C O L L O Q U I U M M A T H E M A T I C U M

VOL. LXV 1993 FASC. 1

FOURIER TRANSFORM OF CHARACTERISTIC FUNCTIONS AND LEBESGUE CONSTANTS FOR MULTIPLE FOURIER SERIES

BY

LUCA BRANDOLINI (MILANO)

Introduction. The rate of decrease at infinity of the Fourier transform of the characteristic function χ of a compact set C has been studied by several authors under various regularity assumptions on ∂C (see [6], [7], [10] and [11]). If C is also convex, then there exist precise estimates, depending on the Gauss curvature, of the behavior of $\widehat{\chi}$ at infinity (see [6] and [11]). In this paper we consider the non-convex N-dimensional case. We produce an asymptotic estimate for $\widehat{\chi}(x)$ as $x \to \infty$. Such an estimate depends on the number of points of the boundary "having normal in the same direction". The estimate holds for a certain direction if that number is finite. More precisely, let $C \subset \mathbb{R}^N$ be a compact set which is the closure of its interior points and whose boundary ∂C is a manifold of class $[N/2]+5$. Consider the normal map $\vec{n} : \partial C \to S_N$, where $S_N = \{\theta \in \mathbb{R}^N : |\theta| = 1\}$, and an open set $A \subset S_N$. Suppose there exist q functions $\sigma_j : A \to \partial C$ of class $[N/2]+4$ such that:

(a) the sets $\sigma_j(A)$ are pairwise disjoint;

(b) for every $\theta \in A$ the Gauss curvature at $\sigma_j(\theta)$ is different from zero;

(c) for every $\theta \in A$ the points $\sigma_1(\theta), \ldots, \sigma_q(\theta)$ are the only points of ∂C having normal in direction θ.

Our main results are the following:

THEOREM 1. *Let C satisfy the above conditions. Let χ be the characteristic function of C and $\widehat{\chi}$ be its Fourier transform. Then, for every compact set $K \subset A$, $\theta \in K$ and $r > 0$,*

$$(1)\qquad \widehat{\chi}(r\theta) = -\frac{1}{2\pi i} r^{-(N+1)/2} \times \sum_{j=1}^{q} \exp[-2\pi i r\theta\sigma_j(\theta) + \Gamma(\sigma_j(\theta))\pi i/4] K^{-1/2}(\sigma_j(\theta)) + E_r$$

where $\Gamma(\sigma_j(\theta))$ is the signature of the first fundamental form of the surface ∂C at $\sigma_j(\theta)$, $K(\sigma_j(\theta))$ is the absolute value of the Gauss curvature at $\sigma_j(\theta)$

and $|E_r| \le M_K r^{-N/2-1}$ *for a suitable constant* M_K *depending on* K *but not on* r *and* θ.

As a consequence of Theorem 1 we can obtain precise estimates for the Lebesgue constants, on the torus T^N, associated with C.

THEOREM 2. *Let* C *be a compact subset of* $\mathbb{R}^N$,

$$D_\tau^C(x) = \sum_{m\in Z^N\cap\tau C} \exp[2\pi i m x]$$

be the Dirichlet kernel with respect to C *and let*

$$L_\tau^C = \|D_\tau^C\|_{L^1(T^N)} = \int\limits_{T^N} \Big| \sum_{m\in Z^N\cap\tau C} \exp[2\pi i m x]\Big|\, dx$$

be the Lebesgue constant with respect to C. *Then if* C *satisfies the same assumptions as in Theorem* 1, *there exist positive constants* C_1 *and* C_2 *such that*

$$C_1\tau^{(N-1)/2} \le L_\tau^C \le C_2\tau^{(N-1)/2}$$

for τ *sufficiently large.*

We use the method of stationary phase in the N-dimensional case for the estimate of oscillatory integrals. General references for this method are [4], [9] and [12].

