}\delta(J)\,.\end{aligned}$$

Since the term

$$\frac{(\sin\beta_J^*)^2}{(\beta_J^* - \beta_J)(\beta_J^* + \beta_J)}$$

is bounded, we conclude

$$\lim_{J\to\infty}(1 - x_J)\sum_{j=J}^{\infty}\arctan\omega(\beta_j) = 0\,.$$

This establishes our claim.

We may now apply a theorem of Zygmund [10, p. 135] to conclude that the conjugate $\widetilde{u}$ of u is in the harmonic h^1 space. Hence $F(z) = u(z) + i\widetilde{u}(z)$ is the sum of two h^1 functions and thus F is in H^1.

Next we consider the function G defined by

$$G(z) = \frac{-i}{\pi} \sum_{j=1}^{\infty} \operatorname{Log}\left(\overline{\eta}_j \frac{z-\zeta_j}{z-1}\right) \tag{3.2}$$

where $|\eta_j| = 1$ and $\arg \eta_j = \arg(1+\zeta_j) = (\arg \zeta_j)/2$. The series in (3.2) converges uniformly on compact subsets of D by the same argument as applied to (3.1). For future reference we note the following. Let

$$G_n(z) = \frac{-i}{\pi} \sum_{j=1}^{n+1} \operatorname{Log}\left(\overline{\eta}_j \frac{z-\zeta_j}{z-1}\right).$$

We have

$$\begin{aligned} F_n(z) &= \frac{i}{\pi} \sum_{j=1}^{n+1} \operatorname{Log}\left[\frac{\overline{\zeta_j \eta_j}}{\eta_j} \cdot \frac{z-\zeta_j}{z-1} \cdot \frac{z-1}{z-\overline{\zeta}_j}\right] \\ &= -G_n(z) + \frac{i}{\pi} \sum_{j=1}^{n+1} \operatorname{Log}\left[\overline{\left(\frac{\zeta_j}{\eta_j}\right)} \cdot \frac{z-1}{z-\overline{\zeta}_j}\right]. \end{aligned}$$

Since $\zeta_j/\eta_j^2 = 1$ and since

$$i \operatorname{Log}\left(\overline{\eta}_j \frac{z-1}{z-\overline{\zeta}_j}\right) = \overline{-i \operatorname{Log}\left(\eta_j \frac{\overline{z}-1}{\overline{z}-\zeta_j}\right)} = -\left[\overline{-i \operatorname{Log}\left(\overline{\eta}_j \frac{\overline{z}-\zeta_j}{\overline{z}-1}\right)}\right]$$

we have

$$F_n(z) = -(G_n(z) + \overline{G_n(\overline{z})}). \tag{3.3}$$

Thus

$$F(z) = -(G(z) + \overline{G(\overline{z})}). \tag{3.4}$$

We note at this stage that we may prove that G is in H^1 in exactly the same way that we proved F is in H^1.

For future reference, we note that $\operatorname{Im}(1-z)^2 G'(z) < 0$ for z in D. Thus by a result of Hengartner and Schober [5], G is one-to-one and maps D onto a domain which is convex in the direction of the imaginary axis. An examination of the boundary values shows that G maps D onto the left half plane with line segments of the form $\{z = -j - iy : y \geq m(j)\}$, $j = 1, 2, \ldots$, removed.

We now define

$$f = u + iv = \operatorname{Re} F + i \operatorname{Re} G.$$

We will prove in the next two sections that this is the required mapping function.

4. Boundary correspondence. The L^1 boundary values of u and v are related as follows:

$$-v(e^{i\theta}) = \chi_Q(e^{i\theta})u(e^{i\theta})$$

where

$$\chi_Q(e^{i\theta}) = \begin{cases} 1 & \text{if } e^{i\theta} \in \bigcup_{j=1}^{\infty} I_j, \\ 0 & \text{otherwise.} \end{cases}$$

