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EXISTENCE AND NONEXISTENCE OF SOLUTIONS FOR A MODEL OF GRAVITATIONAL INTERACTION OF PARTICLES, I

BY

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We study the existence of stationary and evolution solutions to a parabolic-elliptic system with natural (no-flux) boundary conditions describing the gravitational interaction of particles.

1. Introduction. We are interested in the parabolic-elliptic system of partial differential equations defined in a bounded domain Ω of $\mathbb{R}^n$,

$$u_t = \Delta u + \nabla\cdot(u\nabla\varphi)\,, \tag{1}$$

$$\Delta\varphi = u\,, \tag{2}$$

with the nonlinear no-flux condition

$$\frac{\partial u}{\partial\nu} + u\frac{\partial\varphi}{\partial\nu} = 0\,, \tag{3}$$

where ν denotes the outward unit normal vector to the $C^{1+\varepsilon}$ ($\varepsilon > 0$) boundary $\partial\Omega$ of the considered domain. For the potential φ we assume either

$$\varphi = 0 \quad \text{on } \partial\Omega\,, \tag{4.1}$$

or instead of the Dirichlet boundary condition above

$$\varphi = E_n * u \tag{4.2}$$

with E_n the fundamental solution of the n-dimensional Laplacian. The initial-boundary value problem is supplemented with the initial condition

$$u(x,0) = u_0(x) \ge 0\,. \tag{5}$$

The physical interpretations of the system (1)–(5) are connected with the evolution version of the Chandrasekhar equation for the gravitational equilibrium of polytropic stars, with the evolution of self-interacting clusters

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of particles, and with nonequilibrium statistical mechanics description of some flows for the Euler equations (see the review article [2], as well as [8], [12], [18], [19], and the references therein).

Namely, if a gas, consisting of gravitationally interacting particles and filling up the reservoir Ω, is in thermodynamical equilibrium, then the potential φ of the forces (after a suitable rescaling) satisfies the equation (2). The density u of the gas is given by the Boltzmann formula $u = M\mu^{-1}\exp(-\varphi/(kT))$, where $\mu = \int_\Omega \exp(-\varphi/(kT))$, $M = \int_\Omega u$ is the total mass of the gas in Ω, k is the Boltzmann constant, and T the absolute temperature, so (2) leads to the equation

$$\Delta\varphi + \sigma\mu^{-1}\exp\varphi = 0 \tag{6}$$

with $\sigma = M/(kT)$. Solutions to these equations are exactly the time-independent solutions of the system (1), (2). Note that (3) is the simplest physically relevant (no-flux) boundary condition, which guarantees the conservation of the integral $\int_\Omega u$ in time. However, (3) causes serious technical difficulties, since some maximum principle arguments may fail for solutions to (1)–(5).

The second model leading to (1)–(5) comes from hydrodynamics, from the analysis of the canonical Gibbs measure associated to an N-vortex system in a bounded two-dimensional domain. In the limit $N \to \infty$, with suitably scaled intensity of the vortices and temperature, the stream function φ corresponding to such vortex solutions of the Euler equation satisfies the equation (6) and the boundary condition (4.1). Again, the evolution system (1)–(4) includes solutions to this mean field equation from equilibrium statistical mechanics as the stationary ones.

Finally, another interpretation of (1)–(5) is linked with the Debye system from the theory of electrolytes studied in [4], [5], [13]. The original Debye system includes the equation $u_t = \Delta u - \nabla\cdot(u\nabla\varphi)$ instead of (1), and this describes the electrostatic repulsion of ions. Replacing the sign $-$ by $+$ is equivalent to modelling the gravitational interaction of particles.

Note that, analogously to [4], [5], [7], the drift coefficient in both equations (1), (3) can be reconstructed from (2) and (4), which leads to a differential-integral parabolic equation of Fokker–Planck type. However, this interpretation does not provide a substantial simplification of the problem.

For simplicity of exposition we only consider two possible conditions that determine uniquely the potential φ satisfying (2), hence reproduced by a potential type operator $\varphi(x) = \int_\Omega K(x,y)u(y)\,dy$. Namely, (4.1) corresponds to $K = G_\Omega$, the Green function of the domain Ω, and in (4.2), $K(x,y) = E_n(x-y)$ is the fundamental solution of Δ: $E_n(x) = -((n-2)\sigma_n)^{-1}|x|^{2-n}$, $n \ge 3$, $E_2(x) = (2\pi)^{-1}\log|x|$. Examples of more general kernels K are given in [7]. Note that the boundary behavior of the kernels K reflects the smoothness properties of $\partial\Omega$.

One may replace φ in the equation (1) by $\varphi + V$ with an external potential V satisfying e.g. $|D^2V|_p < \infty$ for some $p \in [1,\infty]$. Such a generalization has been considered in [18], [19] and [7]. All the reasonings, here for $V \equiv 0$, concerning existence, global existence or nonexistence of solutions etc., can be modified in an inessential way to cover that more general case.

In Section 2, we discuss the existence, uniqueness and nonexistence of stationary solutions to the system (1)–(4). Section 3 deals with the local- and global-in-time existence of solutions to the evolution system in arbitrary domains in $\mathbb{R}^n$ and $\mathbb{R}^2$, resp. Moreover, it is shown that for n-dimensional star-shaped domains ($n > 1$) and large initial conditions there are no solutions global in time: solutions blow up in finite time. Radially symmetric solutions in balls and in annuli/spherical shells are considered in a companion paper [6]. For these particular domains we consider in [6] a more general formulation of the problem (1)–(5) using a new dependent variable: the integrated density $Q(r,t) = \int_{B_r} u(x,t)\,dx$. Of course, for the radial problem in balls of $\mathbb{R}^n$ various conditions for existence/nonexistence of solutions can be made sharp or more precise than for general domains considered in this paper.