Proof of the theorems. Let $\mathbb{R}^n$ denote the n-dimensional euclidean space and T^n the n-dimensional torus. If Ω is an open set in $\mathbb{R}^n$ and X a subset of $\mathbb{R}^k$ we denote by $\mathcal{C}^m(\Omega, X)$ the set of all functions from $\Omega\times X$ to some $\mathbb{R}^h$ having m continuous derivatives with respect to the first n variables. $\mathcal{C}_c^m(\Omega, X)$ will denote the set of all functions in $\mathcal{C}^m(\Omega, X)$ with compact support in $\Omega\times X$. If f is a twice differentiable function, let $H_f(x)$ denote the matrix $[\partial^2 f(x)/\partial x_i\partial x_j]$ and let $\delta_f(x)$ denote the signature of the quadratic form associated with $H_f(x)$.

LEMMA 1. *Let* $\theta_0\in\mathbb{R}^k$, $U(\theta_0)$ *be a neighborhood of* θ_0, $f\in\mathcal{C}^m(\Omega, U(\theta_0))$ *and* $g\in\mathcal{C}_c^{m-1}(\Omega, U(\theta_0))$ (*we suppose* $m\ge 1$). *If* $\|\nabla_x f(x,\theta)\|$ *is bounded away from zero for every* $(x,\theta)\in\operatorname{supp} g$, *then there exists a constant* M *independent of* θ *and* λ *such that*

$$\Big|\int\limits_\Omega \exp[-2\pi i\lambda f(x,\theta)] g(x,\theta)\, dx\Big| \le M\lambda^{-m+1} \tag{2}$$

for every θ *in a suitable neighborhood* $\widetilde{U}(\theta_0)$.

LEMMA 2. *Let* $\theta_0\in\mathbb{R}^k$, $U(\theta_0)$ *be a neighborhood of* θ_0, $g\in\mathcal{C}_c^m(\mathbb{R}^n, U(\theta_0))$ *and* $m = [(n+|l|)/2]+1$, *where* l *is a multi-index. Then there exists a* *constant M independent of θ and λ such that*

$$\Big|\int_{\mathbb{R}^n} \exp\Big[-2\pi i\lambda \sum \pm x_j^2\Big] x^l g(x,\theta)\,dx\Big| \le M\lambda^{-(n+|l|)/2} \tag{3}$$

for every θ in a suitable neighborhood $\widetilde{U}(\theta_0)$.

Proofs for Lemmas 1 and 2 when the functions involved are independent of the parameter θ can be found in the literature. See for example [12] (Proposition 4, p. 316 for Lemma 1, and p. 320, formula (2.4) for Lemma 2). A careful reading of the proofs shows that the estimates are uniform with respect to the parameter θ.

Lemma 3 (Morse's lemma). *Let $U(\theta_0)$ be a neighborhood of $\theta_0 \in \mathbb{R}^k$, $m \ge 2$, Ω be an open subset of $\mathbb{R}^n$ containing the origin and let $f \in \mathcal{C}^m(\Omega, U(\theta_0))$ be such that $\nabla_x f(0,\theta) = 0$ for every $\theta \in U(\theta_0)$. Suppose moreover that the matrix $H_f(0,\theta)$ is non-singular. Then there exist neighborhoods $V(0)$ and $\widetilde{U}(\theta_0)$ and a diffeomorphism $F : V(0) \times \widetilde{U}(\theta_0) \to \Omega$, $F \in \mathcal{C}^{m-2}(V(0), \widetilde{U}(\theta_0))$, depending on the parameter θ, such that*

$$f(F(v,\theta),\theta) = \sum \pm v_j^2 + f(0,\theta) \tag{4}$$

for every $v \in V$ and $\theta \in \widetilde{U}(\theta_0)$. Moreover, the Jacobian of the diffeomorphism F at the point $(0,\theta)$ is given by $|\det H_f(0,\theta)|^{-1/2}$ and the quadratic form on the right hand side of (4) has the same signature as the matrix $H_f(0,\theta)$.

In the original version of Morse's lemma the function f does not depend on the parameter θ. Using the original version we can only ensure that, for every fixed θ, there exist a neighborhood $V(0)$ and a function F defined on $V(0)$, both depending on θ, such that (4) holds. But the local inverse theorem implies that the neighborhood in which the inverse function exists depends continuously on the derivative of the function. A careful reading of the proof of Morse's lemma shows that, if θ belongs to a suitable neighborhood $\widetilde{U}(\theta_0)$, then $V(0)$ can be chosen independent of θ, and $F \in \mathcal{C}^{m-2}$. For the proof of Morse's lemma see for example [8], p. 6.

