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ON A GENERAL BIDIMENSIONAL EXTRAPOLATION PROBLEM

BY

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TO THE MEMORY OF PROFESSOR ANTONI ZYGMUND

Several generalized moment problems in two dimensions are particular cases of the general problem of giving conditions that ensure that two isometries, with domains and ranges contained in the same Hilbert space, have commutative unitary extensions to a space that contains the given one. Some results concerning this problem are presented and applied to the extension of functions of positive type.

I. The problem and some related results

Notation and definitions. We say that V is an isometry that acts in a Hilbert space E if $V \in \mathcal{L}(D, R)$ is a unitary operator such that its domain D and its range R are closed subspaces of E.

In this paper the following notation is kept: E is a Hilbert space and, for $j = 1, 2, V_j$ denotes an isometry that acts in E, with domain D_j and range R_j, and defect subspaces N_j and M_j, the orthogonal complements in E of D_j and R_j, respectively.

We shall say that (U_1, U_2, F) is a *commutative unitary extension* of the pair of isometries (V_1, V_2) that act in E if:

(i) $U_1, U_2 \in \mathcal{L}(F)$ are unitary operators in a Hilbert space F such that E is a closed subspace of F;

(ii) $U_1 U_2 = U_2 U_1$;

(iii) $U_j|_{D_j} = V_j$, $j = 1, 2$.

The extension is *minimal* if, moreover,

(iv) $F = \bigvee\{U_1^m U_2^n E : m, n \in \mathbb{Z}\}$.

(As usual, $F = \bigvee\{\ldots\}$ means that F is the smallest closed subspace such that $F \supset \{\ldots\}$.)

Let $\mathcal{U}$ be the family of all the (U_1, U_2, F) such that (i) to (iv) hold, modulo the following equivalence relation: $(U_1, U_2, F) \approx (U_1', U_2', F')$ in $\mathcal{U}$ if there exists a unitary operator $\tau \in \mathcal{L}(F, F')$ such that its restriction to E is the identity and $\tau U_j = U'_j \tau$ for $j = 1, 2$.

Abstract moment problems. It happens that in several generalized moment problems in $\mathbb{Z}^2$ a Hilbert space E and a pair of isometries (V_1, V_2) that act in it appear naturally, in such a way that the problem has solutions iff $\mathcal{U}$ is nonvoid and, moreover, there is a bijection between the set of all solutions and $\mathcal{U}$. This solution is the bidimensional extension of a fundamental and well known approach to moment problems, as can be seen in [Sa] and [C]. In fact, the last paper suggested the title of this one, where the extrapolation problem is the one of giving conditions for $\mathcal{U}$ to be nonvoid.

The partial results that we have obtained ensure the existence of distinguished elements of $\mathcal{U}$, apparently related with the existence of maximal entropy solutions of moment problems. Let $P_H^G \equiv P_H$ denote the orthogonal projection of the Hilbert space G onto its closed subspace H, and set

$$\mathcal{U}_j = \{(U_1, U_2, F) \in \mathcal{U} : U_j \in \mathcal{L}(F) \text{ is a unitary dilation of } V_j P_{D_j} \in \mathcal{L}(E)\}, \qquad j = 1, 2\,.$$

(Recall that $U \in \mathcal{L}(F)$ is a *unitary dilation* of a contraction $T \in \mathcal{L}(E)$ if U is a unitary operator in $F \supset E$ such that $P_E^F U^n|_E = T^n$ for every $n \geq 0$.) In the next section we shall prove the following extension of a result given in [Ar1]:

THEOREM A. *The following properties are equivalent*:

(a) $P_{R_2}(V_1 P_{D_1})^n (V_2 P_{D_2}) = (V_2 P_{D_2})(V_1 P_{D_1})^n P_{D_2}$, $n = 1, 2, \ldots$;
(b) $\mathcal{U}_1 \neq \emptyset$.

On the extension of functions of positive type. For $0 \leq a, b \leq \infty$ set $\varrho^{(a,b)} = \{(m,n) \in \mathbb{Z}^2 : |m| \leq a,\ |n| \leq b\}$ and $\varrho_+^{(a,b)} = \{(m,n) \in \varrho^{(a,b)} : m, n \geq 0\}$. We consider functions of positive type

$$k : \varrho^{(a,b)} \to \mathcal{L}(G)$$

where G is a Hilbert space. That is, if A is the space of functions with finite support from $\mathbb{Z}^2$ to G and $A^{(a,b)}$ is the set of functions in A with support in $\varrho_+^{(a,b)}$, then

$$\sum \{\langle k(s-t)h(s), h(t)\rangle_G : s, t \in \varrho_+^{(a,b)}\} \geq 0, \qquad \forall h \in A^{(a,b)}\,.$$

In order to avoid technical details, we also assume that $k(0,0) = I$.