We use largely the notation and results of papers [4], [5], [7] relevant to the study of the system (1)–(4). In particular, we use the standard notation $|u|_p$ for the $L^p(\Omega)$ norms of functions, and $\|u\|_s$ for the $H^s(\Omega)$ norms. The constants independent of functions defined on Ω will be denoted by the same letter C, even if they may vary from line to line. For various interpolation inequalities we refer to [1], [10], [11].

2. Stationary solutions. In this section we collect some results concerning stationary solutions of the problem (1)–(4). More detailed information can be found in e.g. [2], [7], [8], [12], [14], [17], [18]. New results are contained in Theorem 1 below.

The stationary solution $\langle U, \Phi\rangle$ of (1)–(2) satisfies the system

$$\Delta U + \nabla\cdot(U\nabla\Phi) = 0\,, \qquad \Delta\Phi = U\,, \tag{7}$$

which is equivalent to

$$\nabla\cdot(\exp(-\Phi)\nabla(\exp(\Phi)U)) = 0 \tag{8}$$

(since in the class of (nonstationary) solutions considered here, in Theorem 2 below, $\Phi \in H^1(\Omega)\cap L^\infty(\Omega)$ is bounded). The boundary condition (3) can be rewritten as

$$\frac{\partial(\exp(\Phi)U)}{\partial\nu} = 0\,. \tag{9}$$

Multiplying (8) by $\exp(\Phi)U$ we obtain

$$\int_\Omega \exp(-\Phi)|\nabla(\exp(\Phi)U)|^2 = 0,$$

so U has the Boltzmann form $U = \alpha\exp(-\Phi)$. The normalization constant α can be determined from the relation $\int_\Omega U = M$ so that $\alpha = M\mu^{-1}$ with $\mu = \int_\Omega \exp(-\Phi)$ and (9) is incorporated in

$$U = M\mu^{-1}\exp(-\Phi)\,. \tag{10}$$

The potential Φ satisfies the integral equation

$$\Phi = \frac{M}{\int_\Omega \exp(-\Phi)} J(\exp(-\Phi))\,, \tag{11}$$

with $J(v) = \int_\Omega K(x,y)v(y)\,dy$. The conditions (4.1) or (4.2) are already encoded in (11). The stationary problem in the form (11) has been studied in [7], [12], [14]. It is worth noting that the existence and uniqueness of solutions of (11) depend strongly on the geometric properties of the domain Ω.

From [12], [7] it follows that (11) has a solution for sufficiently small $M > 0$. Few general uniqueness results are known (cf. [7, Th. 2(iii)]).

Assuming that Ω is a star-shaped domain in $\mathbb{R}^n$, $K = G_\Omega$, it can be proved, using the Pokhozhaev identity (see [8], [14] and [7, Th. 2(iv)]) that (11) has no solution for M large enough. Indeed, from the relation

$$\int_{\partial\Omega} \left(\frac{\partial\Phi}{\partial\nu}\right)^2 (x\cdot\nu) = M\mu^{-1}\int_\Omega (2n(e^{-\Phi}-1) + (n-2)\Phi e^{-\Phi})$$

we infer

$$M^2 = \left(\int_{\partial\Omega}\frac{\partial\Phi}{\partial\nu}\right)^2 \le \int_{\partial\Omega}\left(\frac{\partial\Phi}{\partial\nu}\right)^2(x\cdot\nu)\int_{\partial\Omega}(x\cdot\nu)^{-1}.$$

Since $\int_{\partial\Omega}(x\cdot\nu)^{-1} \le C(\Omega)d^{n-2}$, where $d = \operatorname{diam}\Omega$ and $C(\Omega)$ depends on the shape of Ω only (not on the size of Ω), and by the maximum principle $\Phi \le 0$, we obtain $M < 2nC(\Omega)d^{n-2}$. In particular, for the balls B_R, $M < 2n\sigma_n R^{n-2}$ is a necessary condition for the existence of solutions to (11).

An analogous nonexistence result holds true if Ω is any simply connected bounded domain in the plane (see [14]).

On the other hand, in multiply connected domains, e.g. in annuli, solutions can exist for a larger range of M's, even for all $M > 0$ (cf. [8], [16], [17], [6], [14]).

THEOREM 1. *Let Ω be a bounded domain in $\mathbb{R}^n$.*

(i) *If $n = 2$ and $M \in (0, 8\pi)$ in the case of the boundary condition* (4.1), *or $M \in (0, 4\pi)$ if* (4.2) *is assumed, then there exists a solution $\Phi \in L^\infty(\Omega)$ of* (11). *For all sufficiently small $M > 0$ these solutions are unique in the class of potentials Φ corresponding via* (10) *to densities $U \in L^1(\Omega)$ with $|U|_1 = M$.*

(ii) *If $n \ge 3$ and $M \in (0, M(\Omega))$ with some $M(\Omega) > 0$, then there is a solution $\Phi \in L^\infty(\Omega)$ of* (11). *For all sufficiently small $M > 0$ these solutions are unique in the class of potentials Φ corresponding via* (10) *to densities $U \in M^{n/2}(\Omega)$ of small concentration: $\|U\|_* < \beta_n$, where $(M^{n/2}(\Omega), \|\cdot\|_*)$ is the Morrey space of exponent $n/2$. Its definition will be given in the proof below and the quantities $M(\Omega)$, β_n will be calculated explicitly.*

Remarks. The existence of solutions for small $M > 0$ for a large class of generalizations of (11), where the kernels K are supposed to be dominated by $|E_n|+$ const only, follows from a simple argument involving the Schauder fixed point theorem (see [7, Th. 2(i), (ii)]).