Lemma 4. *Let $U(\theta_0)$ be a neighborhood of $\theta_0 \in \mathbb{R}^k$, Ω be an open subset of $\mathbb{R}^n$, $f \in \mathcal{C}^m(\Omega, U(\theta_0))$ and $g \in \mathcal{C}_c^{m-1}(\Omega, U(\theta_0))$, with $m \ge [(n+1)/2]+5$. Suppose that there exists a continuous function $\phi : U(\theta_0) \to \Omega$ such that for every $\theta \in U(\theta_0)$:*

1) $\nabla_x f(\phi(\theta),\theta) = 0$ *and the matrix $H_f(\phi(\theta),\theta)$ is non-singular*;
2) $\nabla_x f(x,\theta) \ne 0$ *for $x \ne \phi(\theta)$.*

Then there exist a constant M, independent of θ and λ, and a neighborhood

$\widetilde{U}(\theta_0)$ *such that*

$$\begin{aligned} I &= \int\limits_{\Omega} \exp[-2\pi i\lambda f(x,\theta)]g(x,\theta)\,dx \\ &= \lambda^{-n/2}\exp[-2\pi i\lambda f(\phi(\theta),\theta) + \delta_f(\phi(\theta),\theta)\pi i/4] \\ &\quad \times g(\phi(\theta),\theta)\,|\det H_f(\phi(\theta),\theta)|^{-1/2} + E_\lambda \end{aligned}$$

where $|E_\lambda| \le M\lambda^{-(n+1)/2}$ *for every* θ *in* $\widetilde{U}(\theta_0)$.

Proof. Let $B(\phi(\theta), r) \subset \Omega$ be the ball of center $\phi(\theta)$ and radius r. By a proper choice of r and $U(\theta_0)$ we may assume that $B(\phi(\theta), r) \subset \Omega$ for every $\theta \in U(\theta_0)$. Let $\xi \in C_0^\infty(\mathbb{R}^n)$ such that $\xi(x) = 1$ for $|x| \le r/2$ and $\xi(x) = 0$ for $|x| \ge r$. Then $I = I_1 + I_2$ where

$$I_1 = \int\limits_{B(\phi(\theta),r)} \exp[-2\pi i\lambda f(x,\theta)]g(x,\theta)\xi(x-\phi(\theta))\,dx$$

and

$$I_2 = \int\limits_{\Omega} \exp[-2\pi i\lambda f(x,\theta)]g(x,\theta)[1-\xi(x-\phi(\theta))]\,dx\,.$$

Since $\nabla_x f(x,\theta)$ is bounded away from zero on the support of $g(x,\theta)[1 - \xi(x-\phi(\theta))]$, applying Lemma 1 to I_2, we obtain $I_2 \le M_1\lambda^{-m+1}$. Let us consider the integral I_1. By the change of variable $z = x - \phi(\theta)$ we obtain

$$I_1 = \int\limits_{B(0,r)} \exp[-2\pi i\lambda f(z+\phi(\theta),\theta)]g(z+\phi(\theta),\theta)\xi(z)\,dz\,.$$

Since $\nabla_x f(\phi(\theta),\theta) = 0$ we can apply Lemma 3 to the function f. If we choose r and $U(\theta_0)$ sufficiently small, then setting $z = F(v)$, I_1 becomes

$$\begin{aligned} I_1 = {} & \exp[-2\pi i\lambda f(\phi(\theta),\theta)] \\ & \times \int\limits_{G(\theta)} \exp\Big[-2\pi i\lambda \sum \pm v_j^2\Big] g(\phi(\theta)+F(v),\theta)\xi(F(v))J(F)\,dv \end{aligned}$$

where $G(\theta) = F^{-1}(B(0,r),\theta)$ and $J(F)$ is the Jacobian of F. Let $h(x,\theta) = g(\phi(\theta) + F(v),\theta)\xi(F(v))J(F)$ and observe that $h \in \mathcal{C}_{\mathrm{c}}^{m-3}(G(\theta), U(\theta_0))$. Since $G(\theta)$ depends continuously on θ we may suppose, provided that we restrict $U(\theta_0)$, that $G(\theta) \subset Q(0,\varrho)$, where $Q(0,\varrho)$ is a cube of side 2ϱ centered at the origin and ϱ is independent of θ. Let

$$I_3 = \int\limits_{Q(0,\varrho)} \exp\Big[-2\pi i\lambda \sum \pm v_j^2\Big] h(v,\theta)\,dv\,.$$