We denote by $\mathcal{K}$ the set of all positive type extensions $K : \mathbb{Z}^2 \to \mathcal{L}(G)$ of k and consider the problem of giving conditions on k that ensure that $\mathcal{K}$ is nonvoid, which is a version of a well known problem of Krein [K]. Our approach to this problem is based on associating with k a Hilbert space E and two isometries, V_1 and V_2, that act in E.

The vector space $A^{(a,b)}$ and the positive semidefinite sesquilinear form given by

$$(h,h') \to \sum\{\langle k(s-t)h(s), h'(t)\rangle_G : s,t \in \varrho_+^{(a,b)}\}\,, \qquad \forall h,h' \in A^{(a,b)}\,,$$

generate a Hilbert space $E \equiv E^{(a,b)} \equiv E_k^{(a,b)}$ such that $A^{(a,b)}$ is naturally associated with a dense subspace of it. Also, since G can obviously be identified with $A^{(0,0)}$, we may assume that $G \subset E$. Let S_1 and S_2 be the natural shifts in A, i.e., $(S_1h)(m,n) \equiv h(m-1,n)$ and $(S_2h)(m,n) \equiv h(m,n-1)$; restricting S_1 to $A^{(a-1,b)}$ and S_2 to $A^{(a,b-1)}$, we get two isometries $V_1 = V_1^{(a,b)}$ and $V_2 = V_2^{(a,b)}$ that act in E. The construction of E, V_1 and V_2 from k is nothing but the one of Naimark's famous dilation theorem, and can be repeated for any $K \in \mathcal{K}$. In this way it is seen that there exists a bijection from the set $\mathcal{U}$, defined by means of V_1 and V_2 as above, and $\mathcal{K}$ which associates with each $(U_1,U_2,F) \in \mathcal{U}$ the function $K : \mathbb{Z}^2 \to \mathcal{L}(G)$ given by $K(m,n) = P_G^F U_1^m U_2^n|_G$. Thus, as a consequence of Theorem A we get the following:

Theorem B. *Let $k : \varrho^{(a,b)} \to \mathcal{L}(G)$ be a function of positive type such that $k(0,0) = I$ and that given any $v \in G$, n an integer in $[0,b)$ and $\varepsilon > 0$ there exists $h' \in A^{(a-1,b-1)}$ with the property that $h = S_1^a S_2^n v + h'$ satisfies $\sum\{\langle k(s-t)h(s), h(t)\rangle_G : s,t \in \varrho_+^{(a,b)}\} < \varepsilon$. Then $\mathcal{K}$ is nonvoid.*

The proof of Theorem B, its relations with known results and some complements will be given in Section III.

Remark on lifting theorems. Some applications of results concerning the extrapolation problem we are considering to bidimensional versions of the Nagy–Foiaş commutant lifting theorem are given in [Ar1] and [Ar2]. The last is closely related to Ando's theorem on the existence of a commutative unitary dilation of a commutative pair of contractions [An]. In Section IV we point out some connections between our subject and Ando's theorem.

II. On the existence of commutative unitary extensions of isometries

Proof of Theorem A. (i) Let $(U_1,U_2,F) \in \mathcal{U}_1$. Since $P_E^F U_1^n|_E = (V_1P_{D_1})^n$, $n = 1,2,\ldots$, we see that

$$\begin{aligned} P_{R_2}(V_1P_{D_1})^n(V_2P_{D_2}) &= P_{R_2}^E P_E^F U_1^n V_2 P_{D_2} = P_{R_2}^F U_1^n U_2 P_{D_2} = P_{R_2}^F U_2 U_1^n P_{D_2} \\ &= P_{R_2}^F U_2(P_{D_2}^F \oplus P_{F\ominus D_2}^F)U_1^n P_{D_2} = P_{R_2}^F U_2 P_{D_2}^F U_1^n P_{D_2} \\ &= V_2 P_{D_2}^E(P_E^F U_1^n|_E)P_{D_2} = (V_2P_{D_2})(V_1P_{D_1})^n P_{D_2}\,, \end{aligned}$$

so (b) implies (a).