The use of variational methods (applied to the functional W, see (23) below) gives the result of Theorem 1(i) above when $K = G_\Omega$, i.e. when (4.1) is satisfied, in [8] and (for small $M > 0$ only) in [17] (cf. also [16]). The arguments in those papers are rather delicate compared to the Leray–Schauder theorem used in our proof below.

Concerning Theorem 1(ii), it seems that the direct variational methods (requiring a priori regularity of Φ: $\Phi \in H^1_0(\Omega) \cap L^\infty(\Omega)$) are no longer applicable since for $n \ge 3$ the functional W is not bounded from below for any $M > 0$ (compare [18, Lemma 4.4]).

As we will check further, the regularity of solutions and their uniqueness for $n \ge 3$ and small $M > 0$ is expected (and guaranteed) only when the densities U have small concentrations, even in the case of radially symmetric solutions in balls.

The regularity of solutions Φ sought for in $L^\infty(\Omega)$ can be easily deduced from the smoothness properties of the kernels K off the diagonal $\{(x, x) : x \in \Omega\}$ (cf. [7, Th. 2(ii)]). In our approach (aimed at a unified functional framework for stationary and evolution solutions) we exclude unbounded stationary solutions Φ studied in [3].

Proof of Theorem 1. (i) First we consider the case $K = E_2$ (in fact we only use $|K| \le |E_2|$), and we show a uniform $L^\infty(\Omega)$ a priori bound for solutions of (11) with the densities (10). For $M \in (0, 4\pi)$ fix $\beta \in (M, 4\pi)$ and consider $0 \le U \in L^1(\Omega)$, $0 < |U|_1 < \beta$. Using the Jensen inequality for the exponential function we can write, for $\Phi = J(U)$ and $s \in (1, 4\pi/\beta)$,

$$\int_\Omega \exp(s|\Phi|) \le \int_\Omega \exp\Big(s\beta|U|_1^{-1}\int_\Omega |K(x,y)|U(y)\,dy\Big) \tag{12}$$
$$\le \int_\Omega |U|_1^{-1}\Big(\int_\Omega U(y)C|x-y|^{-s\beta/(2\pi)}\,dy\Big)\,dx$$
$$= C\int_\Omega |U|_1^{-1}U(y)\Big(\int_{B(0,2d)} |x|^{-s\beta/(2\pi)}\,dx\Big)\,dy$$
$$= C2\pi(2-s\beta/(2\pi))^{-1}(2d)^{2-s\beta/(2\pi)} < \infty\,,$$

where $d = \operatorname{diam}\Omega$. Moreover, we have

$$|\Omega| \le \mu^{s/(s+1)}\Big(\int_\Omega \exp(s\Phi)\Big)^{1/(s+1)} \le \mu^{s/(s+1)}|\exp(|\Phi|)|_s^{s/(s+1)}\,,$$

which implies

$$\mu^{-1} \le |\Omega|^{-1-1/s}|\exp(|\Phi|)|_s\,. \tag{13}$$

Since $|M\mu^{-1}\exp(-\Phi)|_s \le M|\Omega|^{-1-1/s}|\exp(|\Phi|)|_s^2 \le MC(\beta) < \infty$ we arrive at the bound

$$|M\mu^{-1}J(\exp(-\Phi))|_\infty \le M\sup_{x\in\Omega}\Big(\int_\Omega |K(x,y)|^{s'}\,dy\Big)^{1/s'}C(\beta) < \infty\,, \tag{14}$$

where $1/s + 1/s' = 1$. Having the above a priori estimate we define the operator $\mathcal{T} : L^\infty(\Omega) \to L^\infty(\Omega)$ by $\mathcal{T}(\Phi) = M\mu^{-1}J(\exp(-\Phi))$ whose fixed points solve (11). $\mathcal{T}$ is continuous and compact (cf. [7]). For $M \in (0, 4\pi)$ and arbitrary $\lambda \in [0,1]$ each solution of the equation $\Phi = \lambda\mathcal{T}(\Phi)$ is a priori bounded in $L^\infty(\Omega)$ from (14). The Leray–Schauder theorem applies in this situation and furnishes a fixed point Φ of $\mathcal{T}$ solving (11).

Note that our computation (12) is closely related to that in the proof of the Moser–Trudinger inequality

$$\int_\Omega \exp(|\psi|) \le C\exp\Big(|\Omega|^{-1}\Big|\int_\Omega \psi\Big| + |\nabla\psi|_2^2/(8\pi)\Big) \tag{15.2}$$

valid for all $\psi \in H^1(\Omega)$ (cf. [9, Th. 3, II]).

For the case $K = G_\Omega$ and $M \in (0, 8\pi)$ (so $|K| \le |E_2| + C(\Omega)$ as follows from [10, Ch. II, Sec. 4, Prop. 5, (4.27)]) an analogous argument applies, and the counterpart of (15.2) for the boundary condition (4.1) is

$$\int_\Omega \exp(|\psi|) \le C\exp\Big(|\Omega|^{-1}\Big|\int_\Omega \psi\Big| + |\nabla\psi|_2^2/(16\pi)\Big) \tag{15.1}$$

valid for all $\psi \in H_0^1(\Omega)$ (cf. [8]). The constants 8π, 16π in the denominators are optimal (cf. [9, Sec. III]).