Then $I_1 = \exp[-2\pi i f(\phi(\theta),\theta)]I_3$. We choose $\beta \in C_0^\infty(\mathbb{R})$ such that $\beta(t) = 1$ for $|t| \le \varrho/2$ and $\beta(t) = 0$ for $|t| \ge \varrho$ and $B(v) = \beta(v_1)\beta(v_2)\ldots\beta(v_n)$. So we can write $I_3 = I_4 + I_5$ where

$$I_4 = \int\limits_{Q(0,\varrho)} \exp\Big[-2\pi i\lambda \sum \pm v_j^2\Big] h(v,\theta)B(v)\,dv$$

and

$$I_5 = \int\limits_{Q(0,\varrho)} \exp\Big[-2\pi i\lambda \sum \pm v_j^2\Big] h(v,\theta)[1-B(v)]\,dv\,.$$

Lemma 1 is applicable to the integral I_5 and so $|I_5| \le M_2\lambda^{-m+1}$. For the integral I_4 we write $h(x,\theta) = h(0,\theta)+\sum_k v_k h_k(v,\theta)$, for suitable $h_j \in \mathcal{C}^{m-4}$, and we split I_4 into the sum $I_4 = h(0,\theta)I_6 + \sum_k I_k'$ where

$$I_6 = \int\limits_{Q(0,\varrho)} \exp\Big[-2\pi i\lambda \sum \pm v_j^2\Big] B(v)\,dv$$

and

$$I_k' = \int\limits_{Q(0,\varrho)} \exp\Big[-2\pi i\lambda \sum \pm v_j^2\Big] v_k h_k(v,\theta)B(v)\,dv\,.$$

We have

$$I_6 = \prod_{j=1}^{n} \int\limits_{-\varrho}^{\varrho} \exp[\pm 2\pi i\lambda t^2]\beta(t)\,dt$$

but

$$\begin{aligned}\int\limits_{-\varrho}^{\varrho} &\exp[\pm 2\pi i\lambda t^2]\beta(t)\,dt\\ &= \int\limits_{-\varrho}^{\varrho} \exp[\pm 2\pi i\lambda t^2]\,dt \int\limits_{-\varrho}^{\varrho} \exp[\pm 2\pi i\lambda t^2][1-\beta(t)]\,dt\\ &= \frac{1}{\sqrt{2\lambda}}\exp[\pm\pi i/4] + O(\lambda^{-1})\end{aligned}$$

(see [1] for details) and so

$$I_6 = 2^{-n/2}\lambda^{-n/2}\exp[\delta_f(\phi(\theta),\theta)\pi i/4] + O(\lambda^{-(n+1)/2})$$

(remember that the quadratic form $\sum \pm x_j^2$ has the same signature as the matrix $H_f(\phi(\theta),\theta)$). Applying Lemma 2 to the integrals I_k' we obtain $|I_k'| \le M_3\lambda^{-(n+1)/2}$. Finally,

$$I_1 = 2^{-n/2}\lambda^{-n/2}\exp[-2\pi i\lambda f(\phi(\theta),\theta) + \delta_f(\phi(\theta),\theta)\pi i/4]h(0,\theta) + E_\lambda$$

where $|E_\lambda| \le M_4\lambda^{-(n+1)/2}$ for a suitable constant M_4 independent of λ and θ.