(ii) Conversely, assume that (a) holds. Let $W_1 \in \mathcal{L}(F_1)$ be the essentially unique minimal unitary dilation of $V_1 P_{D_1} \in \mathcal{L}(E)$. For (U_1, U_2, F) to belong to $\mathcal{U}_1$ it is necessary that $U_2 W_1^m d = U_2 U_1^m d = U_1^m U_2 d = W_1^m V_2 d$, $\forall d \in D_2$, $m \in \mathbb{Z}$. So, we set $\widetilde{D}_2 = \bigvee\{W_1^m D_2 : m \in \mathbb{Z}\}$ and in $\widetilde{D}_2 \subset F_1$ we define an operator $\widetilde{V}_2$ by setting $\widetilde{V}_2 W_1^m d = W_1^m V_2 d$, $\forall d \in D_2$, $m \in \mathbb{Z}$. If $d_1, \ldots, d_k \in D_2$ and $m_1, \ldots, m_k \in \mathbb{Z}$, (a) shows that

$$\begin{aligned}
&\Big\| \sum\{W_1^{m_j} V_2 d_j : 1 \le j \le k\}\Big\|^2 \\
&= \sum\{\langle P_{R_2} W_1^{m_j - m_{j'}} V_2 d_j, V_2 d_{j'}\rangle : 1 \le j, j' \le k,\ m_j \ge m_{j'}\} \\
&\quad + \sum\{\langle V_2 d_j, P_{R_2} W_1^{m_{j'} - m_j} V_2 d_{j'}\rangle : 1 \le j, j' \le k,\ m_j < m_{j'}\} \\
&= \sum\{\langle P_{R_2} (V_1 P_{D_1})^{m_j - m_{j'}} (V_2 P_{D_2}) d_j, V_2 d_{j'}\rangle : 1 \le j, j' \le k,\ m_j \ge m_{j'}\} \\
&\quad + \sum\{\langle V_2 d_j, P_{R_2} (V_1 P_{D_1})^{m_{j'} - m_j} (V_2 P_{D_2}) d_{j'}\rangle : 1 \le j, j' \le k,\ m_j < m_{j'}\} \\
&= \sum\{\langle (V_1 P_{D_1})^{m_j - m_{j'}} d_j, d_{j'}\rangle : 1 \le j, j' \le k,\ m_j \ge m_{j'}\} \\
&\quad + \sum\{\langle d_j (V_1 P_{D_1})^{m_{j'} - m_j} d_{j'}\rangle : 1 \le j, j' \le k,\ m_j < m_{j'}\} \\
&= \Big\| \sum\{W_1^{m_j} d_j : 1 \le j \le k\}\Big\|^2 .
\end{aligned}$$

Thus $\widetilde{V}_2$ is a well defined isometry in $\widetilde{D}_2$. Clearly, $W_1 \widetilde{D}_2 = \widetilde{D}_2$ and $\widetilde{V}_2 W_1 f = W_1 \widetilde{V}_2 f$, $\forall f \in \widetilde{D}_2$.

(iii) Replacing E, V_1 and V_2 by F_1, W_1 and $\widetilde{V}_2$ we may assume that the isometries V_1 and V_2 acting in E are such that V_1 is a unitary operator such that $V_1 D_2 = D_2$ and $V_1 V_2 f = V_2 V_1 f$, $\forall f \in D_2$. Let N be the orthogonal complement of D_2 in E; thus, V_1 commutes with P_N and P_{D_2}. Set $F' = D_2 \oplus N \oplus \ldots \oplus N \oplus \ldots$ and let $V_1', V_2' \in \mathcal{L}(F')$ be defined by $V_1'(d, n_1, \ldots) = (V_1 d, V_1 n_1, \ldots)$ and $V_2'(d, n_1, n_2, \ldots) = (P_{D_2} V_2 d, P_N V_2 d, n_1, n_2, \ldots)$; clearly, V_1' is a unitary operator, V_2' is an isometry and $V_1' V_2' = V_1' V_2'$. Then it is known (see [N–F]) that there exist two commuting unitary operators that extend V_1', V_2' to a space containing F'. The proof of Theorem A is complete. ∎

(1) Corollary. *If $D_2 \cup R_2 \subset D_1$, $V_1(D_2 \cup R_2) \subset D_1, \ldots, V_1^n(D_2 \cup R_2) \subset D_1, \ldots$ then the following properties are equivalent*:

(a) $P_{R_2} V_1^n V_2 P_{D_2} = V_2 P_{D_2} V_1^n P_{D_2}$, $n = 1, 2, \ldots$;
(b) $\mathcal{U}_1 \neq \emptyset$;
(c) $\mathcal{U} \neq \emptyset$.