This implies $0 < -v(z) \le u(z)$ for $z \in D$. If $e^{i\theta}$ is in I_j, then $-v(e^{i\theta}) = u(e^{i\theta}) = j$. If $e^{i\theta}$ is not in $(\bigcup_{j=1}^{\infty} I_j) \cup (\bigcup_{j=1}^{\infty} \bar{I}_j)$, then $u(e^{i\theta}) = v(e^{i\theta}) = 0$ and $u(e^{i\theta}) = j$ and $v(e^{i\theta}) = 0$, if $e^{i\theta}$ is in $\bar{I}_j$. To see what happens at the end points of I_{j_0} and $\bar{I}_{j_0}$ we first note that we can write

$$u(z) = \sum_{j<j_0} \omega_j(z) + \omega_{j_0}(z) + \sum_{j_0<j} \omega_j(z) \tag{4.1}$$

where ω_j is the harmonic measure of the arc K_j on Π, containing $z = 1$ and joining $\bar{\zeta}_j$ to ζ_j. If z tends to ζ_{j_0}, the first term on the right side of (4.1) tends to $j_0 - 1$ and the third term tends to zero. If the sequence z_n tends to ζ_{j_0} along a chord making an angle γ with the tangent line to Π at ζ_{j_0}, then $w_{j_0}(z_n)$ tends to γ/π. A similar statement holds for $v(z)$. If we let $C(f, \zeta_j)$ be the cluster set of f at ζ_j, then we have

$$C(f, \zeta_{j_0}) = \{w = u - iu : j_0 - 1 \le u \le j_0\}$$

for $j_0 = 1, 2, \ldots$

Similarly we obtain

$$C(f, \bar{\zeta}_{j_0}) = \{w = u : j_0 - 1 \le u \le j_0\}\,.$$

5. Univalence. In this section we prove that f is one-to-one in D. Our proof is based on a technique used by Kneser [9] to give a proof of Choquet's Theorem [2]. Let $f = h + \overline{g}$ where $h = (F + iG)/2$ and $g = (F - iG)/2$. As pointed out in the last paragraph of Section 3, $h - g = iG$ is convex in the direction of the real axis. According to Clunie and Sheil-Small [4, Theorem 5.3], f will be one-to-one provided it is locally one-to-one and sense preserving. Thus by remarks made in Section 2, to prove that f is one-to-one it is sufficient to prove that $|g'(z)| < |h'(z)|$ for z in D. This in turn is equivalent to proving that the Jacobian of f, $J(f)$, is positive in D. Since $F'(0) > 0$ and $\mathrm{Re}(iG'(0)) > 0$ it follows that $J(f)(0) = |h'(0)|^2 - |g'(0)|^2 = |F'(0) + iG'(0)|^2 - |F'(0) - iG'(0)|^2 > 0$. Thus it will be sufficient to prove that $J(f) \neq 0$ in D.

With each pair of real numbers (r, s) we associate the linear function

$$L_{r,s}(u, v) = L(u, v) = ru + sv\,.$$

The composition

$$q_{r,s}(z) = q(z) = L_{r,s} \circ f(z) = ru(z) + sv(z)$$

is harmonic in D. To complete the proof, we will prove that for each such L and each real constant c, the level set

$$\Gamma(c, L) = \{z : L \circ f(z) = q(z) = c\}$$

is a Jordan curve in D. This implies that there is no point P in D such that $\frac{\partial q}{\partial x}(P) = 0$ and $\frac{\partial q}{\partial y}(P) = 0$. This in turn implies that $J(f) \neq 0$ in D, which is sufficient to prove that f is one-to-one.

There are several cases to consider. In each case, if branching occurs in the level set $\Gamma(c, L)$ we will conclude that $q(z)$ is identically a constant, which is an obvious contradiction.

Case 1. Assume the function L is given and the straight line $L = c$ meets ∂K at two points, say $(l, 0)$ and $(p, -p)$ where l and p are not integers and $j_0 < l < j_0 + 1$, $k_0 - 1 < p < k_0$, where j_0 and k_0 are positive integers. By the results of Section 4 the level set $\Gamma(c, L)$ can accumulate on Π at the two points ζ_{k_0} and $\overline{\zeta}_{j_0}$. If $\Gamma(c, L)$ branches at an internal point $z_0 \in \Gamma(c, L)$, then there are at least three components, say γ_1, γ_2 and γ_3, in a small neighborhood of z_0. Further, at least two of them accumulate at one of the points ζ_{k_0} or $\overline{\zeta}_{j_0}$. Assume without loss of generality that γ_1 and γ_2 end at ζ_{k_0}. The curves γ_1 and γ_2 cannot meet in D and they bound a simply connected domain $\mathcal{R}$. But q identically c on $\partial\mathcal{R}$ and q harmonic in $\mathcal{R}$ implies that q is identically c in $\mathcal{R}$ and hence in D, which is a contradiction. Thus in this case $\Gamma(c, L)$ is a Jordan curve with end points ζ_{k_0} and $\overline{\zeta}_{j_0}$ on Π.