The uniqueness of solutions in $L^\infty(\Omega)$ for $M > 0$ small enough follows by a standard contraction argument in [7, Th. 2(iii)] whenever an a priori bound on $\Phi \in L^\infty(\Omega)$ is available. The uniqueness in a class of more regular potentials Φ for the problem with $K = G_\Omega$, Ω simply connected, $\partial\Omega$ sufficiently smooth, and $M \in (0, 8\pi)$ can be deduced from [16]. In the variational approach minimizers of the functional W in (23) are unique in the same range of M's (see [8]).

(ii) The assumption $U \in L^1(\Omega)$, even for small $M = |U|_1$, does not guarantee a priori bounds on $\Phi \in L^\infty(\Omega)$. For instance, $U(r) = 2(n-2)r^{-2}$ in the ball $B(0,R)$, solving (10) with $M = 2\sigma_n R^{n-2}$, leads to an unbounded potential Φ. Such a solution U belongs to $L^p(\Omega)\setminus L^{n/2}(\Omega)$ for every $p < n/2$ and, as we will see in Theorem 2, does not enter into the framework for weak solutions of the evolution problem. For a discussion of singular nonradial solutions, we refer the reader to [3]. A natural restriction on the densities is that of a *low concentration*. We mean by that the assumption

$$U \in M^{n/2}(\Omega)\,, \qquad \|U\|_* < \beta_n := 4\sigma_n/(ne)\,. \tag{16}$$

The definition of the Morrey space of exponent $n/2$ (see [11, Sec. 7.9]), reads $M^{n/2}(\Omega) = \{U : \|U\|_* < \infty\}$, where $\|U\|_* = \sup(R^{2-n}\int_{B_R\cap\Omega}|U|) < \infty$ with supremum taken over all balls of radii $R > 0$, $B_R \subset \mathbb{R}^n$. It is well known that $L^{n/2}(\Omega) \subset L^{n/2}_{\mathrm{weak}}(\Omega) \subset M^{n/2}(\Omega)$, and $\|U\|_* \le |U|_{n/2}(\sigma_n/n)^{1-2/n}$ since $\int_{B_R\cap\Omega}|U| \le (\int_\Omega |U|^{n/2})^{2/n}(\omega_n R^n)^{1-2/n}$. Remark that the Morrey norm of $U(r) = 2(n-2)r^{-2}$ is large compared to β_n: $\|U\|_* = 2\sigma_n > \beta_n$. For solutions U to (10) of low concentration, i.e. those satisfying (16), we will show an a priori bound for $\Phi \in L^\infty(\Omega)$.

Fix $\beta \in (0, \beta_n)$ and estimate the right hand side of (10) with $\|U\|_* < \beta$ in $L^{n/2+\varepsilon}(\Omega)$ for any sufficiently small $\varepsilon > 0$. This is possible since [11, Lemma 7.20] implies

$$\int_\Omega \exp\left(\frac{(n-2)\sigma_n}{\alpha\|U\|_*}|\Phi|\right) \le C(\alpha,\beta)d^n\,, \qquad d = \operatorname{diam}\Omega\,, \tag{17}$$

for any $\alpha > e(n/2-1)$. Indeed, we use [11, Lemma 7.20] with $g = (n-2)\sigma_n|E_n| * U$, $|K| \le |E_n| + C(\Omega)$ (for $K = G_\Omega$ this is a consequence of [10, Ch. II, Sec. 4, Prop. 5, (4.27)]). Choosing a suitable $\alpha = \alpha(\beta)$ from (17) we obtain $\exp(|\Phi|) \in L^{n/2+\varepsilon}(\Omega)$ with an upper bound $C(\beta)d^n$ for the power of its norm.

In particular, μ is well defined and, as in (13), $\mu^{-1} \le |\Omega|^{-1-1/(n/2+\varepsilon)} \times |\exp(|\Phi|)|_{n/2+\varepsilon}$. Finally, we obtain

$$\begin{aligned}|M\mu^{-1}\exp(-\Phi)|_{n/2+\varepsilon} &\le M|\Omega|^{-1-1/(n/2+\varepsilon)}|\exp(|\Phi|)|^2_{n/2+\varepsilon}\\ &\le MC(\beta,\Omega)\,,\end{aligned}$$

and since $J : L^{n/2+\varepsilon}(\Omega) \to L^\infty(\Omega)$ ($|E_n| \in L^{n/(n-2)}_{\text{weak}}$ is a convolution kernel),

$$|M\mu^{-1}J(\exp(-\Phi))|_\infty \le MC(\beta,\Omega) < \infty\,.$$

Now consider $M \in (0, M(\Omega))$, $\varepsilon > 0$, both sufficiently small so that

$$\|M(\Omega)\mu^{-1}\exp(-\Phi)\|_* \le M(\Omega)(\sigma_n/n)^{1-2/n}C(\beta)^{4/n}|\Omega|^{-1-2/n}d^4 \le \beta\,, \tag{18}$$

and define the nonlinear integral operator $\mathcal{T} : L^\infty(\Omega) \to L^\infty(\Omega)$ by

$$\mathcal{T}(\Phi) = J(\chi(M\mu^{-1}\exp(-\Phi))),$$

where $\chi(v) = \min(1, \beta/\|v\|_*)v$ normalizes v with large $\|v\|_*$ to $\|\chi(v)\|_* \le \beta$. The operator $\mathcal{T}$ satisfies all the assumptions of the Leray–Schauder theorem, which can be checked as in (i). The condition (18) guarantees that all the fixed points $\Phi = \mathcal{T}(\Phi)$ satisfy (11) because such a Φ equals $J(U)$ for some U with $\|U\|_* \le \beta$, and χ does not cut down these U's.