P r o o f. It is enough to see that (c) implies (a). In fact, if $(U_1, U_2, F) \subset \mathcal{U}$, then

$$P_{R_2}V_1^nV_2P_{D_2} = P_{R_2}U_1^nU_2P_{D_2} = P_{R_2}U_2U_1^nP_{D_2} = P_{R_2}U_2V_1^nP_{D_2}$$
$$= (P_{R_2}V_2P_{D_2} + P_{R_2}U_2P_{N_2})V_1^nP_{D_2} = P_{R_2}V_2P_{D_2}V_1^nP_{D_2}\,. \ \blacksquare$$

(2) Corollary. *If $D_2 \cup R_2 \subset D_1$ and $V_1D_2 \subset D_2$ then the following properties are equivalent*:

(a) $V_1V_2d = V_2V_1d$, $\forall d \in D_2$;
(b) $\mathcal{U}_1 \neq \emptyset$;
(c) $\mathcal{U} \neq \emptyset$.

Proof. (c) implies (a), which in turn implies $V_1^nV_2d = V_2V_1^nd$, for every $n \geq 1$ and $d \in D_2$, as is seen by induction. In fact, from the last equality it follows that $V_1^nV_2d \in R_2 \subset D_1$ and (since $V_1D_2 \subset D_2 \subset D_1$ shows that $V_1^nd \in D_2$)

$$V_1^{n+1}V_2d = V_1(V_1^nV_2d) = (V_1V_2)V_1^nd = V_2V_1^{n+1}d\,. \ \blacksquare$$

(3) Proposition. *Let D be a closed subspace of E such that $D \subset D_1 \cap D_2$, $V_1D \subset D_2$, $V_2D \subset D_1$ and $V_1V_2|_D = V_2V_1|_D$. Then $\mathcal{U} \neq \emptyset$ whenever one of the following equalities holds*:

$$D = D_2, \quad D = D_1, \quad V_1D = D_2, \quad V_2D = D_1\,.$$

Proof. If $D = D_2$, the assertion is a straightforward consequence of (2). If $V_1D = D_2$, set $D_1' = V_1D_1$, $V_1' = V_1^{-1}$ and $D' = V_1D$; then, by the previous case, with V_1', V_2 and D' instead of V_1, V_2 and D, the result follows. $\blacksquare$

III. Applications to a problem of Krein

Proof of Theorem B. The property of k that is assumed in the statement of Theorem B is the same as saying that $E^{(a,b-1)}$ equals the domain $D_1^{(a,b-1)} \equiv E^{(a-1,b-1)}$ of $V_1^{(a,b-1)}$. Now, the last holds iff $V_1^{(a,b-1)}$ is unitary, as follows from the consideration of the unitary operator $J = J^{(a,b)} \in \mathcal{L}(E^{(a,b)})$ given by $J(S_1^mS_2^nv) \equiv S_1^{a-m}S_2^{b-n}v$, because $JD_j = R_j$, $j = 1, 2$. Consequently, Theorem B follows from

(1) Proposition. *If there exists an integer $s \in [0, b)$ such that $V_1^{(a,s)}$ is unitary, then*:

(i) *$\mathcal{K}$ is nonvoid*;
(ii) *there exists only one function $k' : \varrho^{(\infty,s)} \to \mathcal{L}(G)$ of positive type that extends $k|_{\varrho^{(a,s)}}$.*

Proof. (i) We know that it is enough to prove that $\mathcal{U}$ is nonvoid. The hypothesis implies that $V_1 \equiv V_1^{(a,b)}$ is a unitary operator in $E \equiv E^{(a,b)}$ and that the same happens with its restriction $V_1^{(a,b-1)}$, so $E^{(a,b-1)} =$

$D_1^{(a,b-1)} \equiv E^{(a-1,b-1)}$; thus, $D := E^{(a-1,b-1)}$ equals $D_2 \equiv D_2^{(a,b)}$. Then Proposition (II.3) may be applied, because $V_1V_2d = V_2V_1d$ holds for every $d \in \bigvee\{V_1^mV_2^nv : 0 \le m < a,\ 0 \le n < b,\ v \in G\} \equiv E^{(a-1,b-1)}$.