Case 2. Assume the line $L = c$ meets ∂K at points $(k, -k)$ and $(l, 0)$ where k is a positive integer and $j_0 < l < j_0 + 1$ where j_0 is a positive integer. In this case $\Gamma(c, L)$ accumulates at $\overline{\zeta}_{j_0}$ and on the closure of the arc I_k. If $\Gamma(c, L)$ were to branch at $z_0 \in D \cap \Gamma(c, L)$, there would again be at least 3 arcs emanating from z_0, say γ_1, γ_2 and γ_3. Let us assume that γ_1 and γ_2 accumulate on the closure of I_k. Again γ_1 and γ_2 do not intersect and together with a suitable subarc of $\overline{I}_k$ bound a simply connected domain $\mathcal{R}$. Let ϕ be the Riemann map of D onto $\mathcal{R}$. The bounded harmonic function $q \circ \phi$ is constant a.e. on Π and so it is constant on D. Again this leads to an absurdity. The case of two of the three arcs γ_1, γ_2 or γ_3 ending at $\overline{\zeta}_{j_0}$ is already covered in Case 1.

Case 3. Assume the line $L = c$ meets ∂K at one finite point and at ∞. If $\Gamma(c, L)$ branches at a point $z_0 \in D \cap \Gamma(c, L)$, then again there are at least three arcs γ_1, γ_2 and γ_3 emanating from z_0. The only case we need to consider is where two of them, say γ_1 and γ_2, join to 1. Again they cannot meet and they form a simply connected domain $\mathcal{R}$. Map D onto $\mathcal{R}$, by an analytic function ϕ. Since $q(z) = ru(z) + sv(z)$, the conjugate $\widetilde{q}$ of q is given by $\widetilde{q}(z) = r\widetilde{u}(z) + s\widetilde{v}(z)$. Thus, the analytic function $M(z) = q(z) + i\widetilde{q}(z) = rF(z) + sG(z)$ is in H^1, and therefore there exists $m(z)$ harmonic in D such that $|M(z)| \le m(z)$ in D. Thus $|M(\phi(w))| \le m(\phi(w))$ and so $M \circ \phi$ is also in H^1. Since $\operatorname{Re}(M \circ \phi) = q \circ \phi$ and the Poisson formula reproduces this harmonic function from its boundary values, we must have $q \circ \phi$ identically c in $\mathcal{R}$. Again we reach an absurdity.

The remaining cases are similar. This completes the proof of univalence.

Combining the results of Sections 4 and 5 we have the following theorem.

THEOREM 1. *The constructed mapping function f is a nonbasic harmonic mapping of D onto K.*

6. The coefficient S. If $f = h + \overline{g}$ is a harmonic mapping from D into $\mathbb{C}$, it can be viewed as a solution of the elliptic partial differential equation

$$\overline{f_{\overline{z}}} = S(z) f_z$$

where the function $S(z) = g'(z)/h'(z)$ is analytic in D and satisfies $|S(z)| < 1$ for z in D. Recall that with our notation,

$$S(z) = \frac{g'(z)}{h'(z)} = \frac{F'(z) - iG'(z)}{F'(z) + iG'(z)}\,.$$

THEOREM 2. *For the mapping function f constructed from D onto K, S is an inner function.*

Proof. Recall that $F(z) = -(G(z) + \overline{G(\overline{z})})$, so that

$$S(z) = \frac{1 + i + \dfrac{\overline{G'(\overline{z})}}{G'(z)}}{1 - i + \dfrac{\overline{G'(\overline{z})}}{G'(z)}}\,. \tag{6.1}$$

For z in D,

$$\frac{\overline{G'(\overline{z})}}{G'(z)} = \frac{-\displaystyle\sum_{j=1}^{\infty} \frac{\overline{\zeta}_j - 1}{z - \overline{\zeta}_j}}{\displaystyle\sum_{j=1}^{\infty} \frac{\zeta_j - 1}{z - \zeta_j}}\,. \tag{6.2}$$