Proof of Theorem 1. Clearly it suffices to prove the estimate (1) in a suitable neighborhood of every $\theta \in A$. We choose $\theta_0 \in A$ and consider a neighborhood $U(\theta_0)$. Let $h \in C_0^\infty(\mathbb{R}^N)$ be such that $h(x) = 1$ for $|x| \le a/2$ and $h(x) = 0$ for $|x| \ge a$. If $h_j(x,\theta) = h(x - \sigma_j(\theta))$ we can choose a and $U(\theta_0)$ so that the supports of h_j are pairwise disjoint. Set $h_0(x,\theta) = 1 - \sum_{j=1}^q h_j(x,\theta)$. Then, by the divergence theorem,

$$\widehat{\chi}(r\theta) = \int_C \exp[-2\pi i r\theta x]\,dx = -\frac{1}{2\pi i r}\int_{\partial C} \exp[-2\pi i r\theta x]\theta\vec{n}(x)\,dS$$

$$= -\frac{1}{2\pi i r}\sum_{j=0}^{q}\int_{\partial C} \exp[-2\pi i r\theta x]\theta\vec{n}(x)h_j(x,\theta)\,dS\,.$$

Let

$$I_j = \int_{\partial C} \exp[-2\pi i r\theta x]\theta\vec{n}(x)h_j(x,\theta)\,dS\,.$$

We shall estimate separately I_0 and I_j for $j > 0$. Let ξ_k be a partition of unity such that the support of every ξ_k lies in a part of the surface with a representation $\phi : \Omega \subset \mathbb{R}^{N-1} \to \mathbb{R}^N$. Let $h_{0k} = h_0\xi_k$ and consider the integral

$$I_{0k} = \int_{\Omega} \exp[-2\pi i r\theta x]h_{0k}(\phi(u),\theta)\,\theta\vec{n}(\phi(u))\,\frac{\partial S}{\partial u}\,du$$

where $\partial S/\partial u$ is the surface element of ∂C. Applying Lemma 1 we obtain $I_{0k} \le M_1 r^{-(N+1)/2}$ and so $I_0 \le M_2 r^{-(N+1)/2}$. Consider now the integrals I_j. We may suppose, by a suitable choice of the parameter a in the definition of the function h, that the support of h_j lies in a part of the surface having a representation $\phi : \Omega \subset \mathbb{R}^{N-1} \to \mathbb{R}^N$. So

$$I_j = \int_{\Omega} \exp[-2\pi i r\theta\phi(u)]h_j(\phi(u),\theta)\theta\vec{n}(\phi(u))\frac{\partial S}{\partial u}\,du.$$

Let us observe that Lemma 4 is applicable to the integrals I_j since $\nabla_u\theta\phi(u) = 0$ means that θ has the same direction as the normal to the surface ∂C at $\phi(u)$. Moreover, the condition of $H_{\theta\phi}$ being non-singular is satisfied since the Gauss curvature is not zero. So

$$I_j = r^{-(N-1)/2}\exp[-2\pi i r\theta\sigma_j(\theta) + \Gamma(\sigma_j(\theta))\pi i/4]\frac{\partial S}{\partial u}$$
$$\times\,|\det\theta H_\phi(\phi^{-1}(\sigma_j(\theta)))|^{-1/2} + E_r$$

where $|E_r| \le M_3 r^{-N/2}$ for every $\theta \in U(\theta_0)$. Since $(\det\theta H_\phi)[\partial S/\partial u]^{-2}$ is the Gauss curvature we obtain (1).

Using Theorem 1 we can now extend Theorem 1 of [1] to the N-dimensional case.

LEMMA 5. *Let $C \subset \mathbb{R}^N$ satisfy the same assumptions as in Theorem* 1. *Then if $\widehat{\psi}_\tau$ is the Fourier transform of the characteristic function of τC, there exist a measurable set $F_\varepsilon \subset T^N$ and positive constants M_ε (depending on ε), M_1 and M_2 (independent of ε) such that*

1) $\int\limits_{F_\varepsilon} |\widehat{\psi}_\tau(x)|\,dx \ge M_1\varepsilon^{(N-1)/2}\tau^{(N-1)/2} - M_\varepsilon\tau^{N/2-1}$,

2) $\operatorname{meas} F_\varepsilon \le M_2\varepsilon^N$.