(ii) Let $K \in \mathcal{K}$ and let k' be its restriction to $\varrho^{(\infty,s)}$. We have to show that k' is well determined by k, i.e., by V_1 and V_2. Now, $E^{(\infty,s)} \equiv E^{(\infty,s)}(k')$ contains $E = E^{(a,s)}$ as a closed subspace. Since $V_1^{(a,s)}$ is unitary, $D_1^{(a,s)} \equiv E^{(a-1,s)} = E^{(a,s)}$ and it follows that $D_1^{(a,s)} = E^{(a+k,s)}$ for $k = 0, 1, \ldots$ Then, for $v, w \in G$ and $m \ge 0$, we have $\langle k'(m,n)v, w\rangle_G \equiv \langle k'(-m,-n)^*v, w\rangle_G = \langle V_1^mV_2^nv, w\rangle_G$ if $0 \le n \le s$ and $\langle k'(m,n)v, w\rangle_G \equiv \langle k'(-m,-n)^*v, w\rangle_G = \langle V_1^mv, V_2^{-n}w\rangle_G$ if $0 \ge n \ge -s$. ∎

Remark on related results. If, in the statement of Proposition (1), $s = 0$, then (ii) is equivalent to $V_1^{(a,s)}$ being unitary. Thus, Theorem B extends the following result [Ar1]: if there exists only one extension of positive type $k' : \mathbb{Z} \to \mathcal{L}(G)$ of the one-dimensional restriction $k|_{\varrho^{(a,0)}}$ of k then $\mathcal{K} \ne \emptyset$. The last result is the discrete version of a theorem of Livsic on the continuous Krein problem (see [B]) and it extends a theorem of Devinatz [D], according to which if both one-dimensional restrictions of k ($k|_{\varrho^{(a,0)}}$ and $k|_{\varrho^{(0,b)}}$) have only one extension of positive type then $\mathcal{K} \ne \emptyset$.

A more general condition. Assume the hypothesis of Proposition (1). Its proof shows that $\mathcal{U}_1 \ne \emptyset$. Then Theorem A implies that

$$\begin{aligned}(2)\qquad &\langle (V_1P_{D_1})^rV_1^mV_2^{n+1}v, V_2^{n'+1}w\rangle_E = \langle (V_1P_{D_1})^rV_1^mV_2^nv, V_2^{n'}w\rangle_E\,,\\ &\qquad\qquad \forall r \ge 0,\ 0 \le m \le a,\ 0 \le n, n' < b,\ v, w \in G\,.\end{aligned}$$

Now, the restriction $K|_{\varrho^{(\infty,b)}}$ of any $K \in \mathcal{K}$ given by $K(m,n) \equiv P_G^FU_1^mU_2^n|_G$ with $(U_1, U_2, F) \in \mathcal{U}_1$ is completely determined by k. In fact, if $0 \le n \le b$, then $K(m,n) = P_G^EP_E^FU_1^mU_2^n|_G = P_G^E(P_E^{F_1}W_1^m)V_2^n|_G$, with $W_1 \in \mathcal{L}(F_1)$ the minimal unitary dilation of $V_1P_{D_1} \in \mathcal{L}(E)$. That is, $K|_{\varrho^{(\infty,b)}}$ is the same as the function $K_1 : \varrho^{(\infty,b)} \to \mathcal{L}(G)$ defined by

$$(3)\qquad \begin{aligned}K_1(m,n) = K_1(-m,-n)^* &= P_G^E(V_1P_{D_1})^mV_2^n|_G\\ &\qquad \text{if } (m,n) \in \varrho_+^{(\infty,b)}\,,\\ K_1(m,n) = K_1(-m,-n)^* &= P_G^E(V_1P_{D_1})^{*-m}V_2^n|_G\\ &\qquad \text{if } (-m,-n) \in \varrho_+^{(\infty,b)}\,.\end{aligned}$$

