COLLOQUIUM MATHEMATICUM

VOL. LXV 1993 FASC. 1

AN EXTREMAL SET OF UNIQUENESS?

BY

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Let $\mathbb{T}$ denote the group $[0,1)$ with addition modulo one, let $\mathbb{Z}$ denote the integers, and let E be a subset of $\mathbb{T}$. E is a *set of uniqueness* if the only trigonometric series $\sum_{n=-\infty}^{\infty} c(n)e^{2\pi inx}$ on $\mathbb{T}$ which converges to zero for all x outside E is the zero series; E is an *H-set* if there exists a nonempty open interval I in $\mathbb{T}$ such that $N(E;I)=\{n\in\mathbb{Z}\mid nx\notin I \text{ for all } x\in E\}$ is infinite; E is a *Dirichlet set* if $N(E;(\varepsilon,1-\varepsilon))$ is infinite for all $\varepsilon>0$. Let $E^{(0)}=E$ and inductively define $E^{(n)}$ as the set of limit points of $E^{(n-1)}$. If there exists a positive integer n such that $E^{(n)}$ is empty then E has *finite* (Cantor–Bendixson) *rank*; in this case, the least such integer n is the *rank* of E.

Every finite subset of $\mathbb{T}$ is a Dirichlet set [3], every Dirichlet set is clearly an H-set, and every H-set is a set of uniqueness [5]. Cantor [2] showed that any set of finite rank is a set of uniqueness, and a similar argument shows that every countable closed set E in $\mathbb{T}$ is a set of uniqueness [4, p. 32]. By a result of W. H. Young [7], the hypothesis that E is closed can be deleted without changing the conclusion. For an introduction to the vast literature on sets of uniqueness see [1], [8], and [4].

The purpose of this note is an elementary construction of a closed set S of rational numbers in $\mathbb{T}$ which necessarily is a set of uniqueness, but which cannot be expressed as the union of two H-sets. We conjecture, moreover, that S is not the union of a finite number of H-sets. In this case, S would be extremal among the closed subsets of $\mathbb{T}$ which are expressible as a countable union of H-sets. (By a nonconstructive argument [4, pp. 127–128], it is known that possibly uncountable extremal sets of this type exist.) The extremality of S, consequently, would provide insight into the long-standing problem of characterizing the closed sets of uniqueness in $\mathbb{T}$.

Given x in $\mathbb{T}$, let $x=\sum_{k=0}^{\infty} x_k 2^{-k}$, $x_k\in\{0,1\}$, denote its binary expansion, and write $x=x_0.x_1x_2x_3\ldots$; this expression for x is unique if the terminating expansion is chosen whenever possible. Let $S_{-1}=\{0\}$ and, for each nonnegative integer n, let S_n signify the set of all $x=x_0.x_1x_2x_3\ldots$ in $\mathbb{T}$ such that $\sum_{j=0}^{\infty} x_j=n+1$ and $x_j=0$ if $0\le j\le n$.

THEOREM. *The set* $S = \bigcup_{n=-1}^{\infty} S_n$ *is a closed set of rational numbers in* $\mathbb{T}$ *whose rank is infinite and which cannot be expressed as the union of two* H*-sets.*

Proof. By construction, S consists of rational points. To see that it is closed, let $\{x^{(k)}\}$ be a sequence of points from S with $x^{(k)} \to x$ as $k \to \infty$. If there exists a positive integer n such that infinitely many points of $\{x^{(k)}\}$ belong to $\bigcup_{j=-1}^{n} S_j$, then x belongs to this closed set; if no such integer n exists then $x = 0$. In either case, x belongs to S.

It is not hard to see that S has infinite rank; for this purpose, corresponding to each nonnegative integer n, define a mapping Π_n from $\mathbb{T}$ into $\mathbb{T}$ by $\Pi_n(x_0.x_1x_2x_3\ldots) = y_0.y_1y_2y_3\ldots$ where $y_j = 0$ if $0 \le j \le n$ and $y_j = x_j$ if $j \ge n+1$. By convention, $\Pi_{-1} = 0$. It is easy to verify that $S_n^{(1)} = \bigcup_{j=-1}^{n-1} \Pi_n(S_j)$ and $(\Pi_n(S_k))^{(1)} = \Pi_n(S_k^{(1)})$ for all $n \ge k \ge 0$. Induction then yields $S_n^{(n)} = \{0\} \cup \Pi_n(S_0)$, and consequently $S^{(n)} \supseteq S_n^{(n)} \ne \emptyset$, for each $n \ge 0$.

Suppose, by way of contradiction, that $S = E \cup F$ where E and F are H-sets. Then there exist integers r, μ, and ν, where $r \ge 2$ and $\mu, \nu \in [1, 2^r - 1)$, and infinite sequences of positive integers $m_1 < n_1 < m_2 < n_2 < \ldots$, with $n_k/m_k \to \infty$ as $k \to \infty$, such that $m_k x \notin (\mu 2^{-r}, (\mu+1)2^{-r})$ and $n_k y \notin (\nu 2^{-r}, (\nu+1)2^{-r})$ for all $x \in E$, $y \in F$, and integers $k \ge 1$. Fix a positive integer k and let r_k be the nonnegative integer such that $2^{r_k} \le m_k < 2^{r_k+1}$; without loss of generality, $r_k \ge 3r+4$. Let $l = l(k)$ denote the smallest positive integer such that the real number $t_*^{(k)} = (4\mu + l)2^{-(r_k+r+3)}$ belongs to $(\mu 2^{-r} m_k^{-1}, (\mu+1)2^{-r} m_k^{-1})$, and let $T(k; t_*^{(k)})$ denote the set of all points in $\mathbb{T}$ of the form