By inspection of the proof of [11, Lemma 7.20] we get for $\beta = \tau\beta_n$, $\tau \in (0,1)$,

$$\begin{aligned}\max \beta C(\beta)^{-4/n} &\ge 4\sigma_n^{1-4/n}(n-2)^{4/n}(ne)^{-1}\max_{\tau\in(0,1)} \tau\Big(\sum_{m=0}^\infty \tau^m\Big)^{-4/n}\\ &= 4^{1+4/n}(n-2)^{4/n}\sigma_n^{1-4/n}(e(n+4)^{1+4/n})^{-1} =: \gamma_n\,.\end{aligned}$$

This leads to an explicit estimate of M's; specifically, we may take

$$M(\Omega) = \gamma_n n^{1-2/n}\sigma_n^{2/n-1}|\Omega|^{1+2/n}d^{-4}$$

and for balls even $M(B_R) = 2^{2+8/n}(n-2)^{4/n}(e(n+4)^{1+4/n})^{-1}\sigma_n R^{n-2}$.

Let us remark that for negative kernels K, e.g. $K = E_n$, or $K = G_\Omega$, we have $\Phi \le 0$, so $\mu \ge |\Omega|$. This improves our estimate of $M(\Omega)$. For instance, $M(\Omega) = 4^{1+1/n}n((n+2)e)^{-1}((n-2)/n(n+2))^{2/n}|\Omega|d^{-2}$ works well in this situation.

The uniqueness of $L^\infty(\Omega)$ solutions to (11) with low concentration is obtained for sufficiently small $M > 0$ by a standard reasoning (see [7, Th. 2(iii)]). Note that these stationary solutions fall under the evolution theory of Section 3.

3. Solutions to the evolution problem. For arbitrary bounded smooth domains in $\mathbb{R}^2$ or in $\mathbb{R}^3$ the local-in-time existence of solutions in $L^2(\Omega)$ follows from the proof of Theorem 1 in [4], which concerns the electrolytic case, but the existence argument does work for both equations $u_t = \Delta u \mp \nabla\cdot(u\nabla\varphi)$. This is a standard argument involving the Schauder fixed point theorem in a suitable space of vector-valued ($L^2(\Omega)$) functions (cf. [7] for a class of more general systems).

A similar construction gives the existence in the n-dimensional case when the initial data are in $L^p(\Omega)$, $p > n$. For less regular initial conditions in $L^p(\Omega)$, $p > n/2$, solutions can be approximated by those emanating from regularized u_0's.

We collect all the results on the local existence which can be proved using the same arguments as in [4, Theorems 1, Proposition 1, Remark 8], [5, Theorems 1 and 2], and [7, Theorem 1] in parts (i), (ii) of the theorem below.

Here *weak* $H^1(\Omega)$ solutions of the problem (1)–(5) on $\Omega \times (0,T)$ are understood as functions $u \in L^\infty((0,T); L^2(\Omega)) \cap L^2((0,T); H^1(\Omega))$ which satisfy, for every test function $\eta \in H^1(\Omega \times (0,T))$ and for a.e. $t \in (0,T)$, the integral identity

$$\int_\Omega u(x,t)\eta(x,t)\,dx - \int_0^t\int_\Omega u\eta_t + \int_0^t\int_\Omega (\nabla u + u\nabla\varphi)\cdot\nabla\eta = \int_\Omega u_0(x)\eta(x,0)\,dx\,. \tag{19}$$

Moreover, we require that for a.e. $t \in (0,T)$, $\varphi = \varphi(\cdot,t)$ is a weak solution of (2) with (4.1), i.e.

$$\varphi \in H^1_0(\Omega)\,, \qquad \int_\Omega \nabla\varphi\cdot\nabla\xi + \int_\Omega u\xi = 0 \quad \text{for each } \xi \in H^1_0(\Omega)\,,$$

or

$$\varphi \in H^1(\Omega) \quad \text{with} \quad \varphi = E_n * u$$

when (4.2) is assumed.

These definitions are a natural extension of standard ones in [15, Ch. III, Secs. 1, 4, 5] taking into account the no-flux condition (3), and the self-consistent character of the field $\nabla\varphi$ in (1) determined by u itself from (2) and (4).

It is a routine calculation (cf. [15, Ch. III, Sec. 2], [4], [7]) that such a weak solution satisfies $u_t \in L^2((0,T); H^{-1}(\Omega))$, $u \in C([0,T]; L^2(\Omega))$, and the energy (in)equality

$$\frac{1}{2}\int_\Omega u^2(x,t)\,dx + \int_0^t\int_\Omega (\nabla u + u\nabla\varphi)\cdot\nabla u = \frac{1}{2}\int_\Omega u_0^2(x)\,dx$$

for all $t \in [0,T]$. We will write this briefly as a differential (in)equality

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + |\nabla u|_2^2 = -\int_\Omega u\nabla\varphi\cdot\nabla u \tag{20}$$

whose *formal* derivation consists in multiplying (1) by u and integrating by parts. In the sequel other integral inequalities following from the definition of weak solutions will be written in this shorthand differential notation, but understood in their original, proper, integral form.

THEOREM 2. *Assume that Ω is a bounded domain in $\mathbb{R}^n$ with $C^{1+\varepsilon}$ boundary*, $\varepsilon > 0$.