The extension K_1 of a function of positive type $k : \varrho^{(a,b)} \to \mathcal{L}(G)$ is well defined even if $\mathcal{K} = \emptyset$. If K_1 itself is of positive type, then, considering K_1 instead of k, we define the isometries $V_1^{(\infty,b)}$ and $V_2^{(\infty,b)}$ that act in $E^{(\infty,b)}$, and $V_1^{(\infty,b)}$ is unitary; thus, by Corollary (II.2), there exist two commutative unitary operators $U_1, U_2 \in \mathcal{L}(F)$ such that $F \supset E^{(\infty,b)}$, $U_1|_{E^{(\infty,b)}} = V_1^{(\infty,b)}$

and $U_2|_{D_2^{(\infty,b)}} = V_2^{(\infty,b)}$. Thus $K(m,n) \equiv P_G^F U_1^m U_2^n|_E$ is such that $K \in \mathcal{K}$ and $K|_{\varrho^{(\infty,b)}} = K_1$.

If there exists $(U_1, U_2, F) \in \mathcal{U}_1$ then

$$(4) \qquad \langle K_1(r+m-m', n-n')v, w\rangle_G = \langle (V_1 P_{D_1})^r V_1^m V_2^n v, V_1^{m'} V_2^{n'} w\rangle_E \\ \forall r \geq 0,\ 0 \leq m, m' \leq a,\ 0 \leq n, n' \leq b,\ v, w \in G\,.$$

In fact, under such conditions we have

$$\begin{aligned} \langle P_G^F U_1^{r+m-m'} U_2^{n-n'} v, w\rangle_G &= \langle U_1^r U_1^m U_2^n v, U_1^{m'} U_2^{n'} w\rangle_F \\ &= \langle (P_E^F U_1^r|_E) V_1^m V_2^n v, V_1^{m'} V_2^{n'} w\rangle_G \\ &= \langle (V_1 P_{D_1})^r V_1^m V_2^n v, V_1^{m'} V_2^{n'} w\rangle_E\,. \end{aligned}$$

Conversely, if (4) holds, the definition (3) of K_1 shows that also (2) holds. Summing up:

(5) LEMMA. *K_1 is of positive type $\Leftrightarrow$ (4) holds $\Leftrightarrow$ $\mathcal{U}_1 \neq \emptyset \Rightarrow\ \exists K \in \mathcal{K}$ that extends K_1.*

So we get the following extension of Theorem B.

(6) THEOREM C. *Let $k : \varrho^{(a,b)} \to \mathcal{L}(G)$ be a function of positive type such that $k(0) = I$.*

(a) *If one of the restrictions $k|_{\varrho^{(a,0)}}$ and $k|_{\varrho^{(0,b)}}$ has only one extension of positive type to $\mathbb{Z}$, then the hypothesis of Theorem* B *holds.*

(b) *Let E, V_1 and V_2 be the Hilbert space and the isometries associated with k. If the hypothesis of Theorem* B *holds, the extension $K_1 : \varrho^{(\infty,b)} \to \mathcal{L}(G)$ of k defined by* (3) *is of positive type.*

(c) *K_1 is of positive type iff* (4) *holds.*

(d) *If K_1 is of positive type, then $\mathcal{K} \neq \emptyset$ and there exists $K \in \mathcal{K}$ such that K extends K_1.*

IV. On the relations with a theorem of Ando

A similar approach to another extension problem. Let us now consider another problem concerning the existence of extensions of positive type of a given function. Set $\mathbb{Z}_+^2 = \{(m,n) \in \mathbb{Z}^2 : m, n \geq 0\}$ and let $h : \mathbb{Z}_+^2 \to \mathcal{L}(G)$ be a given function such that $h(0,0) = I$. We want to give conditions that ensure that the set $\mathcal{H}$ of positive type extensions $H : \mathbb{Z}^2 \to \mathcal{L}(G)$ of h is nonvoid.