We claim that both series in (6.2) converge absolutely for $|z| = 1$, $z \ne \zeta_j$ and $z \ne \overline{\zeta}_j$, $j = 1, 2, \ldots$ To see this, note that

$$G'(0) = -\frac{i}{\pi} \sum_{j=1}^{\infty} (1 - \overline{\zeta}_j)\,.$$

Since $|\zeta_j - 1| \le |\operatorname{Re}\zeta_j - 1| + |\operatorname{Im}\zeta_j| = (1 - \operatorname{Re}\zeta_j) - \operatorname{Im}\zeta_j$ for all j, it follows that $\sum_{j=1}^{\infty} |\zeta_j - 1|$ converges. If $|z| = 1$, $z \ne \zeta_j$, $j = 1, 2, \ldots$, there exists $M > 0$ so that $|z - \zeta_j| > M > 0$ for all j. Thus $|(\zeta_j - 1)/(z - \zeta_j)| \le (1/M)|\zeta_j - 1|$ and therefore $\sum_{j=1}^{\infty}(\zeta_j - 1)/(z - \zeta_j)$ converges absolutely. Similarly $\sum_{j=1}^{\infty}(\overline{\zeta}_j - 1)/(z - \overline{\zeta}_j)$ converges absolutely if $z \ne \overline{\zeta}_j$.

From (6.1), we see that in order to prove that S is an inner function, it is sufficient to prove that $\overline{G'(\overline{z})}/G'(z)$ is real for $|z| = 1$, $z \ne \zeta_j$ and $z \ne \overline{\zeta}_j$, $j = 1, 2, \ldots$ First, consider one term of the series in the denominator of (6.2). If $z = e^{i\theta}$ then

$$\begin{aligned}
\frac{\zeta_j - 1}{z - \zeta_j} &= \frac{[\cos(\beta_j - \theta) - \cos\theta - 1 + \cos\beta_j] + i[\sin(\beta_j - \theta) + \sin\theta - \sin\beta_j]}{|z - \zeta_j|^2} \\
&= \frac{\cos\beta_j\cos\theta + \sin\beta_j\sin\theta - \cos\theta - 1 + \cos\beta_j}{|z - \zeta_j|^2} \\
&\quad + i\,\frac{\sin\beta_j\cos\theta - \sin\theta\cos\beta_j + \sin\theta - \sin\beta_j}{|z - \zeta_j|^2} \\
&= \left[\frac{\sin\theta}{1 - \cos\theta} - i\right] \frac{\sin\theta\,(\cos\beta_j - 1) + (1 - \cos\theta)\sin\beta_j}{|z - \zeta_j|^2}\,.
\end{aligned}$$

Now we compute one term in the numerator of (6.2):

$$\begin{aligned}
&\frac{\overline{\zeta}_j - 1}{z - \overline{\zeta}_j} \\
&\quad = \frac{(\cos(\theta + \beta_j) - \cos\theta - 1 + \cos\beta_j) + i(-\sin(\theta + \beta_j) + \sin\theta + \sin\beta_j)}{|z - \overline{\zeta}_j|^2} \\
&\quad = \left[\frac{\sin\theta}{1 - \cos\theta} - i\right] \frac{(1 + \cos\theta)(\cos\beta_j - 1) - \sin\beta_j\sin\theta}{|z - \overline{\zeta}_j|^2}\,.
\end{aligned}$$

Thus the common factors $\sin\theta/(1 - \cos\theta) - i$ cancel and we have the ratio of two real numbers in (6.2).

7. ***S* as a limit of Blaschke products.** In this section we will prove that the inner function $S(z)$ is the limit of a sequence of finite Blaschke products. Let

$$F_n(z) = \frac{i}{\pi}\sum_{j=1}^{n+1} \operatorname{Log}\left(\overline{\zeta}_j\,\frac{z - \zeta_j}{z - \overline{\zeta}_j}\right)$$

and

$$G_n(z) = -\frac{i}{\pi}\sum_{j=1}^{n+1} \operatorname{Log}\left(\overline{\eta}_j\,\frac{z - \zeta_j}{z - \overline{\zeta}_j}\right).$$

Then F_n converges to F and G_n converges to G uniformly on compact subsets of D. It can be verified that $\operatorname{Re}(1-z^2)F_n'(z) > 0$ and that $\operatorname{Im}(1-z)^2G_n'(z) < 0$ in D. According to results of Hengartner and Schober [5], F_n and G_n are univalent in D and map D onto domains which are convex in the direction of the imaginary axis. An examination of boundary values indicates that F_n maps D onto the strip $\{w : 0 < \operatorname{Re} w < n+1\}$ with slits of the form $\{w = j + iy : |y| \geq n(j)\}$, $j = 1, \dots, n$, removed and G_n maps D onto the strip $\{w : -(n+1) < \operatorname{Re} w < 0\}$ with slits of the form $\{w = -j - iy : y \geq m(j)\}$, $j = 1, \dots, n$, removed. Using the techniques of Sections 4 and 5 it can be shown that the function $f_n = \operatorname{Re} F_n + i \operatorname{Re} G_n$ is a harmonic map of D onto the triangular region $\{w = u - iv : 0 < u < n+1,\ 0 < v < u\}$. It follows then that the function

$$S_n(z) = \frac{F_n'(z) - iG_n'(z)}{F_n'(z) + iG_n'(z)}$$

is analytic in D and satisfies $|S_n(z)| < 1$ for z in D. Also S_n converges to S uniformly on compact subsets of D. We claim that S_n is a finite Blaschke product.