$$t_*^{(k)} + \sum_{j=r_k+r+4}^{\infty} t_j 2^{-j}$$

where $t_j \in \{0,1\}$ for all j and $\sum_{j=r_k+r+4}^{\infty} t_j \le r_k - r - 2$. Note that $T(k; t_*^{(k)})$ is contained in $S \cap (\mu 2^{-r} m_k^{-1}, (\mu+1)2^{-r} m_k^{-1})$; since $E \cap (\mu 2^{-r} m_k^{-1}, (\mu+1)2^{-r} m_k^{-1})$ is empty, it follows that $T(k; t_*^{(k)})$ is a subset of F. Observe that $\{t \in \mathbb{T} \mid n_j t \in (\alpha, \beta)\} \cap F$ is empty for all $j \ge 1$, where $\alpha = \nu 2^{-r}$ and $\beta = (\nu+1)2^{-r}$; in particular,

$$\{t \in \mathbb{T} \mid n_k t \in (\alpha, \beta)\} \cap T(k; t_*^{(k)}) = \emptyset\,. \tag{1}$$

We assert that (1) implies

$$n_k / 2^{r_k} < 2^{r+7}\,, \tag{2}$$

in contradiction to $n_k/m_k \to \infty$, which would establish the theorem. In order to prove (2), let λ_k be the nonnegative integer satisfying $2^{\lambda_k} \le n_k < 2^{\lambda_k+1}$. Let $m = m(k)$ denote the largest integer such that the real number

$(\beta+m)n_k^{-1}$ does not exceed $t_*^{(k)}$, let $m' = m'(k)$ denote the largest integer such that $(\alpha+m')2^{-(\lambda_k+r+1)}$ is less than $(\beta+m)n_k^{-1}$, and let $m'' = m''(k)$ denote the smallest integer such that $(\alpha+m'+m'')2^{-(\lambda_k+r+1)}$ is greater than $(\alpha+m+1)n_k^{-1}$. The definitions of m'' and λ_k yield

$$\begin{aligned}(\alpha+m+1)n_k^{-1} &< (\alpha+m'+m'')2^{-(\lambda_k+r+1)} \\ &= (\alpha+m'+m''-1+1)2^{-(\lambda_k+r+1)} \\ &< (\alpha+m+1)n_k^{-1} + 2^{-r}n_k^{-1} = (\beta+m+1)n_k^{-1},\end{aligned}$$

that is,

$$(\alpha+m'+m'')2^{-(\lambda_k+r+1)} \in ((\alpha+m+1)n_k^{-1}, (\beta+m+1)n_k^{-1}). \tag{3}$$

The definition of m implies $(\beta+m)n_k^{-1} \le t_*^{(k)} < (\beta+m+1)n_k^{-1}$, and $t_*^{(k)} \notin ((\alpha+m+1)n_k^{-1}, (\beta+m+1)n_k^{-1})$ by (1); consequently,

$$(\beta+m)n_k^{-1} \le t_*^{(k)} \le (\alpha+m+1)n_k^{-1}. \tag{4}$$

Using (4) and the definitions of m' and m'',

$$\begin{aligned}m' &< \alpha+m' < 2^{\lambda_k+r+1}(\beta+m)n_k^{-1} \le 2^{\lambda_k+r+1}t_*^{(k)} \\ &\le 2^{\lambda_k+r+1}(\alpha+m+1)n_k^{-1} < \alpha+m'+m'' < m'+m''+1.\end{aligned}$$

From the definitions of λ_k and m',

$$\begin{aligned}(\alpha+m'+2^{r+1}+1)2^{-(\lambda_k+r+1)} &\ge (\beta+m)n_k^{-1} + 2^{-\lambda_k} \ge (\beta+m+1)n_k^{-1} \\ &> (\alpha+m+1)n_k^{-1},\end{aligned}$$

and thus $m'' \le 2^{r+1}+1$. Again from the definitions of m' and m'',

$$\begin{aligned}(\alpha+m')2^{-(\lambda_k+r+1)} &< (\beta+m)n_k^{-1} < (\alpha+m+1)n_k^{-1} \\ &< (\alpha+m'+m'')2^{-(\lambda_k+r+1)}\end{aligned}$$

so that $m'' > 0$. In summary,

$$2^{\lambda_k+r+1}t_*^{(k)} \in (m', m'+m''+1), \quad \text{and} \quad m'' \in (0, 2^{r+1}+2).$$

Suppose that (2) is violated. Then the integer $2^{\lambda_k+r+1}t_*^{(k)}$ is equal to $m'+s$ for some integer $s \in [1, m'']$. However,

$$(\alpha+m'+m'')2^{-(\lambda_k+r+1)} = t_*^{(k)} + (\alpha+m''-s)2^{-(\lambda_k+r+1)},$$

where $(\alpha+m''-s)2^{-(\lambda_k+r+1)}$ has at most $2r$ ones in its binary expansion and at least λ_k-2 leading zeros, and it follows that $(\alpha+m'+m'')2^{-(\lambda_k+r+1)}$ belongs to $T(k; t_*^{(k)})$. But this, together with (3), contradicts (1). ∎

The fact that S has infinite rank is necessary for our conjecture that S is not a finite union of H-sets. Indeed, by an argument of Salinger [6], if E is a subset of $\mathbb{T}$ with finite rank n then E is the union of at most 2^n Dirichlet sets.

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Reçu par la Rédaction le 19.2.1992;
en version modifiée le 15.10.1992