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LOWER BOUNDS FOR THE SOLUTIONS IN THE SECOND CASE OF FERMAT'S EQUATION WITH PRIME POWER EXPONENTS

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Let p be an odd prime, and let n be a positive integer. Further, let x, y, z be integers satisfying

$$x^{p^n}+y^{p^n}=z^{p^n},\quad p\,|\,xyz,\quad 0<x<y<z,\quad \gcd(x,y)=1\,. \tag{1}$$

Recently, Zhong [2] proved that $y>p^{3np^n-n}/2$ and $z-x>p^{3np^n-n-1}/4$. In this note we partly improve the above result as follows:

THEOREM. *If* $p\equiv 3 \pmod 4$, *then* $y>p^{6np^n-3n^2-2n+3}/2^{1/p^n}$ *and* $z-x>p^{6np^n-3n^2-3n+3}/2^{1-1/p^n}$.

P r o o f. It is a well known fact that (1) is impossible for $p=3$, so we may assume that $p>3$.

We first deal with the case that $p\,|\,x$. Let $p^\alpha\,\|\,x$. Then from (1) we get

$$z-y=p^{\alpha p^n-n}x_0^{p^n}\,, \tag{2}$$

$$\frac{z^{p^i}-y^{p^i}}{z^{p^{i-1}}-y^{p^{i-1}}}=px_i^{p^n}\,,\qquad i=1,\ldots,n\,, \tag{3}$$

where $x_0,x_1,\ldots,x_n$ are positive integers satisfying $p\nmid x_0x_1\ldots x_n$ and

$$x=p^\alpha x_0x_1\ldots x_n\,. \tag{4}$$

For any coprime integers X, Y, by the proof of the Theorem in [1], we find that if $p\equiv 3 \pmod 4$ then $(X^p-Y^p)/(X-Y)=A^2+pB^2$, where A, B are integers satisfying $\gcd(A,B)=1$ and $A\equiv 0 \pmod{(X-Y)}$. Hence, by (3), we have

$$\frac{z^{p^i}-y^{p^i}}{z^{p^{i-1}}-y^{p^{i-1}}}=A_i^2+pB_i^2=px_i^{p^n}\,,\qquad i=1,\ldots,n\,,$$

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whence we get

$$B_i^2 + p\left(\frac{A_i}{p}\right)^2 = x_i^{p^n}\,, \quad i = 1, \ldots, n\,, \tag{5}$$

where A_i, B_i $(i = 1, \ldots, n)$ are integers satisfying $\gcd(A_i, B_i) = 1$ and

$$A_i \equiv 0 \pmod{z^{p^{i-1}} - y^{p^{i-1}}}\,, \quad i = 1, \ldots, n\,.$$

Further, by (2), A_i/p $(i = 1, \ldots, n)$ are integers satisfying

$$\frac{A_i}{p} \equiv 0 \pmod{p^{\alpha p^n - n + i - 2}}\,, \quad i = 1, \ldots, n\,. \tag{6}$$

Notice that $p > 3$ and the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$ is less than p. By an argument similar to the proof of the Theorem in [1], we see from (5) that there exist integers X_i, Y_i $(i = 1, \ldots, n)$ satisfying

$$x_i = X_i^2 + pY_i^2\,, \quad \gcd(X_i, Y_i) = 1\,, \quad i = 1, \ldots, n\,, \tag{7}$$

and

$$B_i + \frac{A_i}{p}\sqrt{-p} = (X_i + Y_i\sqrt{-p})^{p^n}\,, \quad i = 1, \ldots, n\,. \tag{8}$$

From (8),

$$\frac{A_i}{p} = p^n Y_i \sum_{j=0}^{(p^n-1)/2} (-1)^j \binom{p^n}{2j+1} p^{j-n} X_i^{p^n-2j-1} Y_i^{2j}\,, \quad i = 1, \ldots, n\,. \tag{9}$$

Notice that if $p > 3$ and $j > 0$, then $j > (\log(2j+1))/\log p$ and

$$\binom{p^n}{2j+1} p^{j-n} = \binom{p^n - 1}{2j} \frac{p^j}{2j+1} \equiv 0 \pmod{p}\,.$$

Since $p \nmid x_i$ $(i = 1, \ldots, n)$, we have $p \nmid X_i$ $(i = 1, \ldots, n)$ by (7), and hence

$$Y_i \equiv 0 \pmod{p^{\alpha p^n - 2n + i - 2}}\,, \quad i = 1, \ldots, n\,, \tag{10}$$

by (6) and (9). Since $x_i > 1$ $(i = 1, \ldots, n)$, we have $Y_i \neq 0$ $(i = 1, \ldots, n)$ by (7). Thus, we obtain

$$x_i > p^{2\alpha p^n - 4n + 2i - 3}\,, \quad i = 1, \ldots, n\,,$$

by (7) and (10), and hence

$$x > p^{\alpha + \Sigma_{i=1}^n (2\alpha p^n - 4n + 2i - 3)} = p^{2\alpha n p^n - 3n^2 - 2n + \alpha} \tag{11}$$

by (4). Notice that $\alpha \geq 3$ by [2]. We get $x > p^{6np^n - 3n^2 - 2n + 3}$ by (11).

Using the same method, we can prove that $y > p^{6np^n - 3n^2 - 2n + 3}$ and $z > p^{6np^n - 3n^2 - 2n + 3}$ correspond to $p \,|\, y$ and $p \,|\, z$ respectively. Thus, $y >$

$p^{6np^n-3n^2-2n+3}/2^{1/p^n}$ since $2^{1/p^n}y > z$. Simultaneously, we have

$$z - x = \frac{y^{p^n}}{z^{p^n-1} + xz^{p^n-2} + \ldots + x^{p^n-1}} > \frac{y^{p^n}}{p^n z^{p^n-1}}$$

$$> \frac{y^{p^n}}{p^n (2^{1/p^n} y)^{p^n-1}} = \frac{y}{2^{1-1/p^n} p^n} > p^{6np^n-3n^2-3n+3}/2^{1-1/p^n}\,.$$

The theorem is proved.

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