With $\alpha_j = \operatorname{Im} \zeta_j$ we have

$$S_n(z) = \frac{-\dfrac{2}{\pi}\displaystyle\sum_{j=1}^{n+1} \frac{\alpha_j(1-z)^2}{(z-\zeta_j)(z-\overline{\zeta}_j)} + \frac{1}{\pi}\sum_{j=1}^{n+1} \frac{(\zeta_j - 1)(1-z)}{z-\zeta_j}}{-\dfrac{2}{\pi}\displaystyle\sum_{j=1}^{n+1} \frac{\alpha_j(1-z)^2}{(z-\zeta_j)(z-\overline{\zeta}_j)} - \frac{1}{\pi}\sum_{j=1}^{n+1} \frac{(\zeta_j - 1)(1-z)}{z-\zeta_j}} = \frac{P(z)}{Q(z)}$$

where $P(z)$ and $Q(z)$ are polynomials given by

$$\begin{aligned} P(z) = &-2\sum_{j=1}^{n+1} \alpha_j(1-z) \prod_{\substack{k=1\\k\neq j}}^{n+1} (z-\zeta_k)(z-\overline{\zeta}_k) \\ &+ \sum_{j=1}^{n+1} (\zeta_j - 1) \prod_{\substack{k=1\\k\neq j}}^{n+1} (z-\zeta_k) \prod_{t=1}^{n+1} (z-\overline{\zeta}_t), \\ Q(z) = &-2\sum_{j=1}^{n+1} \alpha_j(1-z) \prod_{\substack{k=1\\k\neq j}}^{n+1} (z-\zeta_k)(z-\overline{\zeta}_k) \\ &- \sum_{j=1}^{n+1} (\zeta_j - 1) \prod_{\substack{k=1\\k\neq j}}^{n+1} (z-\zeta_k) \prod_{t=1}^{n+1} (z-\overline{\zeta}_t). \end{aligned}$$

Straightforward computations give

$$P(z) = -z^{2n+1}\overline{Q\left(\frac{1}{\overline{z}}\right)},$$

and therefore for $z \neq 0$, $P(z) = 0$ if and only if $Q(1/\overline{z}) = 0$. At this stage we assume that the common factors of $P(z)$ and $Q(z)$ have been removed. After removing common factors we still have $P(z) = 0$ if and only if $Q(1/\overline{z}) = 0$. Since $S_n(z)$ is analytic in D, Q has no zeros in D and thus the zeros of P must be in the closure of D. The zeros of $P(z)$ and $Q(z)$ which lie on $|z| = 1$ cancel in the quotient $P(z)/Q(z)$, and therefore $P(z)/Q(z)$ is analytic for $|z| \leq 1$. Also from (7.1), if $|z| = 1$, then $|P(z)| = |Q(z)|$. Thus $|S_n(z)| = 1$ on $|z| = 1$. It follows then that $S_n(z)$ is a finite Blaschke product for each n.

8. Another example. The example below was shown to us by Don Marshall. Let R be the infinite strip

$$R = \{z = x + iy : 0 < x,\ 0 < y < 1\}.$$

The function $g(z) = x + iyx$ is a harmonic map of R onto $Q = \{(x, y) : x > 0,\ 0 < y < x\}$. Now let ϕ be a Riemann map of D onto R, say $\phi(w) = x(w) + iy(w)$. The function

$$\phi(w) = g(\phi(w)) = x(w) + iy(w)x(w)$$

is a harmonic map of D onto Q. In this case

$$F = \phi \quad \text{and} \quad G = -i\frac{\phi^2}{2}.$$

Thus

$$S = \frac{1 - \phi}{1 + \phi}.$$

If Γ is the arc mapped by ϕ onto $\{z = iy : 0 < y < 1\}$, then $|S| = 1$ on Γ, but S is not an inner function.

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