(i) *If* $n = 2$ *or* $n = 3$, *and* $0 \le u_0 \in L^2(\Omega)$, *then there exists* $T = T(|u_0|_2)$ *such that the problem* (1)–(5) *has a unique weak solution* $u \in L^\infty((0,T); L^2(\Omega)) \cap L^2((0,T); H^1(\Omega))$. *Moreover*, $u_t \in L^2((0,T); H^{-1}(\Omega))$, $u(x,t) \ge 0$ *for a.e.* $x \in \Omega$ *and* $t \ge 0$, *and* $\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)\,dx$.

(ii) *If* $n \ge 2$ *and* $0 \le u_0 \in L^p(\Omega)$, $p > n/2$, *then there is* $T = T(p, |u_0|_p) > 0$ *and a weak solution* u *such that* $u \in L^\infty((0,T); L^p(\Omega))$ *and* $u^{p/2} \in L^2((0,T); H^1(\Omega))$.

These solutions are unique when $p > n$, *and regular when* $p > n/2$ *in the sense that* $u \in L^\infty_{\mathrm{loc}}((0,T); L^\infty(\Omega))$.

(iii) *If* $n > 2$, $p > n/2$, *and* $|u_0|_p$ *is sufficiently small*, *then the local solution constructed in* (ii) *can be extended to a global one defined for all* $t \ge 0$.

(iv) *If* $n = 2$, $|u_0|_1 = M$ *and either* $M \in [0, 8\pi)$ *for* (4.1), *or* $M \in [0, 4\pi)$ *for* (4.2), *then the solution constructed in* (i) *can be extended to the whole half-line* $(0,\infty)$. *Moreover*, $\sup_{t\ge 0} |u(t)|_2 < \infty$ *and for a stationary solution* U *with* $|U|_1 = |u_0|_1 = M$ *we have* $\lim_{t\to\infty} |u(t) - U|_\infty = 0$.

(v) *If* $\Omega \subset \mathbb{R}^n$, $n \ge 2$, *is a star-shaped domain* (*with respect to* $0 \in \Omega$), *then for* $|u_0|_1 = M > 2n\sigma_n d^{n-2}$ ($d = \operatorname{diam} \Omega$) *there are no global solutions to* (1)–(3), (4.2), (5).

P r o o f. We refer the reader to [4], [5], [7] for the proofs of (i) and (ii).

(iii) We give a formal argument which shows that $|u(t)|_p$, $n/2 < p < n$, is a priori bounded for all $t \ge 0$. The computations below can be justified along the lines of [5, Ths. 2(ii) and 5]. Let us multiply (1) by u^{p-1} and integrate by parts to get

$$
\begin{aligned}
(21)\qquad \frac{d}{dt}|u|_p^p + \frac{4(p-1)}{p}|\nabla(u^{p/2})|_2^2 &= -p \int_\Omega \nabla(u^{p-1}) \cdot (u\nabla\varphi) \\
&= -2(p-1) \int_\Omega u^{p/2}\nabla(u^{p/2}) \cdot \nabla\varphi \\
&\le 2(p-1)|\nabla(u^{p/2})|_2 |u^{p/2}|_q |\nabla\varphi|_r
\end{aligned}
$$

with $1/r = 1/p - 1/n$ and $1/q = 1/2 + 1/n - 1/p$. Using the interpolation inequality for Sobolev norms

$$
|u^{p/2}|_q \le C\|u^{p/2}\|_1^\alpha |u^{p/2}|_2^{1-\alpha}
$$

with $\alpha = n/p - 1$, and $|\nabla\varphi|_r \le C|u|_p$, from (21) we obtain

$$\frac{d}{dt}|u|_p^p + \frac{4(p-1)}{p}|\nabla(u^{p/2})|_2^2 \le \frac{1}{2}\|u^{p/2}\|_1^2 + C(|u|_p^p)^\beta \tag{22}$$

with $\beta = (2p+2-n)/(2p-n) > 1$. Again from the Sobolev inequalities we have

$$\begin{aligned}3|u|_p^p &\le C(\|u^{p/2}\|_1^\gamma |u^{p/2}|_1^{1-\gamma})^2\\ &\le \frac{1}{2}\|u^{p/2}\|_1^2 + C|u^{p/2}|_1^2 \le \frac{1}{2}\|u^{p/2}\|_1^2 + C(|u|_p^\delta |u|_1^{1-\delta})^p\\ &\qquad\qquad \text{(with } \gamma = n/(n+2) \text{ and } \delta = (p-2)/(p-1))\\ &\le \frac{1}{2}\|u^{p/2}\|_1^2 + |u|_p^p + C|u|_1^p.\end{aligned}$$

Finally, (21) and (22) lead to

$$\frac{d}{dt}|u|_p^p + 3|u|_p^p + \frac{4(p-1)}{p}|\nabla(u^{p/2})|_2^2 \le \|u^{p/2}\|_1^2 + |u|_p^p + C|u|_1^p + C(|u|_p^p)^\beta,$$

hence

$$\frac{d}{dt}|u|_p^p + |u|_p^p \le C(|u|_p^p)^\beta + CM^p$$

since $|u|_1 = M$. For $w(t) = |u(t)|_p^p$, the above inequality reads

$$\frac{dw}{dt} + w \le C(w^\beta + M^p)\,.$$

Obviously, for $w(0) = |u_0|_p^p$ (and therefore M) sufficiently small, $\sup_{t\ge 0} w(t) < \infty$ because $C(w^\beta + M^p) - w$ has a (small) root $w_0 > 0$ and

$$\int_0^{w_0} (C(w^\beta + M^p) - w)^{-1}\, dw = \infty\,.$$

This uniform $L^p(\Omega)$ bound is sufficient to prove the global existence of a solution with u_0 as the initial condition.