If $\exists H \in \mathcal{H}$, Naimark's dilation theorem says that there exists a unitary representation $W : \mathbb{Z}^2 \to \mathcal{L}(F)$ such that $F \supset G$, $H(m,n) \equiv P_G^F W(m,n)|_G$ and $F = \bigvee\{W(m,n)G : (m,n) \in \mathbb{Z}^2\}$. Defining $h_1, h_2 : \mathbb{Z} \to \mathcal{L}(g)$ by setting, for every $m \geq 0$,

$$h_1(m) = h_1(-m)^* = h(m,0)\,,$$
$$h_2(m) = h_2(-m)^* = h(0,m)\,,$$

it follows that

$$\begin{aligned}(1)\qquad &\Big|\sum\{\langle h(m,n)v_1(m), v_2(n)\rangle_G : m,n \geq 0\}\Big|^2 \\ &\leq \sum\{\langle h_1(m-n)v_1(m), v_1(n)\rangle_G : m,n\geq 0\} \\ &\times \sum\{\langle h_2(n-m)v_2(m), v_2(n)\rangle_G : m,n\geq 0\} \\ &\text{for all } v_1, v_2 : \{m\in\mathbb{Z} : m\geq 0\} \to G \text{ with finite support}\,,\end{aligned}$$

is a necessary condition on h, which we assume from now on. Then, for $j = 1,2$, h_j is of positive type; let $U_j \in \mathcal{L}(F_j)$ be its minimal unitary dilation. Set also $G_1 = \bigvee\{U_1^m G : m \geq 0\}$, $V_1 = U_1|_{G_1}$, $\widetilde{G}_2 = \bigvee\{U_2^m G : m \leq 0\}$, $\widetilde{V}_2 = U_2^*|_{\widetilde{G}_2}$. If there exists $W : \mathbb{Z}^2 \to \mathcal{L}(F)$ as above it may be assumed that $F \supset F_1, F_2$ and that, if E is the span in F of G_1 and $\widetilde{G}_2$, then

$$(2)\qquad \langle h(m,n)u, w\rangle_G = \langle V_1^m u, \widetilde{V}_2^n w\rangle_E, \qquad \forall m,n \geq 0,\ u,w\in G\,.$$

Now, E can be defined directly from h. In fact, (2) shows that there exists a well determined continuous positive sesquilinear form B in the vector space $G_1 \times \widetilde{G}_2$ such that, for all $m,n,m',n' \geq 0$, $u,w,u',w' \in G$,

$$\begin{aligned}B[(V_1^m u, \widetilde{V}_2^n w), (V_1^{m'} u', \widetilde{V}_2^{n'} w')] = \langle V_1^m u, V_1^{m'} u'\rangle_{G_1} + \langle h(m,n')u, w'\rangle_G \\ + \langle u', h(m',n)w'\rangle_G + \langle \widetilde{V}_2^n w, \widetilde{V}_2^{n'} w'\rangle_{\widetilde{G}_2}\,.\end{aligned}$$

Thus, the vector space $G_1 \times \widetilde{G}_2$ and the form B generate a Hilbert space E such that we may assume that $G_1, \widetilde{G}_2 \subset E$, $G_1 \vee \widetilde{G}_2 = E$, and also $G \subset E$. So we have associated with h a Hilbert space E and two isometries, V_1 and $\widetilde{V}_2$, that act in E, such that (2) holds. Then it is easy to check the following

(3) PROPOSITION. *Let $h : \mathbb{Z}_+^2 \to \mathcal{L}(G)$ be such that $h(0) = 1$ and that* (1) *holds. If E, V_1 and $\widetilde{V}_2$ are the Hilbert space and the isometries acting in it associated with h, there exists a bijection between the set $\mathcal{H}$ of positive type extensions $H : \mathbb{Z}^2 \to \mathcal{L}(G)$ of h and the set $\mathcal{U}$ of minimal commutative unitary extensions of V_1 and $\widetilde{V}_2$. That bijection associates with each $(U_1, \widetilde{U}_2, F) \in \mathcal{U}$ the function $H : \mathbb{Z}^2 \to \mathcal{L}(G)$ given by $H(m,n) \equiv P_G^F U_1^m \widetilde{U}_2^{-n}|_G$. In particular, $\mathcal{H}$ is nonvoid iff $\mathcal{U}$ is nonvoid.*

An example. Let (T_1, T_2) be a commuting pair of contractions in the Hilbert space G. A fundamental theorem due to Ando (see [An] or [N–F]) states that there exists a commuting pair (U_1, U_2) of unitary operators in a Hilbert space $F \supset G$ such that $T_1^m T_2^n = P_G^F U_1^m U_2^n|_G$, $\forall m,n \geq 0$. Following