Proof. Let $U(\theta_0)$ be a neighborhood in which Theorem 1 is applicable and $x = |x|\theta$ be such that $\theta \in U(\theta_0)$. Then

$$\begin{aligned}\widehat{\psi}_\tau(x) &= \int\limits_{\tau C} \exp[-2\pi ixy]\,dy = \tau^N \int\limits_{C} \exp[-2\pi i\tau xy]\,dy\\ &= -\frac{1}{2\pi i}\tau^{(N-1)/2}|x|^{-(N+1)/2}\\ &\quad\times\sum_{j=1}^{q}\exp[-2\pi i\tau x\theta\sigma_j(\theta)+\Gamma(\sigma_j(\theta))\pi i/4]K^{-1/2}(\sigma_j(\theta))+\tau^N E_{\tau|x|}\,.\end{aligned}$$

Set $A_j = K^{-1/2}(\sigma_j(\theta))$ and $B_j = \exp[-2\pi i\tau|x|\theta\sigma_j(\theta)+\Gamma(\sigma_j(\theta))\pi i/4]$. Then

$$\widehat{\psi}_\tau(x) = \frac{\tau^{(N-1)/2}}{2\pi i}|x|^{-(N+1)/2}\{A_1\exp[B_1] + \ldots + A_q\exp[B_q]\} + \tau^N E_{\tau|x|}\,.$$

Let Γ be the cone with vertex at the origin such that $\Gamma \cap S_N = U(\theta_0)$. We choose a cube $F \subset \Gamma$ with sides parallel to the axes and set $F_\varepsilon = \varepsilon F$. Since $|x| \le M_3\varepsilon$ for all $x \in F_\varepsilon$, we have

$$\begin{aligned}\int\limits_{F_\varepsilon} |\widehat{\psi}_\tau(x)|\,dx &\ge M_4\tau^{(N-1)/2}\varepsilon^{-(N-1)/2}\\ &\quad\times\int\limits_{F_\varepsilon} |A_1\exp[B_1] + \ldots + A_q\exp[B_q]|\,dx - M_\varepsilon\tau^{N/2-1}\,.\end{aligned}$$

Arguing as in [1] (p. 238) we claim that there exists a positive constant M_5 such that for every $\varepsilon > 0$ sufficiently small and for every τ sufficiently large

$$\int\limits_{F_\varepsilon} |A_1\exp[B_1] + \ldots + A_q\exp[B_q]|\,dx \ge M_5 \operatorname{meas} F_\varepsilon\,.$$

Let ε_n and τ_n be as in [1]. The proof follows in the same way as in [1] if we can show that

$$\frac{1}{\operatorname{meas} F_{\varepsilon_n}} \int\limits_{F_{\varepsilon_n}} A_j\exp[B_j - B_1]\,dx$$

tends to zero. If we change variable and put

$$G(y) = y[\sigma_j(\theta) - \sigma_1(\theta)]$$

the integral becomes

$$(5)\quad \frac{1}{\operatorname{meas} F}\int\limits_{F} A_j \exp[-2\pi i\tau_n\varepsilon_n G(y)+\Gamma(\sigma_j(\theta))\pi i/4-\Gamma(\sigma_1(\theta))\pi i/4]\,dx\,.$$

Observe that

$$\frac{\partial G}{\partial y_k}=e_k(\sigma_j(\theta)-\sigma_1(\theta))+y\frac{\partial\sigma_j}{\partial y_k}-y\frac{\partial\sigma_1}{\partial y_k}$$

where $\{e_k\}$ is the standard basis of $\mathbb{R}^N$. But $y\partial\sigma_j/\partial y_k=y\partial\sigma_1/\partial y_k=0$ since y is normal to the surface and the $\partial\sigma_j/\partial y_k$ are tangent. So $\partial G/\partial y_k=e_k(\sigma_j(\theta)-\sigma_1(\theta))$. Since $\sigma_j(\theta)\neq\sigma_1(\theta)$ we may suppose $\nabla G\neq 0$. Integration by parts shows that (5) tends to zero.