(iv) This is a generalization of a result for radial solutions on balls in $\mathbb{R}^2$ in [19, Th. 3.2] to the case of weak solutions in arbitrary planar domains.

It is relatively simple to show that if $n = 2$, the boundary condition (4.1) is assumed, and the initial condition $u_0 \in L^2(\Omega)$ has a sufficiently small mass $|u_0|_1$, then the solution constructed in Theorem 1 can be continued to a global-in-time solution. The crucial estimate for (20) with the right hand side integrated once by parts is

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u|_2^2 + |\nabla u|_2^2 &= -\frac{1}{2}\int_\Omega \nabla(u^2)\cdot\nabla\varphi\\ &= -\frac{1}{2}\int_{\partial\Omega} u^2\frac{\partial\varphi}{\partial\nu} + \frac{1}{2}\int_\Omega u^3 \le \frac{1}{2}|u|_3^3\,,\end{aligned}$$

since $\partial\varphi/\partial\nu \ge 0$ on $\partial\Omega$ (recall (2) and (4.1)). Thus we obtain

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + |\nabla u|_2^2 \le \frac{1}{2}|u|_3^3 \le C\|u\|_1^2|u|_1$$

with $C = C(\Omega)$, so if $|u_0|_1 \le 1/C$, then $\frac{d}{dt}|u|_2^2 \le 2|u|_2^2$. The Gronwall inequality permits us to control $|u(t)|_2$ locally uniformly in time for $t \ge 0$, hence the solution u admits a continuation to a global-in-time weak solution.

For the general case a more subtle argument is needed. Let us begin with the observation that (1)–(4) has a Lyapunov function, a natural counterpart of those used in [4, Lemma 3], [5, Proposition 3] (cf. [7, (6)]):

$$W(t) = \int_\Omega \left(u(x,t)\log u(x,t) + \frac{1}{2}u(x,t)\varphi(x,t)\right) dx\,. \tag{23}$$

For the boundary condition (4.1) we have

$$\int_\Omega u\varphi = -\int_\Omega |\nabla\varphi|^2\,, \tag{24.1}$$

and for (4.2),

$$\Big|\int_\Omega u\varphi + \int_\Omega |\nabla\varphi|^2\Big| \le C(\Omega, |u_0|_1)\,. \tag{24.2}$$

Indeed, $\Delta\varphi = u$ in Ω and we may extend φ to the whole plane putting $K(x,y) = E_2(x-y)$ for all $x,y \in \mathbb{R}^2$, and $\varphi(x) = \int K(x,y)u(y)\,dy$. If $\Omega \subset B_{R_0}$ for some R_0, then for $R = 2R_0$ consider a smooth function $\chi(r) = 1$ if $r \le R$, and $\chi(r) = 0$ if $r \ge 2R$. Integrating by parts we obtain

$$\begin{aligned}
-\int_\Omega u\varphi &= -\int_\Omega \chi(|x|)u\varphi = -\int_{\mathbb{R}^2} \Delta\varphi\chi(|x|)\varphi \\
&\ge \int_\Omega \chi(|x|)|\nabla\varphi|^2 - \sup|\chi'|\int_{B_{2R}\setminus B_R} |\varphi||\nabla\varphi| \\
&\ge \int_\Omega |\nabla\varphi|^2 - C(\Omega, |u|_1)\,,
\end{aligned}$$

because for compactly supported densities u with $|u|_1 = M$, we have $\sup_{B_{2R}\setminus B_R}(|\varphi| + |\nabla\varphi|) \le C(R)M$.

Although W decreases in time along the trajectories of (1)–(4) (which can be checked as in [4, Lemma 3]), in general W is not suitable to control the $L\log L$ norm of u and $|\nabla\varphi|_2^2$ (cf. [7, (6)]), as was done for the electrolytic case in [4], [5].

Nevertheless, for $M = |u|_1$ small, W can be estimated from below as in [18, Lemma 4.3]. At fixed time t, define an auxiliary function $\widetilde{u} =$

$M\mu^{-1}\exp(-\varphi)$. From the Jensen inequality with the convex function $\tau \mapsto -\log\tau$ and $u \ge 0$ we have

$$0 = -\log\Big(|u|_1^{-1}\int_\Omega u(\widetilde{u}/u)\Big) \le |u|_1^{-1}\int_\Omega u\log(u/\widetilde{u})\,.$$

Therefore

$$\begin{aligned}
(25)\qquad 0 &\le \int_\Omega u\log(u/\widetilde{u}) = \int_\Omega u\log(\mu M^{-1}u\exp\varphi)\\
&= \int_\Omega u\log\mu - \int_\Omega u\log M + \int_\Omega u\log u + \int_\Omega u\varphi\\
&= M\log\mu - M\log M + \frac{1}{2}\int_\Omega u\varphi + W\,.
\end{aligned}$$

Begin with the case (4.2). From the Moser–Trudinger inequality (15.2), for every $\beta \in (0, 8\pi)$,

$$\begin{aligned}
\mu &\le \int_\Omega \exp(|\varphi|) \le C\exp\Big(|\Omega|^{-1}\Big|\int_\Omega \varphi\Big| + |\nabla\varphi|_2^2/\beta\Big)\\
&\le C(\Omega, M)\exp(|\nabla\varphi|_2^2/\beta)
\end{aligned}$$

because $|\varphi|_1 = |J(u)|_1 \le C|u|_1 = CM$. Taking logarithms we arrive at $M\log\mu \le C + M\beta^{-1}|\nabla\varphi|_2^2$. Now, from (24.2) and (25) we infer

$$(1/2 - M\beta^{-1})|\nabla\varphi|_2^2 \le W + C(\beta)$$

and choosing for $M \in (0, 4\pi)$ a $\beta \in (2M, 8\pi)$ we see that W controls $|\nabla\varphi|_2^2$ from above, so

$$(26)\qquad \sup_{t\ge 0}(|u\log u|_1 + |\nabla\varphi|_2^2) \le C(\beta)(W(0) + 1)$$

is a priori bounded.