[C–S], we say that such a (U_1, U_2) is an *Ando dilation* of (T_1, T_2); let $\mathcal{A}$ be the set of all minimal Ando dilations, modulo unitary equivalences, of (T_1, T_2). Define $h : \mathbb{Z}_+^2 \to \mathcal{L}(G)$ by $h(m,n) \equiv T_1^m T_2^n$; then (1) holds and, as before, a Hilbert space E and two isometries acting in E, V_1 and $\widetilde{V}_2$, are associated with the commuting pair of contractions (T_1, T_2). In this case, the definition of the scalar product in E shows that

$$\begin{aligned}\langle (g_1, g_2), (g_1', g_2') \rangle_E = \langle g_1, g_1' \rangle_{G_1} + \langle P_G^{G_1} g_1, g_2' \rangle_{\widetilde{G}_2} + \langle P_G^{G_1} g_1', g_2 \rangle_{\widetilde{G}_2} \\ + \langle g_2, g_2' \rangle_{\widetilde{G}_2}, \qquad \forall (g_1, g_2), (g_1', g_2') \in G_1 \times \widetilde{G}_2 ,\end{aligned}$$

so $P_{\widetilde{G}_2}^E|_{G_1} = P_G^{G_1}$ and $P_{G_1}^E|_{\widetilde{G}_2} = P_G^{\widetilde{G}_2}$.

Proposition (3) yields

(4) Corollary. *Let (T_1, T_2) be a commuting pair of contractions in a Hilbert space G, and V_1 and $\widetilde{V}_2$ the isometries acting in E associated with (T_1, T_2). There is a bijection between $\mathcal{A}$, the set of all minimal Ando dilations modulo unitary equivalences of (T_1, T_2), and $\mathcal{U}$, the set of minimal commutative unitary extensions of V_1 and $\widetilde{V}_2$ such that $(U_1, U_2) \subset \mathcal{L}(F)$ belongs to $\mathcal{A}$ iff (U_1, U_2^*, F) belongs to $\mathcal{U}$.*

In order to be complete, let us sketch in this context the proof of Ando's theorem by means of the commutant lifting theorem (see [C–S]). The last says that $\exists \varrho_2 \in \mathcal{L}(G_1)$ such that $\varrho_2 V_1 = V_1 \varrho_2$ and $P_G^{G_1} \varrho_2 = T_2 P_G^{G_1}$. Let $\mu_2 \in \mathcal{L}(J)$ be the minimal unitary dilation of ϱ_2; then μ_2 is a unitary dilation of T_2, so we may assume that $J \supset \widetilde{G}_2$ and $\mu_2^*|_{\widetilde{G}_2} = \widetilde{V}_2$. We may also assume that $J \supset E$: in fact, $J \supset G_1$ and, for $m, n \geq 0$ and $u, w \in G$,

$$\begin{aligned}\langle V_1^m u, \widetilde{V}_2^n w \rangle_J = \langle V_1^m u, \mu_2^{-n} w \rangle_J = \langle \mu_2^n V_1^m u, w \rangle_J = \langle \varrho_2^n V_1^m u, w \rangle_{G_1} \\ = \langle T_2^n P_G^{G_1} V_1^m u, w \rangle_E = \langle T_2^n T_1^m u, w \rangle_E = \langle V_1^m u, \widetilde{V}_2^n w \rangle_E ,\end{aligned}$$

by (2). Then, by Theorem A and Corollary (4), there exists an Ando dilation $(U_1, U_2) \subset \mathcal{L}(F)$ such that $F \supset J$ and $U_2|_J = \mu_2$ because $P_{V_1 G_1} \mu_2^n V_1 P_{G_1} = V_1 P_{G_1} \mu_2^n P_{G_1}$ for $n = 1, 2, \ldots$

An application of Ando's theorem to Krein's problem. With the notation of Section III, the following is an obvious consequence of Theorem A.

(5) Proposition. *Let $k : \varrho^{(a,b)} \to \mathcal{L}(G)$ be a function of positive type and E, V_1 and V_2 the Hilbert space and the isometries associated with k. If $(V_1 P_{D_1})(V_2 P_{D_2}) = (V_2 P_{D_2})(V_1 P_{D_1})$, then $\mathcal{K}$ is nonvoid.*

Moreover, applying Ando's theorem to the commutative pair $(V_1 P_{D_1}, V_2 P_{D_2})$ we get elements (U_1, U_2, F) of $\mathcal{U}$ such that the corresponding extensions of k have maximum entropy in the sense of [Se]; this remark will be developed elsewhere. Proposition (5) applies, for example, when $k(s) = 0$ whenever $s \neq 0$.

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