Proof of Theorem 2. The upper estimate is contained in [15]. As for the lower estimate, arguing as in [1] and [2] and using Lemma 5 we have

$$\begin{aligned}L_\tau^C &\geq \int\limits_{F_\varepsilon}|\widehat{\psi}_\tau(x)|\,dx-(\operatorname{meas} F_\varepsilon)^{1/2}\Big(\int\limits_{\mathbb{R}^N}|\widehat{\chi}(x)|^2\,dx\Big)^{1/2}\\ &\geq M_1\varepsilon^{(N-1)/2}\tau^{(N-1)/2}-M_\varepsilon\tau^{N/2-1}-M_2\varepsilon^{N/2}\Big(\int\limits_{\mathbb{R}^N}|\widehat{\chi}(x)|^2\,dx\Big)^{1/2}\end{aligned}$$

and, since the Minkowski upper measure of ∂C is bounded (see [15] for a definition),

$$\begin{aligned}L_\tau^C &\geq M_1\varepsilon^{(N-1)/2}\tau^{(N-1)/2}-M_\varepsilon\tau^{N/2-1}-M_3\varepsilon^{N/2}\tau^{(N-1)/2}\\ &=\tau^{(N-1)/2}\varepsilon^{(N-1)/2}(M_1-M_3\varepsilon^{1/2})-M_\varepsilon\tau^{N/2-1}\,.\end{aligned}$$

Choosing ε such that $M_1-M_3\varepsilon^{1/2}>0$ for τ sufficiently large we have
$L_\tau^C\geq M_4\tau^{(N-1)/2}-M_5\tau^{N/2-1}=\tau^{(N-1)/2}(M_4-M_5\tau^{-1/2})\geq M_6\tau^{(N-1)/2}$.
An analogous extension is possible for Theorem 2 of [1] (see also Theorem A of [3]).

Remark. Only recently have I found, in the Proceedings of the Steklov Institute of Mathematics 180 (1989), 176–177, the announcement, with no proof, of a sharper version of Theorem 2 due to I. R. Liflyand.

REFERENCES

[1] L. Brandolini, *Estimates for Lebesgue constants in dimension two*, Ann. Mat. Pura Appl. (4) 156 (1990), 231–242.

[2] M. Carenini and P. M. Soardi, *Sharp estimates for Lebesgue constants*, Proc. Amer. Math. Soc. 89 (1983), 449–452.

[3] D. I. Cartwright and P. M. Soardi, *Best conditions for the norm convergence of Fourier series*, J. Approx. Theory 38 (1983), 344–353.

[4] E. T. Copson, *Asymptotic Expansions*, The University Press, Cambridge 1965.

[5] S. Giulini and G. Travaglini, *Sharp estimates for Lebesgue constants on compact Lie groups*, J. Funct. Anal. 68 (1986), 106–110.

[6] C. S. Herz, *Fourier transforms related to convex sets*, Ann. of Math. (2) 75 (1962), 81–92.

[7] W. Littman, *Fourier transforms of surface-carried measures and differentiability of surface averages*, Bull. Amer. Math. Soc. 69 (1963), 766–770.

[8] J. Milnor, *Morse Theory*, Princeton University Press, 1963.

[9] F. W. J. Olver, *Introduction to Asymptotics and Special Functions*, Academic Press, New York 1974.

[10] B. Randol, *On the Fourier transform of the indicator function of a planar set*, Trans. Amer. Math. Soc. 139 (1969), 271–278.

[11] —, *On the asymptotic behavior of the Fourier transform of the indicator function of a convex set*, ibid., 279–285.

[12] E. M. Stein, *Oscillatory integrals in Fourier analysis*, in: Beijing Lectures in Harmonic Analysis, Ann. of Math. Stud. 112, Princeton University Press, 1986, 307–355.

[13] —, *Problems in harmonic analysis related to curvature and oscillatory integrals*, in: Proc. Internat. Congress Math. 1986, Vol. I, 196–221.

[14] A. A. Yudin and V. A. Yudin, *Discrete imbedding theorems and Lebesgue constants*, Math. Notes 22 (1977), 702–711.

[15] V. A. Yudin, *Behavior of Lebesgue constants*, ibid. 17 (1975), 233–235.

DIPARTIMENTO DI MATEMATICA DELL'UNIVERSITÀ
VIA C. SALDINI, 50
20133 MILANO, ITALY

Reçu par la Rédaction le 23.9.1992