In the case (4.1) we use a better inequality (15.1) getting an analogous conclusion with $M \in (0, 8\pi)$, $\beta \in (2M, 16\pi)$.

This estimate is sufficient to apply a modification of the reasoning in [5, Theorem 3] leading to the global existence of solutions. For completeness of exposition we sketch the strategy of this demonstration. We proceed as follows: from the energy inequality (20) and the nonlinear imbedding inequality (cf. [5, (22)])

$$(27)\qquad |v|_3^3 \le \varepsilon\|v\|_1^2|v\log|v||_1 + C_\varepsilon|v|_1$$

valid for all $v \in H^1(\Omega)$, $\Omega \subset \mathbb{R}^2$ (given $\varepsilon > 0$ there exists a $C_\varepsilon > 0$), we have

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + |\nabla u|_2^2 \le |\nabla u|_2 |u|_3 |\nabla\varphi|_6$$
$$\le C|\nabla u|_2 |u|_3^{3/2} |\nabla\varphi|_2^{1/2}$$
$$\le C\|u\|_1 (\varepsilon \|u\|_1^2 |u \log u|_1 + C_\varepsilon |u|_1)^{1/2} |\nabla\varphi|_2^{1/2}$$
$$\le \frac{1}{2}\|u\|_1^2 + C|u|_1\,,$$

by (26). The remaining part is fairly standard: from

$$\frac{d}{dt}|u|_2^2 + 2|u|_2^2 + |\nabla u|_2^2 \le 3|u|_2^2 + CM$$
$$\le C\|u\|_1 |u|_1 + CM \le \frac{1}{2}\|u\|_1^2 + C|u|_1^2 + CM$$

the differential inequality $\frac{d}{dt}|u|_2^2 + |u|_2^2 \le CM$ follows, and this gives uniform boundedness of $u(t)$ in $L^2(\Omega)$. This is sufficient to continue local solutions to global ones.

To conclude the proof of (iv) (convergence to steady states) we apply, with minor modifications, the proofs in [4, Theorem 2] and [5, Theorem 6] given for the case of boundary conditions (4.1).

We stress the fact that the global existence of solutions is obtained in both cases for the same range of M's as the existence of stationary solutions.

(v) Define the auxiliary function

$$w(t) = \int_\Omega u(x,t)|x|^2\,dx\,. \tag{28}$$

For any weak solution u we have, with (28),

$$\begin{aligned}\frac{dw}{dt} &= -\int_\Omega (\nabla u + u\nabla\varphi)\cdot\nabla(|x|^2)\\ &= -2\int_\Omega \nabla u\cdot x - 2\int_\Omega u\nabla\varphi\cdot x\\ &= -2\int_{\partial\Omega} u\,x\cdot\nu + 2n\int_\Omega u\\ &\quad -2\iint_{\Omega\times\Omega} u(x,t)(\nabla_x E(x-y))\cdot x\,u(y,t)\,dy\,dx\,.\end{aligned}$$

Since Ω is star-shaped, $x\cdot\nu \ge 0$, so

$$\frac{dw}{dt} \le 2nM - \frac{2}{\sigma_n}\iint_{\Omega\times\Omega} u(x,t)u(y,t)|x-y|^{-n}(|x|^2 - y\cdot x)\,dy\,dx\,. \tag{29}$$

From (29) it follows by symmetry that

$$\begin{aligned}\frac{dw}{dt} &\le 2nM \\ &\quad - \frac{1}{\sigma_n} \iint_{\Omega\times\Omega} u(x,t)u(y,t)|x-y|^{-n}(|x|^2 - y\cdot x - x\cdot y + |y|^2)\,dy\,dx \\ &= 2nM - \frac{1}{\sigma_n} \iint_{\Omega\times\Omega} u(x,t)u(y,t)|x-y|^{-n+2}\,dy\,dx \\ &\le 2nM - \frac{d^{2-n}}{\sigma_n} \iint_{\Omega\times\Omega} u(x,t)u(y,t)\,dy\,dx = 2nM - M^2(\sigma_n d^{n-2})^{-1}\,.\end{aligned}$$

Now it is clear that for $M > 2n\sigma_n d^{n-2}$ the function $w(t)$ becomes negative in a finite time, which is absurd. Hence any weak solution $u(t)$ with $|u_0|_1 = M$ cannot exist globally in time. We note that for $\Omega = B_R$ even $M > 2n\sigma_n R^{n-2}$ leads to the nonexistence of global solutions (cf. [6, Th. 3]).

Observe that near the blow-up time T the density u cannot be bounded. This is a consequence of the regularity of weak solutions in Theorem 2(ii). Informally speaking, Theorem 2(ii) means that the norms $|u(t)|_p$ blow up for all $p \in (n/2, \infty]$ at the same moment. Moreover, there is a conjecture (strongly supported by the preceding analysis of regularity of stationary solutions) that weak solutions to (1)–(5) cannot exist when $u_0 \notin L^p(\Omega)$ with $p > n/2$.

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